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IDENTITIES FOR PRODUCTS AND SQUARES OF FIBONACCI AND LUCAS POLYNOMIALS

Abstract. For products and squares of Fibonacci and Lucas polynomials that generalize Fibonacci and Lucas numbers we describe several identities extending the Cusumano identity for Fibonacci numbers stated long ago as a problem in the journal Fibonacci Quarterly.

1. Fibonacci and Lucas polynomials

The sequence F_0, F_1, F_2, F_3, ... of Fibonacci numbers is defined recursively so that $F_0 = 0$ and $F_1 = 1$ and we require $F_{k+2} = F_{k+1} + F_k$ for every $k \geq 0$. Hence, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, $F_7 = 13$, $F_8 = 21$, $F_9 = 34$, etc.

Quite similar is also the sequence L_0, L_1, L_2, L_3, ... of Lucas numbers when we take $L_0 = 2$ and $L_1 = 1$, while the recursion $L_{k+2} = L_{k+1} + L_k$ for every $k \geq 0$ remains the same. Hence, $L_2 = 3$, $L_3 = 4$, $L_4 = 7$, $L_5 = 11$, $L_6 = 18$, $L_7 = 29$, $L_8 = 47$, $L_9 = 76$, etc.

Since the polynomials (in one variable x with integer coefficients) provide natural extension of integers, the above definitions can be easily modified in order that they hold also for polynomials. For example, the sequence $F_0(x)$, $F_1(x)$, $F_2(x)$, $F_3(x)$, ... of Fibonacci polynomials is defined by $F_0(x) = 0$ and $F_1(x) = 1$ as initial conditions and we require

$$F_{k+2}(x) = x F_{k+1}(x) + F_k(x)$$

for every $k \geq 0$. Hence,

$$\begin{aligned} F_2(x) &= x, \\ F_3(x) &= x^2 + 1, \\ F_4(x) &= x(x^2 + 2), \\ F_5(x) &= x^4 + 3x^2 + 1, \\ F_6(x) &= x(x^2 + 3)(x^2 + 1), \ldots \text{etc.} \end{aligned}$$

It is obvious that $F_k(1) = F_k$ for every $k \geq 0$ (i.e., the value of the k-th Fibonacci polynomial at $x = 1$ is equal to the k-th Fibonacci number). On the other hand, if $x = n$ (natural number), then $F_k(n) = F_k^{(n)}$ for every $k \geq 0$ (i. e., the value of the k-th Fibonacci polynomial at $x = n$ is equal to the k-th term $F_k^{(n)}$ of the n-th metallic Fibonacci number).

In a similar way, the conditions $L_0(x) = 2$, $L_1(x) = x$ and

$$L_{k+2}(x) = x L_{k+1}(x) + L_k(x)$$

for every $k \geq 0$ define Lucas polynomials. Hence,

$$L_2(x) = x^2 + 2,$$
$$L_3(x) = x(x^2 + 3),$$
$$L_4(x) = x^4 + 4x^2 + 2,$$
$$L_5(x) = x(x^4 + 5x^2 + 5),$$
$$L_6(x) = (x^2 + 2)(x^4 + 4x^2 + 1), \ldots \text{etc.}$$

Of course, we have the identity $L_k(1) = L_k$ for every $k \geq 0$.

When for every $k \geq 0$ we require that $f_{k+2}(x) = x\,f_{k+1}(x) - f_k(x)$ and similarly that $\ell_{k+2}(x) = x\,\ell_{k+1}(x) - \ell_k(x)$ (with initial conditions $f_0(x) = 0$, $f_1(x) = 1$ and $\ell_0(x) = 2$, $\ell_1(x) = x$) we shall get so called dual Fibonacci polynomials $f_k(x)$ and dual Lucas polynomials $\ell_k(x)$. Hence,

$$f_2(x) = x,$$
$$f_3(x) = x^2 - 1,$$
$$f_4(x) = x(x^2 - 2),$$
$$f_5(x) = x^4 - 3x^2 + 1,$$
$$f_6(x) = x(x^2 - 3)(x^2 - 1), \ldots \text{etc.}$$

and

$$\ell_2(x) = x^2 - 2,$$
$$\ell_3(x) = x(x^2 - 3),$$
$$\ell_4(x) = x^4 - 4x^2 + 2,$$
$$\ell_5(x) = x(x^4 - 5x^2 + 5),$$
$$\ell_6(x) = (x^2 - 2)(x^4 - 4x^2 + 1), \ldots \text{etc.}$$

We use the following notation: $\Delta = \sqrt{x^2 + 4}$, $\delta = \sqrt{x^2 - 4}$, $A = \frac{x+\Delta}{2}$, $B = \frac{x-\Delta}{2}$, $a = \frac{x+\delta}{2}$, $b = \frac{x-\delta}{2}$. Note the identities $A + B = a + b = x, A - B = \Delta, a - b = \delta, AB = -1$ and $ab = 1$. Using the mathematical induction as was done in the book [2] for Fibonacci and Lucas numbers when $x = 1$ (see also [3]), it is easy to prove that $F_k(x) = \frac{A^k - B^k}{\Delta}$, $L_k(x) = A^k + B^k$, $f_k(x) = \frac{a^k - b^k}{\delta}$ and $\ell_k(x) = a^k + b^k$ for every $k \geq 0$. The right hand sides of the previous identities are well-defined also for negative values of k. Hence, with a little bit of extra care, it is possible to define all of the above polynomials for all integer indices k.

2. Generalization of Cusumano identity

In the Elementary Problem B-896 in the May issue of the journal Fibonacci Quarterly in the year 2000 Andrew Cusumano [1] (see also [4]) proposed the square of the following identity that holds for every integer $i \geq 0$.

(1) $$F_{i+1}F_i - F_{i+1}^2 = -F_i^2 - (-1)^i.$$

We have written it on purpose in the most awkward form with too many minuses, in order to make it easier to recognize that this is a special case of the following identity for Fibonacci and Lucas polynomials

(2) $$F_{i+j}(x)\,F_i(x)\,L_j(x) - F_{i+j}^2(x) = (-1)^j F_i^2(x) - (-1)^i F_j^2(x).$$

This identity holds for all integers i and j, in particular we find (2.1) when $x = 1$ and $j = 1$ since $L_1(x) = L_1(1) = L_1 = 1$ and $(-1)^j = -1$.

Of course, there is a version of the previous formula (2.2) when Lucas numbers have the main role

(3) $$L_{i+j}(x)L_i(x)L_j(x) - L_{i+j}^2(x) = (-1)^j L_i^2(x) + (-1)^i \Delta^2 F_j^2(x).$$

For $j = 1$ and $x = 1$ from (2.3) we obtain the version of the identity (2.1) for Lucas numbers

(4) $$L_{i+1}L_i - L_{i+1}^2 = -L_i^2 + 5\,(-1)^i.$$

It is somewhat surprising that for dual Fibonacci and Lucas polynomials the versions of the identities (2.2) and (2.3) are simpler

(5) $$f_{i+j}(x)\,f_i(x)\,\ell_j(x) - f_{i+j}^2(x) = f_i^2(x) - f_j^2(x),$$

(6) $$\ell_{i+j}(x)\,\ell_i(x)\,\ell_j(x) - \ell_{i+j}^2(x) = \ell_i^2(x) + \delta^2 f_j^2(x).$$

3. Similar identities for products

Let $M_i(x) = F_i(x)\,L_i(x)$ and $m_i(x) = f_i(x)\,\ell_i(x)$.

In this section instead of the squares $F_{i+j}^2(x)$, $L_{i+j}^2(x)$ etc. we consider the products $M_{i+j}(x)$, $M_i(x)$, etc.

The following two identities for (dual) Fibonacci and (dual) Lucas polynomials holds for all integers i and j

(7) $$M_{i+j}(x) = F_{i+j}(x)\,L_i(x)\,L_j(x) - (-1)^j M_i(x) - (-1)^i M_j(x) = \\ = \Delta^2 F_{i+j}(x)\,F_i(x)\,F_j(x) + (-1)^j M_i(x) + (-1)^i M_j(x),$$

$$(8)\quad m_{i+j}(x) = f_{i+j}(x)\,\ell_i(x)\,\ell_j(x) - m_i(x) - m_j(x) = \\ = \delta^2 f_{i+j}(x)\,f_i(x)\,f_j(x) + m_i(x) + m_j(x).$$

For dual Fibonacci and dual Lucas polynomials these identities are again simpler.

4. Identities for more indices

There is a natural question of the existence of similar identities when instead of two indices i and j we consider three, four, five,... integers. The answer is of course affirmative because, for example, when we take four integers a, b, c and d, it is possible to consider the sum $a+b+c+d$ as $(a+b)+(c+d)$ and apply the above formula with $i = a+b$ and $j = c+d$ which will give us one of such identities. Completely different identity emerges when the sum $a+b+c+d$ is represented as $a+(b+c+d)$ and we apply the above formula with $i = a$ and $j = b+c+d$.

If we would like to sell these identities on the mathematical market it would be difficult to achieve good price once the customers know that there are so many of them and that it is very easy to get them. The only thing that can save us is to offer those identities that are particularly nice. Beauty always attracts customers, in spite of the old claim that only beauty is not enough but it brings at least something. Therefore, we now propose six identities for three integers i, j and k that we would always consider as beautiful. We also invite readers to discover for themselves some of those beautiful identities for (dual) Fibonacci and Lucas polynomials.

Let $z = i+j+k$ (the sum of all), $z_i = j+k$ (the sum of all except i), $z_j = k+i$ and $z_k = i+j$, we have the following identity for Fibonacci polynomials

$$(9)\quad L_{z_k}(x)\,F_{z_j}(x)\,F_{z_i}(x) - F_z^2(x) = \\ = (-1)^{z_k}\,F_k^2(x) - (-1)^{z_j}\,F_j^2(x) - (-1)^{z_i}\,F_i^2(x).$$

For Lucas polynomials we have a similar (nice) formula (in two versions)

$$(10)\quad L_{z_k}(x)\,L_{z_j}(x)\,L_{z_i}(x) - L_z^2(x) = \\ = (-1)^{z_k}\,L_k^2(x) + \Delta^2\left[(-1)^{z_j}\,F_j^2(x) + (-1)^{z_i}\,F_i^2(x)\right] = \\ = (-1)^{z_k}\,L_k^2(x) + (-1)^{z_j}\,L_j^2(x) + (-1)^{z_i}\,L_i^2(x) + 8\,(-1)^z.$$

Analogous identities for dual Fibonacci polynomials and dual Lucas polynomials are even nicer (because the powers of -1 have disappeared)

$$(11)\quad \ell_{z_k}(x)\,f_{z_j}(x)\,f_{z_i}(x) - f_z^2(x) = f_k^2(x) - f_j^2(x) - f_i^2(x),$$

$$(12)\quad \ell_{z_k}(x)\,\ell_{z_j}(x)\,\ell_{z_i}(x)-\ell_z^2(x)= \\ =\ell_k^2(x)+\delta^2\left[f_j^2(x)+f_i^2(x)\right]=\ell_k^2(x)+\ell_j^2(x)+\ell_i^2(x)-8.$$

When in the above four identities we consider $i=0$ and then we rename as i the index k, we get the identities (2.2), (2.3), (2.5) and (2.6), respectively. Hence, beautiful identities for three integer indices contain (beautiful) identities for two integer indices as special cases.

We can make similar extension to more indices of the identities for products in section 3

$$(13)\quad M_z(x)= \\ =\Delta^2 F_{z_i}(x)\,F_{z_j}(x)\,F_{z_k}(x)+(-1)^{z_i}M_i(x)+(-1)^{z_j}M_j(x)+(-1)^{z_k}M_k(x) \\ =F_{z_i}(x)\,L_{z_j}(x)\,L_{z_k}(x)+(-1)^{z_i}M_i(x)-(-1)^{z_j}M_j(x)-(-1)^{z_k}M_k(x).$$

Once again, for dual Fibonacci and Lucas polynomials these identities are less complicated

$$(14)\quad m_z(x)=\delta^2 f_{z_i}(x)\,f_{z_j}(x)\,f_{z_k}(x)+m_i(x)+m_j(x)+m_k(x)= \\ =f_{z_i}(x)\,\ell_{z_j}(x)\,\ell_{z_k}(x)+m_i(x)-m_j(x)-m_k(x).$$

5. Proof of (2.2)

Since $AB=-1$ and also $F_k(x)=\frac{A^k-B^k}{\Delta}$ and $L_k(x)=A^k+B^k$ hold for every integer k, notice first that the square $(\Delta F_k(x))^2$ is equal to

$$(A^k-B^k)^2=A^{2k}-2\,(AB)^k+B^{2k}=A^{2k}-2\,(-1)^k+B^{2k}.$$

Let $M=A^{i+j}-B^{i+j}$. For all integers i and j we have

$$\Delta^2\left[F_{i+j}(x)\,F_i(x)\,L_j(x)-F_{i+j}^2(x)\right]=M\left[\left(A^i-B^i\right)\left(A^j+B^j\right)-M\right]= \\ =M\left(A^iB^j-A^jB^i\right)=(AB)^j\left(A^{2i}+B^{2i}\right)-(AB)^i\left(A^{2j}+B^{2j}\right)= \\ =(-1)^j\left(A^{2i}-2(-1)^i+B^{2i}\right)-(-1)^i\left(A^{2j}-2(-1)^j+B^{2j}\right)= \\ =\Delta^2\left[(-1)^jF_i^2(x)-(-1)^iF_j^2(x)\right].$$

Dividing the above identity with Δ^2 we shall get (2.2).

6. Proof of (2.3)

For every integer k the square $L_k(x)^2$ is equal to

$$(A^k + B^k)^2 = A^{2k} + 2\,(AB)^k + B^{2k} = A^{2k} + 2\,(-1)^k + B^{2k}.$$

Let $N = A^{i+j} + B^{i+j}$. For all integers i and j we have

$$\begin{aligned} L_{i+j}(x)\,L_i(x)\,L_j(x) - L_{i+j}^2(x) &= N\left[\left(A^i + B^i\right)\left(A^j + B^j\right) - N\right] = \\ &= N\left(A^i B^j + A^j B^i\right) = (AB)^j\left(A^{2i} + B^{2i}\right) + (AB)^i\left(A^{2j} + B^{2j}\right) = \\ &= (-1)^j\left(A^{2i} + 2(-1)^i + B^{2i}\right) + (-1)^i\left(A^{2j} - 2(-1)^j + B^{2j}\right) = \\ &= (-1)^j L_i^2(x) + (-1)^i \Delta^2 F_j^2(x). \end{aligned}$$

Hence, we can conclude that the proof for the identity (2.3) differs from the proof for the identity (2.2) only in the change of signs of key terms.

The proofs for the identities (2.5), (2.6), (3.1) and (3.2) are similar to the above proofs so that we leave them as exercises to the reader. Because of the identity $a\,b = 1$ it is clear that the number -1 does not appear.

7. Proof of (4.1)

Let $C_k = A^k + B^k$ and $D_k = A^k - B^k$. For all integers i, j and k we have

$$\begin{aligned} \Delta^2\left[L_{z_k}(x)\,F_{z_j}(x)\,F_{z_i}(x) - F_z^2(x)\right] &= C_{z_k}\,D_{z_j}\,D_{z_i} - D_z^2 = \\ &= 2\,(-1)^z + (A\,B)^{z_k}\,C_{2k} - (A\,B)^{z_j}\,C_{2j} - (A\,B)^{z_i}\,C_{2i} = \\ &= (-1)^{z_k}\left(C_{2k} - 2(-1)^k\right) - (-1)^{z_j}\left(C_{2j} - 2(-1)^j\right) - (-1)^{z_i}\left(C_{2i} - 2(-1)^i\right) = \\ &= \Delta^2\left[(-1)^{z_k}\,F_k^2(x) - (-1)^{z_j}\,F_j^2(x) - (-1)^{z_i}\,F_i^2(x)\right]. \end{aligned}$$

Dividing the above identity with Δ^2 will give us (4.1). The identities (4.2), (4.3), (4.4), (4.5) and (4.6) are proved similarly.

References

[1] Andrew Cusumano, *Problem B-896*, Fibonacci Quarterly, **38** 2 (2000), 181.

[2] V. E. Hoggatt, Jr., *Fibonacci and Lucas numbers*, The Fibonacci Association, Santa Clara, 1979.

[3] R. Knott, *Fibonacci numbers and the Golden Section*, Mathematical Gazette, `http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fib.html`.

[4] Kee-Wai Lau, *Solution of Problem B-896*, Fibonacci Quarterly, **39** 1 (2001), 88.

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