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REMARKS ON THE GEOMETRY OF SURFACES IN THE FOUR-DIMENSIONAL MÖBIUS SPHERE

Dedicated to the memory of a friend: Sergio Console

Abstract. We study the conformal geometry of surfaces immersed in the four-dimensional conformal sphere Q_4, viewed as a homogeneous space under the action of the Möbius group. We introduce the classes of $\pm$ isotropic surfaces and characterize them as those whose conformal Gauss map is antiholomorphic or holomorphic. We then relate these surfaces to Willmore surfaces and prove some interesting vanishing results and some bounds on the Euler characteristic of the surfaces. Finally, we characterize $-$ isotropic surfaces through an Enneper-Weierstrass-type parametrization.

Mathematical subject classification: 53A30, 53B25, 53C40, 14M17, 14M15, 32L05

1. Introduction

In recent years, the study of the geometry of submanifolds of the conformal sphere has considerably flourished. The interest in the subject has various motivations spanning from it being a natural extension of the theory of curves and surfaces in the Euclidean space, to its connections with the theory of integrable systems and general relativity. In particular, the theory of Willmore surfaces has seen a great development in many directions. Among the numerous books and papers on this subject, [6] is undoubtedly worth mentioning and we refer the reader to the references therein for a complete and updated bibliography on the subject.

Of all the different possible approaches that have been employed to deal with these topics, we chose Cartan's method of the moving frame because of its flexibility and intuitiveness and because, when dealing with homogeneous spaces, it seems to us to be the fittest.

This paper studies the geometry of surfaces in the conformal 4-sphere Q_4 and it is organized as follows. After a basic introduction on the generalities of the frame reduction procedure, needed to fix the notation, in Section 4.1 we introduce the conformal Grassmannian of 2-planes in $\mathbb{R}^6$ and its Kahler-Lorentzian structure. We also provide a holomorphic embedding of this Grassmann manifold into a quadric in the complex projective space.

In Section 4 we define the conformal Gauss map of a surface in Q_4 and, inspired by [13], we identify a special class of Willmore surfaces, called isotropic surfaces, that we characterize as those surfaces whose conformal Gauss map is holomorphic or antiholomorphic (in what follows, for the sake brevity, we will write "$-$ holomorphic" to mean antiholomorphic and "$+$ holomorphic" instead of holomorphic). This result is stated in Theorem 6.1 and mirrors the well known characterization of Willmore surfaces as those with harmonic conformal Gauss map. This and other concepts and results studied here have been introduced in the study of minimal surfaces in the Riemannian 4-sphere and even in oriented Riemannian 4-manifolds. An interesting paper in this direction, besides the aforementioned [13], is [4].

We then employ some classical techniques such as Cauchy-Riemann inequalities and Carleman-type estimates that, combined with classical index theorems for vector fields and, more generally, for sections of suitable vector bundles, allow us to deduce an upper bound on the Euler characteristic of a compact, non isotropic surface. This result is stated in Theorem 6.2.

In Section 7 we consider the notion of S-Willmore surface, first introduced by Ejiri in [9]. There, the author proved that, in the Riemannian setting, an S-Willmore surface is a Willmore surface; this holds true also in our setting, as proved in Proposition 87. We also prove some vanishing and holomorphicity results that have nice topological consequences.

In the last part of the paper, we show that, roughly speaking, $-$ isotropic surfaces in the conformal 4-sphere are characterized by their conformal Gauss map: in Theorem 8.1 and Theorem 8.2 we establish a bijection between certain $-$ isotropic, weakly conformal branched immersions of a fixed Riemann surface in Q_4 and holomorphic maps, valued in the conformal Grassmannian, that are solutions of a suitable Pfaffian system.

Finally, we point out that some of our results on isotropic surfaces and their Enneper-Weierstrass representations look similar to those in [6]. However, the authors of [6] seem to be working in a slightly different setting and, in particular, their conformal group does not seem to include the inversions. Moreover, our approach, involving the conformal Grassmannian, seems to be more suitable for possible generalizations to surfaces in spheres of greater codimension.

2. The conformal sphere and its submanifolds

Consider $\mathbb{S}^n$ and $\mathbb{R}^n$ with their standard metrics of constant curvatures, and let $\sigma : \mathbb{S}^n \backslash \{N\} \to \mathbb{R}^n$ be the stereographic projection, where $N = (1, 0, \dots, 0) \in \mathbb{R}^{n+1}$ is the north pole. It is well known that σ is a conformal diffeomorphism. If $n \geqslant 3$, by Liouville's theorem ([8], pp.138-141; [12], pp.52-53, [19], pp. 289-290), every conformal diffeomorphism of $\mathbb{S}^n$ is of the form $\sigma^{-1} \circ g \circ \sigma$, where g is a composition of Euclidean similarities of $\mathbb{R}^n$ with possibly the inversion $\mathbb{R}^n \backslash \{0\} \ni x \mapsto x/|x|^2$. The assertion holds even for $n = 2$, although a proof of this fact relies, for instance, on standard compact Riemann surfaces theory since Liouville's theorem is false for $\mathbb{C}$. We observe that the group of conformal diffeomorphisms of the sphere, $\mathrm{Conf}(\mathbb{S}^2)$, can also be identified with the fractional linear transformations of $\mathbb{C}$, either holomorphic or antiholomorphic. From now on, we let $n \geqslant 2$ and we fix the index convention $1 \leqslant A, B, C \leqslant n$. We denote by Q_n the Darboux hyperquadric

$$Q_n = \left\{ (x^0 : x^A : x^{n+1}) \mid \sum_A (x^A)^2 - 2x^0 x^{n+1} = 0 \right\} \subset \mathbb{P}^{n+1}(\mathbb{R}).$$

The Dirac-Weyl embedding $\chi : \mathbb{R}^n \to Q_n$ is defined by

$$\chi : x \longmapsto \left(1 : x : \frac{1}{2}|x|^2\right) \tag{1}$$

and it extends to a diffeomorphism $\chi \circ \sigma : \mathbb{S}^n \to Q_n$ by setting $\chi \circ \sigma(N) = (0 : 0 : 1)$. The advantage of such a representation for the sphere is that every conformal diffeomorphism of $\mathbb{S}^n$ acts as a linear transformation on the homogeneous coordinates of Q_n, so that $\mathrm{Conf}(\mathbb{S}^n)$ can be viewed as the projectivized of the linear subgroup of $GL(n+2)$ preserving the quadratic form that defines the Darboux hyperquadric.

Endow $\mathbb{R}^{n+2}$ with the Lorentzian metric $\langle\ ,\ \rangle$ represented, with respect to the standard basis $\{\eta_0, \eta_A, \eta_{n+1}\}$, by the matrix

$$S = \begin{pmatrix} 0 & 0 & -1 \\ 0 & I_n & 0 \\ -1 & 0 & 0 \end{pmatrix}, \tag{2}$$

and let L^+ be the positive light cone, that is, $L^+ = \{v = {}^t(v^0, v^A, v^{n+1}) \in \mathbb{R}^{n+2} : {}^t vSv = 0,\ v^0 + v^{n+1} > 0\}$. Note that L^+ projectivizes to Q_n and that $\eta_0, \eta_{n+1} \in L^+$. Moreover, there is a bijection between $\mathrm{Conf}(\mathbb{S}^n)$ and the Lorentz group of $\langle\ ,\ \rangle$ preserving the positive light cone (usually called the orthochronous Lorentz group). This gives a Lie group structure to the conformal group $\mathrm{Conf}(\mathbb{S}^n)$, which can be proved to be unique when the action of $\mathrm{Conf}(\mathbb{S}^n)$ on $\mathbb{S}^n$ is required to be smooth (see [15], pp. 95-98). In particular, the identity component of the Lorentz group is called the **Möbius group**, $\mathrm{Mob}(n)$, and coincides with the subgroup of the orientation preserving elements of $\mathrm{Conf}(\mathbb{S}^n)$. The transitivity of the action of $\mathrm{Mob}(n)$ on the n-sphere gives Q_n a homogeneous space structure, allowing us to identify it with the space of left cosets $\mathrm{Mob}(n)/\mathrm{Mob}(n)_0$, where $\mathrm{Mob}(n)_0$ is the isotropy subgroup of $[\eta_0] \in Q_n$:

$$\mathrm{Mob}(n)_0 = \left\{ \begin{pmatrix} r^{-1} & {}^t xA & \frac{1}{2}r|x|^2 \\ 0 & A & rx \\ 0 & 0 & r \end{pmatrix} \middle| \begin{array}{l} r > 0,\ x \in \mathbb{R}^n, \\ A \in \mathrm{SO}(n) \end{array} \right\}. \tag{3}$$

It follows that the principal bundle projection $\pi : \mathrm{Mob}(n) \to Q_n$ associates to a matrix $G = (g_0|g_A|g_{n+1})$ the point $[G\eta_0] = [g_0] \in Q_n$. From now on, we shall use the Einstein summation convention. Let $\mathfrak{m\ddot{o}b}(n)$ denote the Lie algebra of $\mathrm{Mob}(n)$; the Maurer-Cartan form Φ of $\mathrm{Mob}(n)$ is the $\mathfrak{m\ddot{o}b}(n)$-valued 1-form

$$\Phi = \begin{pmatrix} \Phi^0_0 & \Phi^0_B & 0 \\ \Phi^A_0 & \Phi^A_B & \Phi^A_{n+1} \\ 0 & \Phi^{n+1}_B & \Phi^{n+1}_{n+1} \end{pmatrix},$$

with the symmetry relations

$$\Phi^{n+1}_{n+1} = -\Phi^0_0, \quad \Phi^A_B = -\Phi^B_A, \quad \Phi^A_{n+1} = \Phi^0_A, \quad \Phi^{n+1}_B = \Phi^B_0$$

and satisfying the structure equation $d\Phi + \Phi \wedge \Phi = 0$, which component-wise reads

$$\left\{\begin{array}{rcl} d\Phi^0_0 & = & -\Phi^0_A \wedge \Phi^A_0; \\ d\Phi^A_0 & = & -\Phi^A_0 \wedge \Phi^0_0 - \Phi^A_B \wedge \Phi^B_0; \\ d\Phi^0_A & = & -\Phi^0_0 \wedge \Phi^0_A - \Phi^0_B \wedge \Phi^B_A; \\ d\Phi^A_B & = & -\Phi^A_0 \wedge \Phi^0_B - \Phi^A_C \wedge \Phi^C_B - \Phi^0_A \wedge \Phi^B_0. \end{array}\right. \tag{4}$$

Through a local section $s : U \subset Q_n \to \mathrm{Mob}(n)$, Φ pulls back to a flat Cartan connection $\psi = s^*\Phi = s^{-1}ds$. In particular, the set $\{\psi^A_0\}$ gives a local basis for the cotangent bundle of Q_n. Under a change of section $\widetilde{s} = sK$, where $K : U \subset Q_n \to \mathrm{Mob}(n)_0$, the change of gauge becomes

$$\widetilde{\psi} = \widetilde{s}^{-1}d\widetilde{s} = K^{-1}\psi K + K^{-1}dK. \tag{5}$$

By the expression of $\mathrm{Mob}(n)_0$ in (3), we have in particular

$$(\widetilde{\psi}^A_0) = r^{-1}\,{}^tA(\psi^A_0), \tag{6}$$

where (ψ^A_0) stands for the column vector whose A-th component is ψ^A_0. It follows that

$$\widetilde{\psi}^A_0 \otimes \widetilde{\psi}^A_0 = r^{-2}\psi^A_0 \otimes \psi^A_0, \qquad \widetilde{\psi}^1_0 \wedge \ldots \wedge \widetilde{\psi}^n_0 = r^{-n}\psi^1_0 \wedge \ldots \wedge \psi^n_0, \tag{7}$$

which implies that

$$\Big\{ \big(U, \psi^A_0 \otimes \psi^A_0\big) \ : \ U \subset Q_n \text{ domain of a local section } s : U \to \mathrm{Mob}(n) \Big\}$$

defines a conformal structure on Q_n, that is, a collection of locally defined metrics varying conformally on the intersection of their domains of definition, together with an orientation (locally defined by $\psi^1_0 \wedge \ldots \wedge \psi^n_0$), both preserved by $\mathrm{Mob}(n)$. It is easy to prove that, with this conformal structure, $\chi \circ \sigma : \mathbb{S}^n \to Q_n$ is a conformal diffeomorphism. This gives sense to the whole construction.

Let now M be an m-dimensional, oriented manifold. We fix the index ranges

$$1 \leqslant i, j, \ldots \leqslant m, \qquad m+1 \leqslant \alpha, \beta, \ldots \leqslant n.$$

Let $f : M \to Q_n$ be an immersion. A **zeroth order frame field along** f is a smooth map e defined on an open set $U \subseteq M$ with values in $\mathrm{Mob}(n)$ such that $\pi \circ e = f_{|U}$. From now on, dealing with frames along f, we will omit to specify their domains of definition when no possible confusion will arise. We set

$$\phi = e^*\Phi$$

and observe that under a change of frames $\widetilde{e} = eK$, $\widetilde{\phi} = \widetilde{e}^*\Phi$ expresses in terms of ϕ as in (5). As a consequence, at any point $p \in M$ we can choose a zeroth order frame such that

$$\phi^\alpha_0 = 0. \tag{8}$$

The isotropy subgroup at this point is given by

$$(9)\quad \mathrm{Mob}(n)_1 = \left\{ \begin{pmatrix} r^{-1} & {}^t xA & {}^t yB & \frac{1}{2} r\left(|x|^2+|y|^2\right) \\ 0 & A & 0 & rx \\ 0 & 0 & B & ry \\ 0 & 0 & 0 & r \end{pmatrix} \middle| \begin{array}{l} r\in\mathbb{R}^+, A\in \mathrm{SO}(m), \\ B\in\mathrm{SO}(n-m), \\ x\in\mathbb{R}^m, y\in\mathbb{R}^{n-m} \end{array} \right\}.$$

and since it is independent of p, smooth zeroth order frame fields such that (8) holds can be chosen in an appropriate neighborhood of each point of M by general theory, see [19].

A zeroth order frame field e such that (8) holds on its domain of definition is called **first order frame**. Any two such frame fields are related by $\tilde{e} = eK$, where now K takes values in $\mathrm{Mob}(n)_1$.

It can be easily verified that, with respect to first order frames,

$$(10)\qquad \mathrm{d}s^2 = \sum_i \phi_0^i \otimes \phi_0^i, \qquad \mathrm{d}V = \phi_0^1 \wedge \ldots \wedge \phi_0^m$$

define a conformal structure on M and, with respect to these natural structures, f becomes a conformal immersion.

Differentiating (8) and using the structure equations of $\mathrm{Mob}(n)$ and Cartan's lemma, we find that there exist (locally defined) functions h^α_{ij} such that

$$(11)\qquad \phi_i^\alpha = h^\alpha_{ij}\phi_0^j, \qquad h^\alpha_{ij} = h^\alpha_{ji}.$$

We use (5) and (9) to obtain that, under a change of first order frame fields

$$(12)\qquad \tilde{h}^\alpha_{ij} = rB^\beta_\alpha A^l_j (A^k_i h^\beta_{lk} - A^l_i y^\beta).$$

Taking the trace of (12) with respect to i and j we obtain

$$(13)\qquad \tilde{h}^\alpha_{ii} = rB^\beta_\alpha (h^\beta_{kk} - my^\beta).$$

The next step is therefore to consider at any point $p\in M$ a first order frame such that

$$(14)\qquad h^\alpha_{kk} = 0.$$

The isotropy subgroup is given by

$$(15)\qquad \mathrm{Mob}(n)_D = \left\{ \begin{pmatrix} r^{-1} & {}^t xA & 0 & \frac{1}{2} r|x|^2 \\ 0 & A & 0 & rx \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & r \end{pmatrix} \middle| \begin{array}{l} A\in \mathrm{SO}(m), \\ B\in\mathrm{SO}(n-m), \\ r\in\mathbb{R}^+, x\in\mathbb{R}^m \end{array} \right\},$$

which is again independent of the point p considered, so that first order frames with the above property can be smoothly chosen in an appropriate neighborhood of any point. We define a **Darboux frame field along** f as a first order frame field for which (14) holds.

Any two Darboux frame fields are related again by $\widetilde{e} = eK$ where now K is a smooth function taking values in $\mathrm{Möb}(n)_D$.

We observe that for Darboux frames (12) becomes

$$\widetilde{h}^{\alpha}_{ij} = rB^{\beta}_{\alpha}A^{l}_{j}A^{k}_{i}h^{\beta}_{kl}. \tag{16}$$

For further details on the generality of the frame reduction procedure, we refer the reader to [18], [20], [19].

Differentiating (11), using the structure equations and Cartan's lemma, with respect to a Darboux frame e we have

$$\mathrm{d}h^{\alpha}_{ij} - h^{\alpha}_{ik}\phi^{k}_{j} - h^{\alpha}_{kj}\phi^{k}_{i} + h^{\beta}_{ij}\phi^{\alpha}_{\beta} + h^{\alpha}_{ij}\phi^{0}_{0} + \delta_{ij}\phi^{0}_{\alpha} = h^{\alpha}_{ijk}\phi^{k}_{0}, \tag{17}$$

for some (locally defined) functions h^{α}_{ijk} satisfying, also thanks to (11),

$$h^{\alpha}_{ijk} = h^{\alpha}_{jik} = h^{\alpha}_{jki}. \tag{18}$$

Taking the trace of (17) with respect to i and j and using (14) we obtain

$$\phi^{0}_{\alpha} = p^{\alpha}_{k}\phi^{k}_{0} \tag{19}$$

where we have set

$$p^{\alpha}_{k} = \frac{1}{m}h^{\alpha}_{iik}. \tag{20}$$

We say that a point $p \in M$ is an **umbilical point** if and only if for some (hence any) Darboux frame

$$h^{\alpha}_{ij} = 0 \qquad \text{at } p. \tag{21}$$

A totally umbilical submanifold is actually an m-dimensional sphere, as stated in the following proposition.

PROPOSITION 2.1. *Let $f : M \to Q_n$ be an immersion, M oriented, $m = \dim M \geqslant 2$, for which $h^{\alpha}_{ij} \equiv 0$ at every point. Then, there exists $Q_m \subset Q_n$ such that $f(M) \subseteq Q_m$. Furthermore, if M is compact, f is a diffeomorphism onto Q_m.*

The proof relies on a standard technique in the method of the moving frame and therefore we omit it. Alternatively, one can show the result by comparing the Euclidean and conformal description of submanifolds of Q_n, as done in Appendix II.

The form (15) of the isotropy subgroup $\mathrm{Möb}(n)_D$ of Darboux frames along f suggests the following considerations: let us consider the matrix of 1-forms Ψ defined by

$$\Psi = \begin{pmatrix} \phi^{0}_{0} & \phi^{0}_{i} & 0 \\ \phi^{i}_{0} & \phi^{i}_{j} & \phi^{0}_{i} \\ 0 & \phi^{i}_{0} & -\phi^{0}_{0} \end{pmatrix}. \tag{22}$$

We can clearly think of Ψ as taking values in the Lie algebra of $\mathrm{Mob}(m)$.

Under a change of Darboux frames $\widetilde{e} = eK$, where K takes values in $\mathrm{Mob}(n)_D$, we have

$$\widetilde{\Psi} = \bar{K}^{-1}\Psi\bar{K} + \bar{K}^{-1}d\bar{K},$$

with

$$\bar{K} = \begin{pmatrix} r^{-1} & {}^t xA & \frac{r}{2}|x|^2 \\ 0 & A & rx \\ 0 & 0 & r \end{pmatrix},$$

$x \in \mathbb{R}^m$, $A \in \mathrm{SO}(m)$, $r \in \mathbb{R}^+$.

We therefore conclude that Ψ defines a Cartan conformal connection on M. Taking into account (15) for the expression of the isotropy group, we can define a suitable vector bundle N over M whose role parallels that of the normal bundle of an isometric immersion into a Riemannian manifold. In order to do this, with respect to any Darboux frame, we define the fiber of N to be the $(n-m)$-dimensional vector space generated by $\{e_\alpha\}$. Because of (15), it is trivial to see that the bundle N is well defined and on it there is a naturally defined inner product $(\ ,\ )$ for which $\{e_\alpha\}$ is an orthonormal basis at p. With respect to this inner product we define a metric connection

$$\nabla : \Gamma(N) \to \Gamma(T^*M \otimes N)$$

by setting

$$\nabla e_\alpha = \phi^\beta_\alpha \otimes e_\beta. \tag{23}$$

As a matter of fact, if we consider the space form $\mathbb{R}^n$ (or $\mathbb{S}^n$, or $\mathbb{H}^n$) as a open subset of Q_n in the standard fashion and M as a submanifold of $\mathbb{R}^n$ (or $\mathbb{S}^n$, or $\mathbb{H}^n$) with its standard induced Riemannian structure, (N, ∇) is naturally isomorphic to the (Riemannian) normal bundle N^eM endowed with its normal (Van der Waerden-Bortolotti) connection; in Appendix II, we will review these basic links between the Riemannian and conformal descriptions of $M \to Q_n$ for the sake of completeness.

The curvature forms Λ^α_β of N are defined via the structure equations

$$d\phi^\alpha_\beta = -\phi^\alpha_\gamma \wedge \phi^\gamma_\beta + \Lambda^\alpha_\beta.$$

Using the structure equations of the group $\mathrm{Mob}(n)$ and (11) and setting

$$^\perp\tau^\alpha_{\beta ij} = h^\alpha_{ki}h^\beta_{kj} - h^\alpha_{kj}h^\beta_{ki}, \tag{24}$$

we obtain

$$\Lambda^\alpha_\beta = \frac{1}{2}{}^\perp\tau^\alpha_{\beta ij}\phi^i_0 \wedge \phi^j_0.$$

Observe that we have the symmetry relations

$$^\perp\tau^\alpha_{\beta ij} = -{}^\perp\tau^\alpha_{\beta ji} = -{}^\perp\tau^\beta_{\alpha ij}$$

Moreover, with respect to Darboux frames $\tilde{e}$, e

$$ {}^{\perp}\tilde{\tau}^{\alpha}_{\beta ij} = r^2 B^{\gamma}_{\alpha} B^{\rho}_{\beta} A^{t}_{i} A^{v}_{j} \, {}^{\perp}\tau^{\gamma}_{\rho t v} $$

It follows that we can define a tensor ${}^{\perp}\tau$ by locally setting

$$ {}^{\perp}\tau = {}^{\perp}\tau^{\alpha}_{\beta ij} \phi^{i}_{0} \otimes \phi^{j}_{0} \otimes e_{\alpha} \otimes e_{\beta}. $$

We will call ${}^{\perp}\tau$ the normal curvature tensor.

We conclude this section by defining a key object of our investigation. Because of formulas (10) and (16), the form

$$ (25) \qquad \mathrm{w} = \frac{1}{m} \left[\sum_{i,k,\alpha} (h^{\alpha}_{ik})^2 \right]^{\frac{m}{2}} \phi^{1}_{0} \wedge \ldots \wedge \phi^{m}_{0} $$

turns out to be independent of the chosen Darboux frame, hence it gives rise to a globally defined form.

DEFINITION 2.1. *Let $\Omega \Subset M$ be compact. The functional*

$$ (26) \qquad W_{\Omega}(f) \doteq \int_{\Omega} \mathrm{w} = \frac{1}{m} \int_{\Omega} \left[\sum_{i,k,\alpha} (h^{\alpha}_{ik})^2 \right]^{\frac{m}{2}} \phi^{1}_{0} \wedge \ldots \wedge \phi^{m}_{0} $$

is called the Willmore functional (on Ω). A submanifold $f : M^m \to Q_n$ is called a **Willmore submanifold** *if it is a stationary point for the Willmore functional for each $\Omega \Subset M$.*

When M is compact and $\Omega \equiv M$, we simply write $W(f)$.

3. The conformal Grassmannian

Set $s = n - m \geqslant 1$ and let $\{\varepsilon_0, \ldots, \varepsilon_m, \varepsilon_{m+1}, \ldots, \varepsilon_n, \varepsilon_{n+1}\}$ be the standard basis of $\mathbb{R}^{n+2}$. Fix as an origin in the Grassmann manifold of oriented s-planes in $\mathbb{R}^{n+2}$, $G_s(\mathbb{R}^{n+2})$, the point $O = [\varepsilon_{m+1}, \ldots, \varepsilon_n]$ and consider the orbit $\mathcal{Q}_s(\mathbb{R}^{n+2})$ of the point O under the left action (by matrix multiplication) of the group $\mathrm{Mob}(n)$ onto $G_s(\mathbb{R}^{n+2})$. Then the isotropy subgroup of the action on the orbit at the point O is given by

$$ (27) \qquad H_0 = \left\{ \begin{pmatrix} a & {}^t z & 0 & b \\ x & A & 0 & y \\ 0 & 0 & B & 0 \\ c & {}^t w & 0 & d \end{pmatrix} \middle| \begin{array}{c} \begin{pmatrix} a & {}^t z & b \\ x & A & y \\ c & {}^t w & d \end{pmatrix} \in \mathrm{Mob}(m), \\ B \in \mathrm{SO}(s) \end{array} \right\} \subseteq \mathrm{Mob}(n). $$

Note that, since $H_0 \subseteq \mathrm{Mob}(n)$, $z, w, x, y, a, b, c, d, A$ cannot be chosen arbitrarily but have to satisfy certain compatibility relations between them that will be essential in determining that certain quantities are globally well defined.

Thus $Q_s\big(\mathbb{R}^{n+2}\big)$ is identified with the homogeneous space $\mathrm{Mob}(n)/H_0$ with the canonical projection

$$\widehat{\pi} : \mathrm{Mob}(n) \to Q_s\big(\mathbb{R}^{n+2}\big)$$

given by

$$\widehat{\pi} : P \mapsto [P_{m+1}, \dots, P_n] \tag{28}$$

where P_0, P_A, P_{n+1} are the columns of the matrix P.

On their common domain of definition, two local sections of the bundle $\widehat{\pi}$: $\mathrm{Mob}(n) \to Q_s\big(\mathbb{R}^{n+2}\big)$ are related by $\widetilde{s} = sK$ where K is a function taking values in H_0. Considering the components Φ^0_α, Φ^i_α, Φ^α_0 of the Maurer-Cartan form of $\mathrm{Mob}(n)$ and setting $\varphi = s^*\Phi$, we find that their pull-backs under the sections s, $\widetilde{s}$ are related by the following transformation laws:

$$\begin{cases} \widetilde{\varphi}^0_\alpha = d\,\varphi^0_\beta B^\beta_\alpha - y^i \varphi^i_\beta B^\beta_\alpha + b \varphi^\beta_0 B^\beta_\alpha \\ \widetilde{\varphi}^i_\alpha = -w^i \varphi^0_\beta B^\beta_\alpha + A^k_i \varphi^k_\beta B^\beta_\alpha - z^i \varphi^\beta_0 B^\beta_\alpha \\ \widetilde{\varphi}^\alpha_0 = c\,\varphi^0_\beta B^\beta_\alpha - x^k \varphi^k_\beta B^\beta_\alpha + a \varphi^\beta_0 B^\beta_\alpha \end{cases} \tag{29}$$

where the meaning of d,c,a,b,y,x,w,z,A,B is given in (27). From (29) and the relations defining the group $\mathrm{Mob}(n)$, it is not hard to deduce that the quadratic form dl^2 of signature $(s, s(m+1))$ given by

$$dl^2 = -\varphi^0_\alpha \otimes \varphi^\alpha_0 - \varphi^\alpha_0 \otimes \varphi^0_\alpha + \sum_{i,\alpha} \varphi^i_\alpha \otimes \varphi^i_\alpha \tag{30}$$

is well defined on $Q_s\big(\mathbb{R}^{n+2}\big)$ and determines a pseudo-metric on it. In particular the forms φ^0_α, φ^α_0, φ^i_α constitute a local (non orthonormal) coframe on $Q_s\big(\mathbb{R}^{n+2}\big)$ whose dimension is $s(m+2)$. It is convenient to set

$$\theta^{0,\alpha} = \varphi^\alpha_0, \qquad \theta^{\alpha,0} = \varphi^0_\alpha, \qquad \theta^{\alpha,i} = \varphi^i_\alpha \tag{31}$$

and to order the pairs $(\alpha,0)$, (α,i), $(0,\alpha)$ as

$$\begin{aligned} &(\gamma,0) < (\beta,i) < (0,\alpha) && \forall \alpha,\beta,\gamma,i \\ &(0,\beta) < (0,\alpha) && \text{iff } \beta < \alpha \\ &(\beta,j) < (\alpha,i) && \text{iff } \beta < \alpha \text{ or } \beta = \alpha \text{ and } j < i \\ &(\beta,0) < (\alpha,0) && \text{iff } \beta < \alpha. \end{aligned} \tag{32}$$

Thus, representing with the symbols $\widetilde{A}, \widetilde{B}, \dots$ the $s(m+2)$ indices $(\alpha,0)$, (α,i), $(0,\alpha)$, we can write dl^2 as

$$dl^2 = g_{\widetilde{A}\widetilde{B}} \theta^{\widetilde{A}} \otimes \theta^{\widetilde{B}} \tag{33}$$

with

$$\big(g_{\widetilde{A}\widetilde{B}}\big) = \begin{pmatrix} 0 & 0 & -I_s \\ 0 & I_{sm} & 0 \\ -I_s & 0 & 0 \end{pmatrix} \qquad s = n - m. \tag{34}$$

The Levi-Civita connection forms $\theta^{\widetilde{A}}_{\widetilde{B}}$ with respect to the previous coframe are therefore characterized by the equations

$$\begin{cases} d\theta^{\widetilde{A}} = -\theta^{\widetilde{A}}_{\widetilde{B}} \wedge \theta^{\widetilde{B}} \\ g_{\widetilde{A}\widetilde{C}}\theta^{\widetilde{C}}_{\widetilde{B}} + g_{\widetilde{B}\widetilde{C}}\theta^{\widetilde{C}}_{\widetilde{A}} = 0. \end{cases} \tag{35}$$

This allows us to determine the connection forms by simply taking exterior derivatives of (31) and using the structure equations of the group Mob(n). We obtain

$$\begin{cases} \theta^{\alpha,0}_{\beta,0} = \delta^{\alpha}_{\beta}\varphi^0_0 + \varphi^{\alpha}_{\beta}, & \theta^{\alpha,0}_{\beta,i} = \delta^{\alpha}_{\beta}\varphi^0_i, & \theta^{\alpha,0}_{0,\beta} = 0 \\ \theta^{\alpha,i}_{\beta,0} = \delta^{\alpha}_{\beta}\varphi^i_0, & \theta^{\alpha,i}_{\beta,k} = \delta^{\alpha}_{\beta}\varphi^i_k + \delta^i_k\varphi^{\alpha}_{\beta}, & \theta^{\alpha,i}_{0,\beta} = \delta^{\alpha}_{\beta}\varphi^0_i \\ \theta^{0,\alpha}_{\beta,0} = 0, & \theta^{0,\alpha}_{\beta,i} = \delta^{\alpha}_{\beta}\varphi^i_0, & \theta^{0,\alpha}_{0,\beta} = \varphi^{\alpha}_{\beta} - \delta^{\alpha}_{\beta}\varphi^0_0 \end{cases} \tag{36}$$

and, by a simple computation, one checks the validity of the skew-symmetry relations given by the second of (35).

It is worth considering the special case $s = 2$, that is $m = n-2$. Indeed, starting from the $2n$ independent forms φ^0_{α}, φ^i_{α}, φ^{α}_0 we can construct the n independent forms over $\mathbb{C}$

$$\zeta^0 = \varphi^0_{n-1} + i\varphi^0_n, \qquad \zeta^k = \varphi^k_{n-1} + i\varphi^k_n, \qquad \zeta^{n-1} = \varphi^{n-1}_0 + i\varphi^n_0. \tag{37}$$

Using the structure equations, it is immediate to verify that their differentials belong to the ideal they generate, showing that $\mathcal{Q}_2\left(\mathbb{R}^{n+2}\right)$ is a complex manifold, in fact complex Lorentzian. Indeed the complex structure J induced by the forms (37) is determined by

$$\zeta^0(X+iJX) = \zeta^k(X+iJX) = \zeta^{n-1}(X+iJX) = 0 \qquad \forall X \in T\mathcal{Q}_2\left(\mathbb{R}^{n+2}\right),$$

that is

$$\varphi^0_{n-1}(X) = \varphi^0_n(JX) \qquad \varphi^k_{n-1}(X) = \varphi^k_n(JX) \qquad \varphi^{n-1}_0(X) = \varphi^n_0(JX).$$

It is therefore trivial to verify that the metric dl^2 is Hermitian-Lorentzian:

$$\begin{aligned} dl^2(JX,JY) &= -\varphi^0_{n-1}(JX)\varphi^{n-1}_0(JY) - \varphi^0_n(JX)\varphi^n_0(JY) + \\ &\quad - \varphi^{n-1}_0(JX)\varphi^0_{n-1}(JY) - \varphi^n_0(JX)\varphi^0_n(JY) + \\ &\quad + \varphi^i_{n-1}(JX)\varphi^i_{n-1}(JY) + \varphi^i_n(JX)\varphi^i_n(JY) = \\ &= dl^2(X,Y). \end{aligned}$$

We verify that $\mathcal{Q}_2\left(\mathbb{R}^{n+2}\right)$ is Kähler by showing that the differential of the Kähler form

$$\mathcal{K}(X,Y) = dl^2(JX,Y)$$

vanishes identically. This is a simple exercise using (37) and the Maurer-Cartan structure equations. Indeed we have that

$$\begin{aligned} \mathcal{K} &= -\varphi^0_{n-1} \wedge \varphi^n_0 - \varphi^{n-1}_0 \wedge \varphi^0_n + \varphi^i_{n-1} \wedge \varphi^i_n = \\ &= \frac{i}{2}\left(-\zeta^0 \wedge \overline{\zeta^{n-1}} - \zeta^{n-1} \wedge \overline{\zeta^0} + \zeta^k \wedge \overline{\zeta^k}\right), \end{aligned} \tag{38}$$

therefore $d\mathcal{K} = 0$.

Finally we describe the complex projective structure of the conformal Grassmannian.

Proposition 3.1. *There is a holomorphic embedding of the conformal Grassmannian $\mathcal{Q}_2(\mathbb{R}^{n+2})$ into the hyperquadric of $\mathbb{P}^{n+1}_{\mathbb{C}}$ whose homogeneous equation is*

$$-2x^0x^{n+1} + \sum_{A=1}^{n}(x^A)^2 = 0. \tag{39}$$

Proof. There is a natural injection of $\mathcal{Q}_2(\mathbb{R}^{n+2})$ in $\mathbb{P}^{n+1}_{\mathbb{C}}$ defined as follows. Let $[G\varepsilon_{n-1}, G\varepsilon_n]$, with $G \in \mathrm{Mob}(n)$, be a 2-plane of $\mathcal{Q}_2(\mathbb{R}^{n+2})$. The map sending $[G\varepsilon_{n-1}, G\varepsilon_n]$ to the projectivization of the complex, non-zero vector $G(\varepsilon_{n-1} + i\varepsilon_n)$ is well defined and injective, and thus provides a complex projective representation for the whole conformal Grassmannian of 2-planes in $\mathbb{R}^{n+2}$.

Indeed, let $[G\varepsilon_{n-1}, G\varepsilon_n]$ and $[G'\varepsilon_{n-1}, G'\varepsilon_n]$ be two representatives for the same 2-plane in $\mathcal{Q}_2(\mathbb{R}^{n+2})$, then G and G' must differ by an element of the isotropy subgroup H_0, namely $G' = GH$ for some $H \in H_0$. But H has an expression as in (27), with $B \in \mathrm{SO}(2)$, that is

$$B = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

for some $\theta \in \mathbb{R}$, so we have

$$\begin{aligned} G'(\varepsilon_{n-1} + i\varepsilon_n) &= GH(\varepsilon_{n-1} + i\varepsilon_n) = \\ &= G(\cos\theta\varepsilon_{n-1} + \sin\theta\varepsilon_n - i\sin\theta\varepsilon_{n-1} + i\cos\theta\varepsilon_n) = \\ &= e^{-i\theta}G(\varepsilon_{n-1} + i\varepsilon_n) \end{aligned}$$

which projects to the same complex projective class as $G(\varepsilon_{n-1} + i\varepsilon_n)$. As for injectivity, if $G(\varepsilon_{n-1} + i\varepsilon_n)$ and $G'(\varepsilon_{n-1} + i\varepsilon_n)$ project to the same projective class, then there exists $\rho > 0$ and $\theta \in \mathbb{R}$ such that

$$G'(\varepsilon_{n-1} + i\varepsilon_n) = \rho e^{i\theta}G(\varepsilon_{n-1} + i\varepsilon_n) = \rho GH(\varepsilon_{n-1} + i\varepsilon_n),$$

where

$$H = \begin{pmatrix} I_{n-1} & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

clearly belongs to H_0. So $[G\varepsilon_{n-1}, G\varepsilon_n]$ and $[G'\varepsilon_{n-1}, G'\varepsilon_n]$ are in fact the same 2-plane in $\mathcal{Q}_2(\mathbb{R}^{n+2})$.

We will show that, as a matter of fact, $\mathcal{Q}_2(\mathbb{R}^{n+2})$ can be identified with an open submanifold of the projective quadric of homogeneous equation (39). As we have explained above, the image in $\mathbb{P}^{n+1}_{\mathbb{C}}$ of a 2-plane of $\mathcal{Q}_2(\mathbb{R}^{n+2})$ is the projective class of a complex vector of the form $G(\varepsilon_{n-1} + i\varepsilon_n)$, for some $G \in \mathrm{Mob}(n)$. Now, the vector

$\varepsilon_{n-1}+i\varepsilon_n$ trivially satisfies equation (39), and therefore lies in the quadric. Note that the quadric (39) is represented by the matrix

$$S=\begin{pmatrix} 0 & 0 & -1 \\ 0 & I_n & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

introduced in (2) and, since $G\in \mathrm{M\ddot{o}b}(n)$,

$$\begin{aligned} {}^t[G(\varepsilon_{n-1}+i\varepsilon_n)]S[G(\varepsilon_{n-1}+i\varepsilon_n)] &= {}^t(\varepsilon_{n-1}+i\varepsilon_n){}^tGSG(\varepsilon_{n-1}+i\varepsilon_n)= \\ &= {}^t(\varepsilon_{n-1}+i\varepsilon_n)S(\varepsilon_{n-1}+i\varepsilon_n)=0. \end{aligned}$$

Therefore $G(\varepsilon_{n-1}+i\varepsilon_n)$ lies in the quadric (39).

However, the conformal Grassmannian does not cover the whole quadric. Indeed the points of the quadric coming from a 2-plane in $\mathcal{Q}_2\left(\mathbb{R}^{n+2}\right)$ are those that have a representative $v+iw\in\mathbb{C}^{n+2}$ such that, with respect to the Lorentzian product in $\mathbb{R}^{n+2}$, $\|v\|^2=\|w\|^2>0$. This leaves out the projective classes represented by vectors $v+iw$ where v and w are isotropic and non zero. All such vectors lie in the quadric but cannot be obtained from ε_{n-1} or ε_n through a matrix of $\mathrm{M\ddot{o}b}(n)$, because such matrices preserve the Lorentzian norm defined through the matrix S. □

4. The geometry of surfaces in Q_4

Let $f: M\to Q_4$ be an oriented immersed surface. We let $e: U\subset M\to \mathrm{M\ddot{o}b}(n)$ be a local Darboux frame along f, so that, according to (8),

$$\phi_0^\alpha=0,\quad h_{ii}^\alpha=0 \qquad 3\leqslant\alpha\leqslant 4,$$

and the isotropy subgroup $\mathrm{M\ddot{o}b}(n)_D$ is given by (15). For the ease of computations, the matrices A and B appearing in the matrices of $\mathrm{M\ddot{o}b}(n)_D$ will also be written in the form

$$A=\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix},\qquad B=\begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix}. \tag{40}$$

The induced conformal structure on M is given locally by the metric $\phi_0^1\otimes\phi_0^1+\phi_0^2\otimes\phi_0^2$ and the volume form $\phi_0^1\wedge\phi_0^2$. Fix on M a global Riemannian metric g which belongs to the conformal structure on each frame neighbourhood (this can always be dome via a partition of unity). By setting $\varphi=\phi_0^1+i\phi_0^2\in TM^{\mathbb{C}}$, under a change of Darboux frames and using (40) it holds

$$\tilde{\varphi}=r^{-1}e^{-it}\varphi. \tag{41}$$

Therefore, the prescription that φ spans $TM^{(1,0)}$ induces a globally defined endomorphism $J: TM\to TM$ giving rise to an integrable complex structure that makes M a Riemann surface. If (U,z), $z=x+iy$ is a local complex chart on M, with respect to a prescribed Darboux frame e we have $\mathrm{d}z=\mu\varphi$ for some smooth $\mu\neq 0$, which up to the rotation $z \to -z$ can be chosen to be everywhere positive. Hence, changing e to $\tilde{e} = eK$ with K the $\mathrm{Möb}(n)_D$-valued section having $r = \mu^{-1}$, $x = 0$, $A = B = I$, by (41) it holds $\tilde{\varphi} = \mu\varphi = \mathrm{d}z$. According to the literature, Darboux frames with $\mathrm{d}z = \varphi$ are called **canonical Darboux frames**. In a canonical Darboux frame, $|\mathrm{d}z|^2 = \phi_0^1 \otimes \phi_0^1 + \phi_0^2 \otimes \phi_0^2$. Occasionally, the use of canonical Darboux frames will help computations. This is the case, for instance, of the next

PROPOSITION 4.1. *In a canonical Darboux frame, the form $\phi_0^0 + i\phi_2^1$ is of type* $(1,0)$.

Proof. Let e be a canonical Darboux frame on the domain of a complex chart (U, z). From the structure equations, $\mathrm{d}\varphi = (\phi_0^0 + i\phi_2^1) \wedge \varphi$, and since $\varphi = \mathrm{d}z$ this gives $(\phi_0^0 + i\phi_2^1) \wedge \mathrm{d}z = 0$. Hence, $\phi_0^0 + i\phi_2^1 = \mu \mathrm{d}z$ for some $\mu \in C^\infty(U)$, as claimed. □

Starting from Darboux frames, we are now going to introduce a number of geometric invariants. On the normal bundle N locally spanned by $\{e_3, e_4\}$ we can define an endomorphism $J^\perp$ by setting $J_3^\perp = e_4$ and $J^\perp e_4 = -e_3$. Using the transformation law $\tilde{e}_\alpha = B_\alpha^\beta e_\beta$ under a change of Darboux frames and formula (23), it is easy to check that $J^\perp$ is globally defined and $\nabla J^\perp = 0$, hence J is integrable and gives N the structure of a complex line bundle.

We know from the previous section that

$$\phi_i^\alpha = h_{ij}^\alpha \phi_0^j, \qquad h_{ij}^\alpha = h_{ji}^\alpha \qquad 1 \leqslant i, j \leqslant 2 \tag{42}$$

and we have the transformation laws (12). Let L^α denote the **Hopf transform** of the symmetric matrix (h_{ij}^α), that is

$$L^\alpha = \frac{1}{2}(h_{11}^\alpha - h_{22}^\alpha) - ih_{12}^\alpha = h_{11}^\alpha - ih_{12}^\alpha. \tag{43}$$

Using (42), in terms of L^α we can write

$$\phi_1^\alpha = h_{1j}^\alpha \phi_0^j = \frac{1}{2}L^\alpha \varphi + \frac{1}{2}\overline{L^\alpha}\bar{\varphi}, \qquad \phi_2^\alpha = h_{2j}^\alpha \phi_0^j = \frac{i}{2}L^\alpha \varphi - \frac{i}{2}\overline{L^\alpha}\bar{\varphi}.$$

Under a change of Darboux frames,

$$\tilde{L}^\alpha = re^{2it} B_\alpha^\beta L^\beta, \tag{44}$$

in particular

$$\tilde{L}^3 \pm i\tilde{L}^4 = re^{2it} e^{\mp is} \left(L^3 \pm iL^4\right). \tag{45}$$

Using this, we see that the real, locally defined 2-forms

$$\omega_\pm = \left|L^3 \pm iL^4\right|^2 \phi_0^1 \wedge \phi_0^2, \tag{46}$$

are in fact globally defined and smooth.

DEFINITION 4.1. *Given $f : M \to Q_4$, we say that $p \in M$ is* $+$ **or** $-$ **isotropic** *respectively if $\omega_+(p) = 0$ or $\omega_-(p) = 0$. The immersion f is called $+$ isotropic (resp. $-$ isotropic) if it is so at each point.*

Denote with $I_\pm$, respectively, the set of $\pm$ isotropic points, and with $I = I_+ \cup I_-$. Note that the form w in (25) can be written as

$$\mathrm{w} = (|L^3|^2 + |L^4|^2)\phi_0^1 \wedge \phi_0^2. \tag{47}$$

It follows that, whenever f is both $+$ and $-$ isotropic, $\mathrm{w} \equiv 0$ and thus M is a totally umbilical 2 sphere $Q_2 \hookrightarrow Q_4$ by Proposition 2.1.

Define η according to the identity

$$\mathrm{w} = \omega_\pm \mp \eta. \tag{48}$$

A simple computation, using the definitions of w and $\omega_\pm$ yields

$$\eta = -i\left(L^3\overline{L^4} - L^4\overline{L^3}\right)\phi_0^1 \wedge \phi_0^2. \tag{49}$$

Expressing it in terms of the h^α_{ij}'s and using $h^\alpha_{ii} = 0$ we obtain

$$-i\left(L^3\overline{L^4} - L^4\overline{L^3}\right) = 2(h^3_{11}h^4_{12} - h^3_{12}h^4_{11}).$$

We go back to the normal bundle N introduced in Section 2. The curvature K_N of this bundle is now given by

$$\Lambda^3_4 = \frac{1}{2}{}^\perp\tau^3_{4ij}\phi_0^i \wedge \phi_0^j = K_N\phi_0^1 \wedge \phi_0^2$$

and using (24) we deduce that

$$K_N = -i\left(L^3\bar{L}^4 - L^4\bar{L}^3\right) \tag{50}$$

or, in other words

$$d\phi^3_4 = K_N\phi_0^1 \wedge \phi_0^2 = \eta. \tag{51}$$

Using (48), (51) and the generalized Gauss-Bonnet theorem we deduce the next

THEOREM 4.1. *Let $f : M \to Q_4$ be an immersion of a compact orientable surface; then*

$$W(f) = \int_M \omega_\pm \mp 2\pi\chi(N) \tag{52}$$

where $\chi(N)$ is the Euler number of the normal bundle N.

COROLLARY 4.1. *Let $f : M \to Q_4$ be an immersion of a compact orientable surface. Then*

$$\int_M \omega_\pm \geqslant \pm 2\pi\chi(N)$$

equality holding if and only if $f(M) = Q_2 \subset Q_4$.

Proof. It follows since $W(f) \geqslant 0$ and $W(f) = 0$ if and only if $f(M) = Q_2 \subset Q_4$ by Proposition 2.1. □

REMARK 4.1. *Suppose that M is compact and orientable;* (52) *implies that, if M is either $+$ or $-$ isotropic, then the values of $W(f)$ are quantized.*

Keeping in mind identity (20), set k^α according to

$$\phi^0_\alpha = p^\alpha_k \phi^k_0 = k^\alpha \varphi + \bar{k}^\alpha \bar{\varphi}, \qquad \text{that is,} \quad k^\alpha = \frac{1}{2}(p^\alpha_1 - i p^\alpha_2). \tag{53}$$

Under a change of Darboux frame, k^α obeys the transformation law

$$\widetilde{k}^\alpha = r^2 B^\beta_\alpha e^{it}\left(k^\beta + \frac{1}{2}(x^1 + ix^2)L^\beta\right), \tag{54}$$

whence

$$\widetilde{k}^3 \pm i\widetilde{k}^4 = r^2 e^{it} e^{\mp is}\left\{k^3 \pm ik^4 + \frac{1}{2}(x^1 + ix^2)(L^3 \pm iL^4)\right\}. \tag{55}$$

From (44) and (54) it holds

$$\widetilde{k}^3 \widetilde{L}^4 - \widetilde{k}^4 \widetilde{L}^3 = r^3 e^{3it}(k^3 L^4 - k^4 L^3),$$

thus the section

$$\vartheta \in \Gamma\left(\bigotimes^3 T^* M^{(1,0)}\right), \qquad \vartheta = (k^3 L^4 - k^4 L^3)\varphi \otimes \varphi \otimes \varphi \tag{56}$$

is globally defined on M. We will investigate the form ϑ in Section 5. More generally, if $M^2 \to Q_n$ is an immersion, the section

$$\vartheta \in \Gamma\left(\Lambda^2(NM) \otimes \left(\bigotimes^3 T^* M^{(1,0)}\right)\right), \qquad \vartheta = (L^\alpha e_\alpha) \wedge (k^\beta e_\beta)\varphi \otimes \varphi \otimes \varphi \tag{57}$$

is globally defined. A variant of this section has been studied by Ejiri in [9].

Consider the equality

$$\phi^0_\alpha = p^\alpha_k \phi^k_0. \tag{58}$$

Taking the exterior derivative of the above equation and using the Maurer-Cartan structure equations together with Cartan's lemma, we obtain the existence of $\{p^\alpha_{ik}\}$ satisfying

$$\mathrm{d}p^\alpha_i - p^\alpha_k\phi^k_i + p^\beta_i\phi^\alpha_\beta + 2p^\alpha_i\phi^0_0 - h^\alpha_{ki}\phi^0_k = p^\alpha_{ik}\phi^k_0 \tag{59}$$

with the symmetry

$$p^\alpha_{ik} = p^\alpha_{ki}. \tag{60}$$

THEOREM 4.2 ([17]). *Let $f : M^2 \to Q_4$ be an oriented surface in the Möbius space. Then, f is a Willmore surface if and only if $p^\alpha_{jj} = 0$, where p^α_{jj} are defined in* (59).

REMARK 4.2. *With a simple but tedious computation, one verifies that under a change of Darboux frames we have*

$$\begin{aligned}\widetilde{p}^\alpha_{ij} = {}& r^3B^\beta_\alpha A^k_iA^t_j\left(p^\beta_{kt} + x^lh^\beta_{lkt} - x^tx^lh^\beta_{lk} - x^kx^lh^\beta_{lt} - \frac{1}{2}x^lx^lh^\beta_{kt} - 2x^tp^\beta_k - 2x^kp^\beta_t\right) + \\ & + r^3B^\beta_\alpha\delta_{ij}\left(x^lx^th^\beta_{lt} + x^lp^\beta_l\right)\end{aligned} \tag{61}$$

so that, tracing with respect to i and j and recalling that M has dimension 2,

$$\widetilde{p}^\alpha_{ii} = r^3B^\beta_\alpha p^\beta_{tt}. \tag{62}$$

The above equality shows that the system of equations

$$p^\alpha_{ii} = 0 \tag{63}$$

is conformally invariant.

5. Further properties of Willmore surfaces

We begin by still considering a general surface $f : M^2 \to Q_4$. It is convenient to define the following complex forms:

$$\delta = \phi^0_0 + i\phi^1_2, \qquad \chi = \phi^0_1 + i\phi^0_2. \tag{64}$$

The structure equations then read as

$$\begin{cases} \mathrm{d}\varphi &= \delta\wedge\varphi \\ \mathrm{d}\delta &= \frac{1}{2}L^\alpha\overline{L^\alpha}\varphi\wedge\bar{\varphi} - \bar{\chi}\wedge\varphi \\ \mathrm{d}\chi &= \chi\wedge\bar{\delta} - k^\alpha\overline{L^\alpha}\varphi\wedge\bar{\varphi}. \end{cases} \tag{65}$$

We differentiate the functions L^α and k^α. Using the structure equations we get

$$\begin{aligned}\mathrm{d}L^\alpha &= \left[\frac{1}{2}(h^\alpha_{11k} - h^\alpha_{22k}) - ih^\alpha_{12k}\right]\phi^k_0 - L^\alpha(\phi^0_0 + 2i\phi^1_2) - L^\beta\phi^\alpha_\beta \\ &= \chi^\alpha\varphi + k^\alpha\bar{\varphi} - L^\alpha(\phi^0_0 + 2i\phi^1_2) - L^\beta\phi^\alpha_\beta,\end{aligned} \tag{66}$$

where

$$\begin{aligned} \chi^\alpha &\doteq \tfrac{1}{4}\left[h^\alpha_{111} - 3h^\alpha_{221} + i\,(h^\alpha_{222} - 3h^\alpha_{112})\right] \\ &= \overline{k^\alpha} - i(h^\alpha_{112} - ih^\alpha_{122}). \end{aligned} \tag{67}$$

In particular,

$$\begin{aligned} \mathrm{d}(L^3 \pm iL^4) &= (L^3 \pm iL^4)\left[-\phi^0_0 - i(2\phi^1_2 \pm \phi^4_3)\right] \\ &\quad +(\chi^3 \pm i\chi^4)\varphi + (k^3 \pm ik^4)\bar{\varphi}, \end{aligned} \tag{68}$$

Taking into account Proposition 4.1, in a canonical Darboux frame the $(0,1)$-part of $\mathrm{d}(L^3 \pm iL^4)$ is given by

$$\bar{\partial}(L^3 \pm iL^4) \equiv -i(L^3 \pm iL^4)\left[\phi^1_2 \pm \phi^4_3\right] + (k^3 \pm ik^4)\bar{\varphi} \qquad (\operatorname{mod}\varphi) \tag{69}$$

About $\mathrm{d}k^\alpha$, using also (59) it holds

$$\begin{aligned} \mathrm{d}k^\alpha &= \frac{1}{2}q^\alpha\varphi + \frac{1}{4}(p^\alpha_{11} + p^\alpha_{22})\bar{\varphi} - k^\alpha(2\phi^0_0 + i\phi^1_2) \\ &\quad -k^\beta\phi^\alpha_\beta + \frac{1}{2}L^\alpha(\phi^0_1 + i\phi^0_2), \end{aligned} \tag{70}$$

where we have set

$$q^\alpha = \frac{1}{2}(p^\alpha_{11} - p^\alpha_{22}) - ip^\alpha_{12}. \tag{71}$$

Hence,

$$\begin{aligned} \mathrm{d}(k^3 \pm ik^4) &= \frac{1}{2}(q^3 \pm iq^4)\varphi + \frac{1}{4}\left[p^3_{11} + p^3_{22} \pm i(p^4_{11} + p^4_{22})\right]\bar{\varphi} \\ &\quad -(k^3 \pm ik^4)\left[2\phi^0_0 + i(\phi^1_2 \pm \phi^4_3)\right] + \frac{1}{2}(L^3 \pm iL^4)(\phi^0_1 + i\phi^0_2). \end{aligned} \tag{72}$$

Combining (66) and (70),

$$\begin{aligned} \mathrm{d}(k^3L^4 - k^4L^3) = -3(k^3L^4 - k^4L^3)(\phi^0_0 + i\phi^1_2) + \frac{1}{2}(q^3L^4 - q^4L^3)\varphi + \\ +(k^3\chi^4 - k^4\chi^3)\varphi + \frac{1}{4}(p^3_{kk}L^4 - p^4_{kk}L^3)\bar{\varphi}, \end{aligned} \tag{73}$$

where ζ^α is as in (67). Suppose now that M is a Willmore surface. According to Theorem 4.2, $p^\alpha_{kk} = 0$ for $3 \leqslant \alpha \leqslant 4$. Expression (72), together with Proposition 4.1, implies that, in a canonical Darboux frame,

$$\bar{\partial}(k^3 \pm ik^4) \equiv -(k^3 \pm ik^4)\left[\phi^0_0 \pm i\phi^4_3\right] + \frac{1}{2}(L^3 \pm iL^4)(\phi^0_1 + i\phi^0_2) \qquad (\operatorname{mod}\varphi). \tag{74}$$

Hence, we deduce the next result concerning the section ϑ in (56).

PROPOSITION 5.1. *Let M be a Willmore surface. Then, ϑ in (56) is a holomorphic section of $\otimes^3 T^*M^{(1,0)}$.*

Proof. Since ϑ is independent of the chosen Darboux frame, we can compute $\bar{\partial}\vartheta$ in a canonical Darboux frame, for which, thanks to Proposition 4.1, $\phi_0^0 + i\phi_2^1$ is of type $(1,0)$. By formula (73)

$$\bar{\partial}\vartheta = \bar{\partial}(k^3L^4 - k^4L^3) \otimes \varphi \otimes \varphi \otimes \varphi = \frac{1}{4}(p^3_{kk}L^4 - p^4_{kk}L^3)\overline{\varphi} \otimes \varphi \otimes \varphi \otimes \varphi.$$

Since M is Willmore, we therefore have $\bar{\partial}\vartheta \equiv 0$, as claimed. □

Next, we recall the definition of sections of analytic type. These are quite useful in many different settings, and have therefore been studied thoroughly (see e.g. [2]).

DEFINITION 5.1. *If $E \to M$ is a complex vector bundle over a Riemann surface M, a smooth section σ of E is said to be of* **analytic type** *if it either vanishes identically or, near any zero x, we have*

$$\sigma = z^k \sigma_0$$

for some positive integer k and some continuous section σ_0 with $\sigma_0(x) \neq 0$, where z is any holomorphic chart centered at x. The integer k is called the order of the zero x.

A sufficient condition for σ to be of analytic type is given by the following result due to Eschenburg and Tribuzy (see [10]), that improves on previous work of Chern ([7]).

LEMMA 5.1. *Let $\pi : E \to M$ be a complex Hermitian vector bundle of rank l over a Riemann surface, with a bundle metric $(\,,\,)$ and a compatible connection ∇. Let $\sigma : M \to E$ be a section of E defined on a complex chart (U,z) around some point $x \in U$. Suppose that the Cauchy-Riemann condition*

$$\left|\nabla_{\frac{\partial}{\partial \bar{z}}} \sigma\right| \leqslant \gamma |\sigma| \tag{75}$$

for some $\gamma \in L^p(U)$ with $p > 2$. Then, σ is of analytic type.

REMARK 5.1. *Due to its local nature, via local trivializations it is enough to prove the result for functions $\sigma : U \to \mathbb{C}^l$, which is actually the original setting of Eschenburg-Tribuzy's theorem.*

Assume now M compact. By the Poincaré-Hopf index theorem (see, e.g. [10] and [11]) we have

PROPOSITION 5.2. *Let M be a compact Riemann surface and L a complex line bundle over M. If $s \not\equiv 0$ is a section of L of analytic type, then the Euler number of L, $\chi(L)$, is equal to the sum of the orders of the zeros of s.*

Applying the above to the section ϑ, we deduce the next

PROPOSITION 5.3. *Let $M^2 \to Q_4$ be a compact Willmore surface. Then, either $\vartheta \equiv 0$ or it has finitely many zeros $\{x_1, \dots, x_s\}$ whose orders $\{\mu_j\}_1^s$ satisfy*

$$\sum_{j=1}^{s} \mu_j = -3\chi(M). \tag{76}$$

In particular, if M is a Willmore torus and $\vartheta \not\equiv 0$, then ϑ has no zeros.

Proof. We have shown in Proposition 5.1 that ϑ is a holomorphic section, thus trivially (75) holds and ϑ is of analytic type. Formula (76) follows from a plain application of Poincaré-Hopf index theorem and the identity

$$\chi\left(\bigotimes^{3} T^*M^{(1,0)}\right) = 3\chi(T^*M^{(1,0)}) = -3\chi(M).$$

□

REMARK 5.2. *In case M is a topological sphere, (76) implies that $\vartheta \equiv 0$. Willmore spheres have been studied in [5, 14]. When M is a torus, (76) implies that either $\vartheta \equiv 0$ or ϑ is nonzero at every point. In particular, in the second case M has no umbilic points, so that we can perform a further frame reduction in such a way that h^α_{ij} is diagonalized at each point.*

Observe that under a change of Darboux frames we have

$$\tilde{p}^3_{kk}\tilde{L}^4 - \tilde{p}^4_{kk}\tilde{L}^3 = r^3 e^{3it}\left(p^3_{kk}L^4 - p^4_{kk}L^3\right). \tag{77}$$

For $p > 2$, one can consider the request

$$\exists \gamma \in L^p_{\text{loc}}(M) \quad \text{such that} \quad \left|p^3_{kk}L^4 - p^4_{kk}L^3\right| \leqslant \gamma\left|k^3L^4 - k^4L^3\right| \quad \text{a.e.} \tag{78}$$

A priori, this condition is only local, since the quantities involved do depend on the choice of the Darboux frame. Nevertheless, because of the transformation laws for $p^3_{kk}L^4 - p^4_{kk}L^3$ and $k^3L^4 - k^4L^3$, it turns out that if condition (78) holds for some Darboux frame, then it holds for any Darboux frame, provided we replace γ with a (possibly different) suitable function in $L^p_{\text{loc}}(M)$, so that the request makes sense globally. Hence, a simple application of Lemma 5.1 yields

PROPOSITION 5.4. *Let $f : M \to Q_4$ be an immersion such that (78) holds for some $p > 2$. Then either $\vartheta \equiv 0$ or its zero set is discrete. In this latter case, for M compact we have*

$$z(\vartheta) = -3\chi(M),$$

where $z(\vartheta)$ is the sum of the orders of the zeros of ϑ.

PROPOSITION 5.5. *Let $M^2 \to Q_4$ be a Willmore surface. Then, either M^2 is $+$ isotropic, or the set I_+ of $+$ isotropic points is contained in the union of finitely many embedded smooth curves. An analogous statement holds for $-$ isotropic points.*

Proof. As usual, we use the symbol ± to deal with both cases simultaneously. Let $p \in I_\pm$, and let e be a Darboux frame around p. By inspecting the change of frame formula (55), at p the vanishing or non-vanishing of $k^3 \pm ik^4$ is independent of the chosen Darboux frame. Denote with $\mathcal{V}_\pm$ the set of common zeros of $L^3 \pm iL^4$ and $k^3 \pm ik^4$. Coupling (69) and (74), for Willmore surfaces the following system is satisfied on the domain of e:

(79)
$$\begin{cases} \bar\partial(L^3 \pm iL^4) & \equiv & -i(L^3 \pm iL^4)\left[\phi^1_2 \pm \phi^4_3\right] + (k^3 \pm ik^4)\bar\varphi & (\operatorname{mod}\varphi) \\ \bar\partial(k^3 \pm ik^4) & \equiv & -(k^3 \pm ik^4)\left[\phi^0_0 \pm i\phi^4_3\right] + \dfrac{1}{2}(L^3 \pm iL^4)(\phi^0_1 + i\phi^0_2) & (\operatorname{mod}\varphi). \end{cases}$$

Looking at the first equation in (79), $L^3 \pm iL^4 \equiv 0$ on some open set implies that also $k^3 \pm ik^4 \equiv 0$, thus $\mathcal{V}_\pm \equiv M$ if and only if M is ± isotropic. By Eschenburg-Tribuzy Lemma 5.1, (79) locally defines a complex vector valued function $F_\pm = (L^3 \pm iL^4, k^3 \pm ik^4)$ of analytic type. Hence, either $\mathcal{V}_\pm \equiv M$ or $\mathcal{V}_\pm$ consists of finitely many isolated points, and around each $p \in \mathcal{V}_\pm$, in a local complex chart (U,z), $F_\pm = z^k(g_\pm, h_\pm)$ for some smooth functions $g_\pm, h_\pm$ satisfying $(g_\pm(p), h_\pm(p)) \neq (0,0)$. If $g_\pm(p) \neq 0$, then p is isolated in $I_\pm$. Otherwise, $\mathcal{V}_\pm \subset \{g_\pm = 0\}$ around p. Inserting the local expression of F into (79) and using $h_\pm(p) \neq 0$ together with $\bar\varphi(\partial_{\bar z}) \neq 0$, we get $\partial_{\bar z}(L^3 \pm iL^4) = z^k \partial_{\bar z} g_\pm$ and $\partial_{\bar z} g_\pm(p) \neq 0$. Hence, locally around p, $\mathcal{V}_\pm \cap U$ is contained in the zero set of a smooth function $g_\pm : U \to \mathbb{C}$ whose differential has (real) rank at least 1 at p. By the rank theorem, up to shrinking U, $\mathcal{V}_\pm \cap U$ is contained in a finite union of smooth curves properly embedded in U. Choosing then a finite covering of M by charts (U,z), $\mathcal{V}_\pm$ is globally contained in a union of smooth, embedded curves. Next, let $p \in I_\pm \backslash \mathcal{V}_\pm$, so that $(k^3 \pm ik^4)(p) \neq 0$. The first equation in (79) then gives $\partial_{\bar z}(L^3 \pm iL^4)(p) \neq 0$, and by the same reasoning $I_\pm$ around p is locally contained in a finite union of smooth curves. □

REMARK 5.3. *In the above result, we have used the following basic fact of measure theory: let $f : M^m \to N^n$ be smooth, and for $1 \leqslant j \leqslant \min\{m,n\}$ let $A_j = \{p : f(p) = 0, \operatorname{rk}(\mathrm{d}f)(p) \geqslant j\}$. Then, for each relatively compact open set Ω, $A_j \cap \overline{\Omega}$ is contained in a finite union of smooth embedded submanifolds of dimension $(m-j)$. Indeed, let $x_0 \in A_j \cap \overline{\Omega}$ and consider a local chart (U,x), $U \Subset \Omega$ around x_0. For each multiindex $I = \{1 \leqslant k_1 < k_2 < \ldots < k_j \leqslant m\}$ of length j, set*

$$M_I = \frac{\partial(f^{k_1}, \ldots, f^{k_j})}{\partial(x^{k_1}, \ldots, x^{k_j})}.$$

Then, $A_j \cap U \subset \cup_I V_I$, where $V_I = \{p \in U : f(p) = 0, \det(M_I)(p) \neq 0\}$ is closed in U. We separately analyse each V_I. Up to renaming coordinates, we can assume that $I = \{1, 2, \ldots, j\}$. Let $\bar x \in V_I$. Then, the map $Y(x) = (f^1(x), \ldots, f^j(x), x^{j+1}, \ldots, x^m)$ is locally a diffeomorphism around $\bar x$, and setting $X = Y^{-1}$ it holds $f \circ X(y) = (y^1, \ldots, y^j, g(y))$ for some smooth g. The set V_I is then locally defined by the conditions $y^1 = \ldots = y^j = 0$ and $g(y) = 0$, hence V_I is contained in the slice $y^1 = \ldots = y^j = 0$, an embedded submanifold of dimension $(m-j)$.

6. Isotropy and the conformal Gauss map

Our next goal is to give a geometric interpretation to $+$ and $-$ isotropic immersions. Towards this aim we introduce the conformal Gauss map. We let $\mathcal{Q}_2(\mathbb{R}^6)$ be the conformal Grassmannian of 2-planes introduced in Section 4.1. As we have seen, $\mathcal{Q}_2(\mathbb{R}^6)$ has the structure of a Kähler-Lorentzian manifold with a local basis of $(1,0)$-type forms given by

$$\zeta^0 = \varsigma^*\Phi^0_3 + i\varsigma^*\Phi^0_4, \qquad \zeta^k = \varsigma^*\Phi^k_3 + i\varsigma^*\Phi^k_4, \qquad \zeta^3 = \varsigma^*\Phi^3_0 + i\varsigma^*\Phi^4_0, \tag{80}$$

where ς is any local section of $\hat{\pi}$.

DEFINITION 6.1. *Let $f : M \to Q_4$ be an immersed oriented surface and let e be a (local) Darboux frame along f. The* ***conformal Gauss map*** *$\gamma_f : M \to \mathcal{Q}_2(\mathbb{R}^6)$ is defined by setting*

$$\gamma_f : p \mapsto [e_3, e_4]_p$$

where with $[e_3, e_4]_p$ we denote the oriented 2-plane generated by the vectors e_3, e_4 at the point p.

We observe that, under a change of Darboux frames, γ_f is in fact globally well defined, and the orientation of the 2-plane $[e_3, e_4]$ is also preserved.

THEOREM 6.1. *Let $f : M \to Q_4$ be an immersed oriented Riemann surface. Then f is $\pm$ isotropic if and only if $\gamma_f : M \to \mathcal{Q}_2(\mathbb{R}^6)$ is $\mp$ holomorphic.*

Proof. We recall that, given a Riemann surface M, a map $f : M \to \mathcal{Q}_2(\mathbb{R}^6)$ is respectively $\pm$ holomorphic (that is, holomorphic or antiholomorphic) if the pull-back of the forms ζ^0, ζ^k, ζ^3 in (80) is respectively of type $(1,0)$ or $(0,1)$.

We begin by observing that if e is any Darboux frame along f, then the following diagram is commutative.

$$\begin{tikzcd}
 & \mathrm{Mob}(4) \arrow[ld, "\hat{\pi}"'] \arrow[rd, "\pi"] & \\
\mathcal{Q}_2(\mathbb{R}^6) & & Q_4 \\
 & M \arrow[lu, "\gamma_f"] \arrow[ru, "f"'] \arrow[uu, "e"] &
\end{tikzcd}$$

This fact enables us to compute in a simple way $\gamma_f^*\zeta^0$, $\gamma_f^*\zeta^k$, $\gamma_f^*\zeta^3$. Indeed, setting

$$\theta^{0,\alpha} = \varsigma^*\Phi^\alpha_0, \qquad \theta^{\alpha,0} = \varsigma^*\Phi^0_\alpha, \qquad \theta^{\alpha,i} = \varsigma^*\Phi^i_\alpha \tag{81}$$

and using (20), (53) and (42) we have:

$$\begin{cases} \gamma_f^*\theta^{\alpha,0} = p_k^\alpha \phi_0^k \\ \gamma_f^*\theta^{\alpha,i} = -h_{ik}^\alpha \phi_0^k \\ \gamma_f^*\theta^{0,\alpha} = 0. \end{cases} \tag{82}$$

In order to see this, we observe that

$$\gamma_f^*\varsigma^*\Phi = (\widehat{\pi}\circ e)^*\varsigma^*\Phi = e^*(\varsigma\circ\widehat{\pi})^*\Phi.$$

And since $\widehat{\pi}\circ(\varsigma\circ\widehat{\pi}) = \widehat{\pi}$, then for every g in the inverse image through $\widehat{\pi}$ of the domain of definition of ς, it holds

$$\varsigma(\widehat{\pi}(g)) = g\widetilde{K}(g),$$

where $\widetilde{K}$ is an H_0-valued function. Therefore

$$(\varsigma\circ\widehat{\pi})^*\Phi_g = \widetilde{K}(g)^{-1}g^{-1}dg\widetilde{K}(g) + \widetilde{K}(g)^{-1}d\widetilde{K}_g,$$

and since $\widetilde{K}(g)^{-1}d\widetilde{K}_g$ has values in the Lie algebra of H_0, we deduce that

$$\begin{aligned}
(\varsigma\circ\widehat{\pi})^*\Phi^\alpha_{0\,g_0} &= \left(\widetilde{K}(g_0)^{-1}g_0^{-1}dg_{g_0}\widetilde{K}(g_0)\right)^\alpha_0 \\
(\varsigma\circ\widehat{\pi})^*\Phi^0_{\alpha g_0} &= \left(\widetilde{K}(g_0)^{-1}g_0^{-1}dg_{g_0}\widetilde{K}(g_0)\right)^0_\alpha \\
(\varsigma\circ\widehat{\pi})^*\Phi^\alpha_{i\;g_0} &= \left(\widetilde{K}(g_0)^{-1}g_0^{-1}dg_{g_0}\widetilde{K}(g_0)\right)^\alpha_i.
\end{aligned}$$

If for a fixed $\widetilde{g}$ we replace the section ς with the section $\varsigma\widetilde{K}(\widetilde{g})^{-1}$ obtained multiplying ς by a constant matrix, we will have defined a new section $\widetilde{\varsigma}$ which satisfies, at the point $\widetilde{g}$ (and in general only there), the equality $\widetilde{\varsigma}(\widehat{\pi}(\widetilde{g})) = \widetilde{g}$, and therefore

$$\begin{aligned}
(\widetilde{\varsigma}\circ\widehat{\pi})^*\Phi^\alpha_{0\,\widetilde{g}} &= \left(\widetilde{g}^{-1}dg_{\widetilde{g}}\right)^\alpha_0 = \Phi^\alpha_{0\,\widetilde{g}} \\
(\widetilde{\varsigma}\circ\widehat{\pi})^*\Phi^0_{\alpha\widetilde{g}} &= \left(\widetilde{g}^{-1}dg_{\widetilde{g}}\right)^0_\alpha = \Phi^0_{\alpha\widetilde{g}} \\
(\widetilde{\varsigma}\circ\widehat{\pi})^*\Phi^\alpha_{i\;\widetilde{g}} &= \left(\widetilde{g}^{-1}dg_{\widetilde{g}}\right)^\alpha_i = \Phi^\alpha_{i\;\widetilde{g}}.
\end{aligned}$$

Now let us fix $p_0 \in M$ and set $\widetilde{g} = e(p_0)$. Given a section ς defined in a neighborhood of $\gamma_f(p_0)$, and possibly replacing it with the section $\varsigma\widetilde{K}(e(p_0))^{-1}$, which we shall still call ς, we have at the point p_0

$$\varsigma(\widehat{\pi}(e(p_0))) = e(p_0),$$

and thus

$$\begin{aligned}
\left(\gamma_f^*\varsigma^*\Phi^\alpha_0\right)_{p_0} &= \left(e^*(\varsigma\circ\widehat{\pi})^*\Phi^\alpha_0\right)_{p_0} = \left(e^*\Phi^\alpha_0\right)_{p_0} = \phi^\alpha_{0\;p_0} = 0 \\
\left(\gamma_f^*\varsigma^*\Phi^0_\alpha\right)_{p_0} &= \phi^0_{\alpha p_0} = p^\alpha_k(p_0)\phi^k_{0p_0} \\
\left(\gamma_f^*\varsigma^*\Phi^\alpha_i\right)_{p_0} &= \phi^\alpha_{i\;p_0} = h^\alpha_{ik}(p_0)\phi^k_{0p_0}.
\end{aligned}$$

Hence, setting $\varphi = \phi_0^1 + i\phi_0^2$ and observing that, if α_k, β_k are real-valued functions, one has

$$(\alpha_k + i\beta_k)\phi_0^k = \left\{\frac{\alpha_1+\beta_2}{2} + i\frac{\beta_1-\alpha_2}{2}\right\}\varphi + \left\{\frac{\alpha_1-\beta_2}{2} + i\frac{\beta_1+\alpha_2}{2}\right\}\overline{\varphi}, \tag{83}$$

we get, with the aid of (43), at the point p_0,

$$\begin{aligned}
\gamma_f^*\zeta^0 &= \left(k^3+ik^4\right)\varphi + \left(\overline{k^3-ik^4}\right)\overline{\varphi}\\
\gamma_f^*\zeta^1 &= -\frac{1}{2}\left(L^3+iL^4\right)\varphi - \frac{1}{2}\left(\overline{L^3-iL^4}\right)\overline{\varphi}\\
\gamma_f^*\zeta^2 &= -\frac{i}{2}\left(L^3+iL^4\right)\varphi + \frac{i}{2}\left(\overline{L^3-iL^4}\right)\overline{\varphi}\\
\gamma_f^*\zeta^3 &= 0.
\end{aligned}$$

It is therefore clear, using (46), that if γ_f is ∓ holomorphic, then f is ± isotropic. To prove the converse, we need to show that $L^3 \pm iL^4 = 0$ implies $k^3 \pm ik^4 = 0$. Towards this aim we recall (68) and observe that, under the assumption $L^3 \pm iL^4 = 0$, in particular the coefficient of $\bar{\varphi}$ must vanish, which is the claim. □

Let us now further analyze the quantities k^α defined in (53). For $p > 2$, consider the condition

$$\exists \gamma \in L^p_{\mathrm{loc}}(M) \quad \text{such that} \quad \left|k^3 \pm ik^4\right| \leqslant \gamma\left|L^3 \pm iL^4\right| \qquad \text{a.e.} \tag{84}$$

Of course we have to check that this condition actually makes sense, since the quantities involved depend on the choice of the Darboux frame. To this end we use (55) and (45) and observe that if condition (84) holds for some Darboux frame, then for any other Darboux frame we can estimate

$$\begin{aligned}
\left|\tilde{k}^3 \pm i\tilde{k}^4\right| &= r^2\left|k^3 \pm ik^4 + \frac{1}{2}(x^1+ix^2)(L^3 \pm iL^4)\right| \leqslant\\
&\leqslant r^2\left(\gamma + \frac{1}{2}\left|x^1+ix^2\right|\right)\left|L^3 \pm iL^4\right| = r\left(\gamma + \frac{1}{2}\left|x^1+ix^2\right|\right)\left|\tilde{L}^3 \pm i\tilde{L}^4\right|.
\end{aligned}$$

Therefore condition (84) still holds, provided we replace γ with another suitable function in $L^p_{\mathrm{loc}}(M)$.

Condition (84) can be paired with Lemma 5.1 to obtain many applications in this context, starting with the following

PROPOSITION 6.1. *Let $f : M \to Q_4$ be an immersion satisfying* (84). *Then either γ_f is ± holomorphic or the set $I_\mp$ of ∓ isotropic points of M is discrete.*

Proof. Keeping in mind the expression (68) for the differential of the functions $L^3 \pm iL^4$, we use (84) in order to apply Lemma 5.1 to these functions. The claim then follows readily. □

Let us now consider the canonical projection $p : \mathbb{R}^6 \backslash \{0\} \to \mathbb{P}^5_{\mathbb{R}}$, sending x to its projective class $[x]$. Given two Darboux frames e and $\tilde{e}$ along $f : M \to Q_4$, we have

$$p_{*\tilde{e}_0}\tilde{e}_\alpha = rB^\beta_\alpha p_{*e_0}e_\beta.$$

Indeed, since $p(\lambda x) = p(x)$ for every $\lambda \in \mathbb{R}^*$ and for every $x \in \mathbb{R}^6 \backslash \{0\}$, then $p_{*\lambda x}\lambda_{*x}v = p_{*x}v$, that is $p_{*\lambda x}\lambda v = p_{*x}v$. Therefore

$$p_{*\tilde{e}_0}\tilde{e}_\alpha = p_{*r^{-1}e_0}\tilde{e}_\alpha = p_{*e_0}(r\tilde{e}_\alpha) = rB^\beta_\alpha p_{*e_0}e_\beta.$$

Hence, setting $E_\alpha = p_{*e_0}e_\alpha$, we get

$$\tilde{E}_\alpha = rB^\beta_\alpha E_\beta. \tag{85}$$

It follows that the bundle P over M locally spanned by E_3, E_4 is globally well defined. Let P_c be its complexification and $P_c = P_c^{(1,0)} \oplus P_c^{(0,1)}$ the splitting of P_c into $(1,0)$ and $(0,1)$ parts, locally spanned by $E_3 - iE_4$ and $E_3 + iE_4$ respectively. Observe that under a change of Darboux frames, by virtue of (85) we have

$$\tilde{E}_3 \pm i\tilde{E}_4 = re^{\mp is}(E_3 \pm iE_4). \tag{86}$$

On the other hand, if $\varphi = \phi^1_0 + i\phi^2_0$ is the form that gives M its complex structure, by (45), (86) and (41) we conclude that

$$\mu_\mp = \left(L^3 \mp iL^4\right)(E_3 \pm iE_4) \otimes \varphi \otimes \varphi$$

are sections of the bundles

$$P_c^{(0,1)} \otimes T^*M^{(1,0)} \otimes T^*M^{(1,0)} \quad \text{and} \quad P_c^{(1,0)} \otimes T^*M^{(1,0)} \otimes T^*M^{(1,0)}$$

respectively, which are globally defined on M. Under assumption (84) we can deduce that these sections either vanish identically or have isolated zeros with positive integer multiplicities. Indeed, since φ is a holomorphic section of $T^*M^{(1,0)}$, then

$$D_{\frac{\partial}{\partial \bar{z}}}\mu_\mp = d(L^3 \mp iL^4)\left(\frac{\partial}{\partial \bar{z}}\right)(E^3 \pm iE^4) \otimes \varphi^2 + (L^3 \mp iL^4)D_{\frac{\partial}{\partial \bar{z}}}(E^3 \pm iE^4) \otimes \varphi^2$$

and now, using (68), assumption (84), and the fact that $P_c^{(1,0)}$ and $P_c^{(0,1)}$ are line bundles, we have

$$\left\| D_{\frac{\partial}{\partial \bar{z}}}\mu_\mp \right\| \leqslant \gamma |L^3 \mp iL^4| \left\| E^3 \pm iE^4 \right\| = \gamma \|\mu_\mp\|$$

for some $\gamma \in L^p_{\mathrm{loc}}(M)$. Thus the sections $\mu_\mp$ satisfy a Cauchy-Riemann type inequality; we can therefore apply Lemma 5.1 and deduce that they are of analytic type.

Assume now M compact. By virtue of Proposition 5.2, assuming γ_f not ± holomorphic and letting $z(\mu_\mp)$ be the sum of the orders of the zeros of $\mu_\mp$, using the properties of the Chern classes of line bundles we obtain

$$\begin{cases} z(\mu_-) = -2\chi(M) + \chi\left(P_c^{(0,1)}\right) = -2\chi(M) - \chi(P) \\ z(\mu_+) = -2\chi(M) + \chi\left(P_c^{(1,0)}\right) = -2\chi(M) + \chi(P). \end{cases}$$

We have therefore proved the following

THEOREM 6.2. *Let $f : M \to Q_4$ be an immersed compact surface satisfying* (84). *Then either $\gamma_f : M \to Q_2(\mathbb{R}^6)$ is $\pm$ holomorphic or*

$$2\chi(M) \leqslant -|\chi(P)|.$$

7. Willmore surfaces and S-Willmore surfaces

It is a known fact that an immersed surface $f : M \to Q_n$ is Willmore if and only if its conformal Gauss map is harmonic (as was first proved in [16]). This fact can be proved by directly computing the tension field of the conformal Gauss map, whose vanishing turns out to be equivalent to condition (63). This result is summarized in the following

THEOREM 7.1. *Let $f : M \to Q_n$ be an immersed oriented Riemann surface with conformal Gauss map $\gamma_f : M \to Q_{n-2}(\mathbb{R}^{n+2})$. Then f is a Willmore surface if and only if γ_f is harmonic.*

Let us now go back to surfaces in Q_4. In this context the concepts of harmonicity and $\pm$ holomorphicity of the conformal Gauss map both make sense, and since $\pm$ holomorphicity implies harmonicity, we find that $\pm$ isotropic surfaces in Q_4 are in particular Willmore surfaces.

In [9], Ejiri has introduced the notion of S-Willmore surface. In our setting, with respect to a Darboux frame along f, the notion corresponds to the two following conditions

$$\begin{aligned} &(a) \quad L^\alpha e_\alpha \not\parallel \overline{L^\alpha} e_\alpha \\ &(b) \quad k^\alpha e_\alpha /\!/ L^\alpha e_\alpha, \end{aligned} \tag{87}$$

whose conformal invariance is apparent once we recognize that, at $p \in M$, condition (87*a*) is equivalent to

$$\begin{vmatrix} L^3 & L^4 \\ \overline{L^3} & \overline{L^4} \end{vmatrix} \neq 0 \qquad \text{that is} \qquad L^3\overline{L^4} - \overline{L^3}L^4 \neq 0,$$

and, by (50), this translates to

$$K_N(p) \neq 0.$$

On the other hand, condition (87*b*) can be expressed as

$$k^3L^4 - k^4L^3 = 0,$$

so, by (56), condition (87*b*) is satisfied at $p \in M$ if and only if

$$\vartheta(p) = 0.$$

Ejiri proved that, in the Riemannian setting, an S-Willmore surface is a Willmore surface. This can be easily checked in our setting, too.

PROPOSITION 7.1. *Let $f : M \to Q_4$ be an S-Willmore surface, namely an immersed oriented Riemann surface such that $K_N \neq 0$ and $\vartheta = 0$. Then f is a Willmore surface.*

Proof. Suppose f is S-Willmore. In particular $k^3L^4 - k^4L^3 = 0$ on M. Setting $k^3L^4 - k^4L^3 = 0$ in (73), we can deduce that, in particular,

$$p^3_{kk}L^4 = p^4_{kk}L^3.$$

Assume by contradiction that f is not a Willmore surface, that is, either $p^3_{kk} \neq 0$ or $p^4_{kk} \neq 0$, say $p^3_{kk} \neq 0$. Then we have

$$iK_N = L^3\overline{L^4} - \overline{L^3}L^4 = \frac{p^4_{kk}}{p^3_{kk}}(L^3\overline{L^3} - \overline{L^3}L^3) = 0$$

which contradicts (87a). □

From the proof of Theorem 6.1, we have that γ_f is ± holomorphic if and only if $k^3 = \pm ik^4$ and $L^3 = \pm iL^4$, hence in this case we automatically have $\vartheta = 0$, so that

PROPOSITION 7.2. *Let $f : M \to Q_4$ be a ± isotropic immersed surface. Then f is S-Willmore if and only if $K_N \neq 0$ on M.*

The next result is another application of Lemma 5.1.

PROPOSITION 7.3. *Let $f : M \to Q_4$ be an immersion without umbilical points and such that the set of ± isotropic points is not discrete. If f satisfies condition (84), then f is S-Willmore.*

Proof. By Proposition 6.1, f must be ± isotropic. This implies $\vartheta = 0$ and

$$K_N = -i(L^3\overline{L^4} - \overline{L^3}L^4) = \mp 2\left|L^4\right|^2 = \mp 2\left|L^3\right|^2.$$

Therefore $K_N(p) = 0$ if and only if p is an umbilical point, and the result follows. □

8. Enneper-Weierstrass type representations for surfaces in Q_4

So far we have considered immersions of oriented surfaces in the conformal sphere Q_4 and we have associated to them certain maps with values in the conformal Grassmannian $\mathcal{Q}_2\left(\mathbb{R}^6\right)$, i.e. the conformal Gauss map. This map has some remarkable properties, for instance it is holomorphic if and only if the original immersion is − isotropic. Now we are going to do the converse: starting from a holomorphic map γ with values in $\mathcal{Q}_2\left(\mathbb{R}^6\right)$ we want to see if, and under what conditions, it is possible to retrieve a Q_4-valued map whose conformal Gauss map is exactly the map γ.

First of all, let us observe that, given a − isotropic immersion $f : M \to Q_4$, the conformal Gauss map γ_f is constant if and only if f is totally umbilical, namely $f(M) \subseteq Q_2$, or equivalently $W_K(f) = 0$ for any compact domain $K \subset M$.

Let M be a Riemann surface and $\gamma : M \to \mathcal{Q}_2(\mathbb{R}^6)$ a non constant holomorphic map. Let φ be a (local) $(1,0)$-form defining the complex structure on M and let $s : U \subset \mathcal{Q}_2(\mathbb{R}^6) \to \mathrm{Mob}(4)$ be a local section of $\widehat{\pi}$. Then

$$\gamma^*\zeta^0 = \Lambda^0\varphi, \qquad \gamma^*\zeta^k = \Lambda^k\varphi, \qquad \gamma^*\zeta^3 = \Lambda^3\varphi, \tag{88}$$

where ζ^0, ζ^k and ζ^3 are defined as in (37) with respect to the section s. The vector Λ of components Λ^0, Λ^k, Λ^3 is of analytic type, i.e. it either vanishes identically or has isolated zeros. Indeed, let ω be such that $d\varphi = i\omega \wedge \varphi$; then, differentiating (88) and using (37) and the structure equations, we have

$$\begin{aligned} d(\gamma^*\zeta^0) &= d\Lambda^0 \wedge \varphi + \Lambda^0 d\varphi = \left(d\Lambda^0 + i\Lambda^0\omega\right) \wedge \varphi = \\ &= \gamma^* d\zeta^0 = -\gamma^* s^*(\Phi^0_0 + i\Phi^4_3) \wedge \Lambda^0\varphi - \gamma^* s^*\Phi^0_k \wedge \Lambda^k\varphi. \end{aligned}$$

Hence

$$\left(d\Lambda^0 + i\Lambda^0\omega + \Lambda^0\gamma^* s^*\left(\Phi^0_0 + i\Phi^4_3\right) + \Lambda^k\gamma^* s^*\Phi^0_k\right) \wedge \varphi = 0$$

and similarly for $d(\gamma^*\zeta^k)$ and $d(\gamma^*\zeta^3)$, so that we obtain

$$\begin{cases} d\Lambda^0 = -i\Lambda^0\left(\omega + \gamma^*s^*\Phi^4_3 - i\gamma^*s^*\Phi^0_0\right) - \Lambda^k\gamma^*s^*\Phi^0_k & \mod \varphi \\ d\Lambda^k = -i\Lambda^k\left(\omega + \gamma^*s^*\Phi^4_3\right) - \Lambda^j\gamma^*s^*\Phi^k_j - \Lambda^0\gamma^*s^*\Phi^k_0 - \Lambda^3\gamma^*s^*\Phi^0_k & \mod \varphi \\ d\Lambda^3 = -i\Lambda^3\left(\omega + \gamma^*s^*\Phi^4_3 + i\gamma^*s^*\Phi^0_0\right) - \Lambda^k\gamma^*s^*\Phi^k_0 & \mod \varphi. \end{cases}$$

Thus $d\Lambda^a = \Psi^a_b\Lambda^b$ modulo φ, for some $\mathfrak{gl}(4,\mathbb{C})$-valued 1-form $\Psi = (\Psi^a_b)$, namely the vector Λ is a solution of the system

$$\frac{\partial\Lambda}{\partial\bar{z}} = \Psi\left(\frac{\partial}{\partial\bar{z}}\right)\Lambda$$

and, by Lemma 5.1 (but see also [7] for a direct proof of this case), the claim follows.

Since we assumed γ to be non constant, it follows that the zeros of Λ are isolated, and in a neighborhood of any zero, Λ factorizes as $\Lambda = z^t\widetilde{\Lambda}$, with $\widetilde{\Lambda} \neq 0$, z a local holomorphic chart centered at the zero and $t \in \mathbb{N}$.

Since $\mathcal{Q}_2(\mathbb{R}^6)$ can be identified with an open subset of a quadric in $\mathbb{P}^5_{\mathbb{C}}$, the map γ can be lifted to a smooth, $\mathbb{C}^6\backslash\{0\}$-valued map $\{\gamma\} = e_3 + ie_4$, where $e = s \circ \gamma : U \subset M \to \mathrm{Mob}(4)$ (note that e is not necessarily an immersion, because in general γ is not). Denoting $\phi = e^{-1}de$, we have

$$\begin{aligned} \Lambda^0\varphi &= \gamma^*\zeta^0 = e^*\Phi^0_3 + ie^*\Phi^0_4 = \phi^0_3 + i\phi^0_4, \\ \Lambda^k\varphi &= \gamma^*\zeta^k = \phi^k_3 + i\phi^k_4, \\ \Lambda^3\varphi &= \gamma^*\zeta^3 = \phi^5_3 + i\phi^5_4, \end{aligned}$$

and since $de = e\phi$,

$$\begin{aligned} d\{\gamma\} &= i(e_3 + ie_4)\phi^3_4 + e_0(\phi^0_3 + i\phi^0_4) + e_k(\phi^k_3 + i\phi^k_4) + e_5(\phi^5_3 + i\phi^5_4) = \\ &= i\{\gamma\}\phi^3_4 + (\Lambda^0 e_0 + \Lambda^k e_k + \Lambda^3 e_5)\varphi \end{aligned}$$

If $p:\mathbb{C}^6\backslash\{0\}\to\mathbb{P}^5_{\mathbb{C}}$ is the canonical projection, then $\gamma = p\circ\{\gamma\}$ and

$$d\gamma_x = \gamma_{*x} = p_{*\{\gamma\}(x)}\{\gamma\}_{*x} = \varphi\, p_{*\{\gamma\}(x)}\Big(\Lambda^0 e_0 + \Lambda^k e_k + \Lambda^3 e_5\Big).$$

The complex tangent line to the curve $\gamma(M)$ at the point $\gamma(x)$ is therefore the vector space spanned over $\mathbb{C}$ by the non-zero vector $p_{*\{\gamma\}(x)}(\Lambda^0 e_0 + \Lambda^k e_k + \Lambda^3 e_5)$. This prompts us to define a new map, called the "derivative" of γ, $\gamma' : M \to \mathbb{P}^5_{\mathbb{C}}$ which associates to the point $x\in M$ the projectivization of the non-zero vector $\Lambda^0 e_0 + \Lambda^k e_k + \Lambda^3 e_5$. This map is trivially well defined and does not depend on the choice of the section s.

We will need to add the further assumption that γ' be valued in the quadric $\mathcal{Q}_2\left(\mathbb{R}^6\right)$; this happens if and only if the vector ${}^t(\Lambda^0,\Lambda^k,0,0,\Lambda^3)$ satisfies the equation

$$-2\Lambda^0\Lambda^3 + \Lambda^k\Lambda^k = 0.$$

DEFINITION 8.1. *A map $\gamma : M \to \mathcal{Q}_2\left(\mathbb{R}^6\right)$ will be called a* ***totally isotropic holomorphic map*** *if it is holomorphic, non constant, and if γ' is valued in $\mathcal{Q}_2\left(\mathbb{R}^6\right)$.*

Let $\tilde{s}$ be another local section of the bundle $\hat{\pi} : \mathrm{Mob}(4) \to \mathcal{Q}_2\left(\mathbb{R}^6\right)$, and $\tilde{e} = \tilde{s}\circ\gamma$. Then $\tilde{e} = eK$ where K takes values in H_0 as defined in (27). At any point $p\in M$ we can therefore choose a section such that $\Lambda^3 = 0$, hence $\Lambda^0 = a$, $\Lambda^1 = \lambda$ and $\Lambda^2 = i\lambda$, for some $a,\lambda\in\mathbb{C}$. Since Λ is of analytic type, such sections can be locally smoothly chosen in a neighborhood of p. The frame e corresponding to such section will be called an **isotropic frame**, and the isotropy subgroup for such frames is exactly $\mathrm{Mob}(4)_D$ as defined in (15). With this choice of frame, (88) rewrites as

$$\gamma^*\zeta^0 = a\varphi, \quad \gamma^*\zeta^1 = \lambda\varphi, \quad \gamma^*\zeta^2 = i\lambda\varphi, \quad \gamma^*\zeta^3 = 0. \tag{89}$$

We can associate, to any totally isotropic holomorphic map γ, a map $J_\gamma : M \to Q_4$ defined as follows. Let e be any isotropic frame along γ and set $J_\gamma = [e_0]$. In this way J_γ is well defined, because isotropic frames change by matrices in $\mathrm{Mob}(4)_D$. Differentiating the second and third equalities of (89), we obtain

$$\begin{aligned} d(\gamma^*\zeta^1) &= -\phi^1_0\wedge\gamma^*\zeta^0 - \phi^1_2\wedge\gamma^*\zeta^2 - i\phi^4_3\wedge\gamma^*\zeta^1 - \phi^0_1\wedge\gamma^*\zeta^3 = \\ &= \left(-a\phi^1_0 - i\lambda\phi^1_2 - i\lambda\phi^4_3\right)\wedge\varphi, \\ d(\gamma^*\zeta^2) &= (-a\phi^2_0 - \lambda\phi^2_1 + \lambda\phi^4_3)\wedge\varphi, \end{aligned}$$

but on the other hand $\gamma^*\zeta^2 = i\gamma^*\zeta^1$, so we have

$$\left(-ia\phi^1_0 + \lambda\phi^1_2 + \lambda\phi^4_3\right)\wedge\varphi = \left(-a\phi^2_0 - \lambda\phi^2_1 + \lambda\phi^4_3\right)\wedge\varphi$$

that is, $ia\left(\phi^1_0 + i\phi^2_0\right)\wedge\varphi = 0$. Differentiating the last of (89) we get

$$0 = d(\gamma^*\zeta^3) = \left(-\lambda\phi^1_0 - i\lambda\phi^2_0\right)\wedge\varphi.$$

Therefore we have obtained

$$\begin{aligned} a\left(\phi^1_0 + i\phi^2_0\right)\wedge\varphi &= 0 \\ \lambda\left(\phi^1_0 + i\phi^2_0\right)\wedge\varphi &= 0 \end{aligned}$$

Since Λ is of analytic type, outside a discrete set (the set of zeros of a and λ), we must have

$$\phi_0^1 + i\phi_0^2 = \mu\varphi \tag{90}$$

for some locally defined complex function μ, whose vanishing is independent of the choice of the isotropic frame. Differentiating (90), we have

$$d\mu \wedge \varphi + i\mu\omega \wedge \varphi = d\phi_0^1 + id\phi_0^2 = \mu\phi_0^0 \wedge \varphi + i\mu\phi_2^1 \wedge \varphi,$$

that is

$$d\mu = -i\mu\big(\omega - \phi_2^1 + i\phi_0^0\big) \qquad \operatorname{mod} \varphi.$$

Therefore μ is of analytic type, and so it either vanishes identically or has isolated zeros.

Let us now consider an open set $U \subset M$ where μ is nonzero and let e be an isotropic frame along γ defined on U. Then e is trivially a zeroth order frame along J_γ, since $\pi \circ e = J_\gamma$. Moreover, it is a first order frame, since from (89)

$$0 = \gamma^*\zeta^3 = \phi_0^3 + i\phi_0^4,$$

so $\phi_0^\alpha = 0$. Also, J_γ is a conformal immersion on U, since the only points where J_γ is not an immersion are the zeros of μ. In the case of μ vanishing identically, then J_γ is constant. Indeed in this case not only $\phi_0^\alpha = 0$, but also $\phi_0^1 = \phi_0^2 = 0$. So

$$dJ_\gamma = p_* de_0 = p_*(e_0\phi_0^0 + e_A\phi_0^A) = \phi_0^A p_* e_A = 0$$

where $p : \mathbb{R}^6 \backslash \{0\} \to \mathbb{P}^5_{\mathbb{R}}$ is the canonical projection.

Thus, either J_γ is constant on M or it is a weakly conformal branched immersion. Assume to be in this latter case; we will prove that an isotropic frame e along γ is a Darboux frame along J_γ.

To this end we use (89) to deduce that

$$\gamma^*\zeta^2 = i\gamma^*\zeta^1. \tag{91}$$

Now we set, as usual, $\phi_i^\alpha = h_{ij}^\alpha \phi_0^j$, $h_{ij}^\alpha = h_{ji}^\alpha$, and observe that

$$\gamma^*\zeta^k = e^*(\Phi_3^k + i\Phi_4^k) = -\phi_k^3 - i\phi_k^4 = -(h_{kj}^3 + ih_{kj}^4)\phi_0^j$$

and equation (91) is equivalent to the following system

$$\begin{cases} h_{1j}^3 = h_{2j}^4 \\ h_{2j}^3 = -h_{1j}^4 \end{cases}$$

which gives

$$h_{11}^3 = h_{21}^4 = -h_{22}^3, \qquad h_{11}^4 = -h_{21}^3 = -h_{22}^4.$$

Moreover, it is trivial to see that, outside the branch points of J_γ, we have $\gamma_{J_\gamma} = \gamma$, and J_γ is $-$ isotropic, since γ_{J_γ} is holomorphic by assumption.

On the other hand, consider a weakly conformal branched immersion $f : M \to Q_4$ with the property that its Gauss map γ_f can be continuously extended to the branch points, and let e be any Darboux frame along f. If f is $-$ isotropic (outside the branch points), then γ_f is holomorphic, and in this case, with the notations of (88), we have

$$\Lambda^0 = k^3 + ik^4, \qquad \Lambda^1 = -\frac{1}{2}\left(L^3 + iL^4\right), \qquad \Lambda^2 = -\frac{i}{2}\left(L^3 + iL^4\right), \qquad \Lambda^3 = 0,$$

so that

$$-2\Lambda^0\Lambda^3 + \sum_k \Lambda^k\Lambda^k = 0$$

and γ_f is a totally isotropic map. Furthermore, $J_{\gamma_f} = f$.

We have therefore proved the following

THEOREM 8.1. *Let M be a Riemann surface. There is a bijective correspondence between $-$ isotropic, non totally umbilical, weakly conformal branched immersions $f : M \to Q_4$, whose conformal Gauss map can be continuously extended at the branch points, and non constant, holomorphic, totally isotropic maps $\gamma : M \to \mathcal{Q}_2\left(\mathbb{R}^6\right)$ with non constant associated map J_γ. The bijection is realized via the conformal Gauss map.*

Using an appropriate Grassmann bundle, we can extend the previous result so as to include the totally umbilical surfaces.

Let us consider the product manifold $Q_4 \times \mathcal{Q}_2\left(\mathbb{R}^6\right)$ and define $\mathcal{Q}_2(Q_4)$ as the orbit of the point $([\eta_0], [\varepsilon_3, \varepsilon_4]) \in Q_4 \times \mathcal{Q}_2\left(\mathbb{R}^6\right)$ with respect to the natural left action (defined componentwise) of the group $\mathrm{Mob}(4)$. In other words

$$\mathcal{Q}_2(Q_4) = \{([\eta], [s_1, s_2]) \mid \eta = P\eta_0,\ s_1 = P\varepsilon_3,\ s_2 = P\varepsilon_4,\ P \in \mathrm{Mob}(4)\}. \tag{92}$$

It is trivial to see that $\mathrm{Mob}(4)$ acts transitively on $\mathcal{Q}_2(Q_4)$, the action being given, for $P \in \mathrm{Mob}(4)$ and $([\eta], [s_1, s_2]) \in \mathcal{Q}_2(Q_4)$, by

$$P([\eta], [s_1, s_2]) = ([P\eta], [Ps_1, Ps_2]).$$

Let us compute the isotropy subgroup of the point $([\eta_0], [\varepsilon_3, \varepsilon_4])$. If $P \in \mathrm{Mob}(4)$ fixes the point $([\eta_0], [\varepsilon_3, \varepsilon_4])$, then in particular it must fix the first component, hence P must be an element of G_0, defined in (3), so it is bound to be of the form

$$P = \begin{pmatrix} r^{-1} & {}^t xA & \frac{1}{2}r|x|^2 \\ 0 & A & rx \\ 0 & 0 & r \end{pmatrix}.$$

But, for $P[\varepsilon_3]$ to belong to $[\varepsilon_3, \varepsilon_4]$, we must have $x^3 = 0$, $A^1_3 = A^2_3 = 0$ and analogously, imposing $P[\varepsilon_4] \in [\varepsilon_3, \varepsilon_4]$, we deduce $x^4 = 0$ and $A^1_4 = A^2_4 = 0$. Putting these conditions together we find that $P \in \mathrm{Mob}(4)_D$. Since in turn any element of $\mathrm{Mob}(4)_D$ fixes $([\eta_0], [\varepsilon_3, \varepsilon_4])$, we can conclude that the isotropy subgroup is exactly $\mathrm{Mob}(4)_D$. Hence $\mathcal{Q}_2(Q_4) \simeq \mathrm{Mob}(4)/\mathrm{Mob}(4)_D$ is realized as a homogeneous space with projection

$$\bar{\pi} : \mathrm{Mob}(4) \to \mathcal{Q}_2(Q_4)$$

given by

$$\bar{\pi} : P \mapsto ([P\eta_0], [P\varepsilon_3, P\varepsilon_4]),$$

that is, $\bar{\pi} = \pi \times \hat{\pi}$. Also, we will denote by $\check{\pi} : Q_2(Q_4) \to Q_4$ the canonical projection

$$\check{\pi} : ([\eta], [s_1, s_2]) \mapsto [\eta].$$

Observe that $Q_2(Q_4)$ has a natural integrable complex structure defined as follows: let ξ be a local section of the bundle $\bar{\pi} : \mathrm{Mob}(4) \to Q_2(Q_4)$; then we declare the forms

$$\begin{aligned} \sigma^{-1} &= \xi^*\Phi^1_0 + i\xi^*\Phi^2_0, \\ \sigma^0 &= \xi^*\Phi^0_3 + i\xi^*\Phi^0_4, \\ \sigma^k &= \xi^*\Phi^k_3 + i\xi^*\Phi^k_4, \\ \sigma^3 &= \xi^*\Phi^3_0 + i\xi^*\Phi^4_0 \end{aligned} \tag{93}$$

a local basis of the space of the forms of type $(1,0)$ over $Q_2(Q_4)$. In order to do this, first we need to check that the ideal they generate is differential. Setting, for the sake of simplicity, $\varphi = \xi^*\Phi$ and using the structure equations, we have

$$\begin{aligned} d\sigma^{-1} = &-\sigma^{-1} \wedge (\varphi^0_0 + i\varphi^1_2) - \varphi^1_3 \wedge \varphi^3_0 - \varphi^1_4 \wedge \varphi^4_0 - i\varphi^2_3 \wedge \varphi^3_0 - i\varphi^2_4 \wedge \varphi^4_0 = \\ = &-\sigma^{-1} \wedge (\varphi^0_0 + i\varphi^1_2) + i\sigma^1 \wedge \varphi^4_0 + i\sigma^2 \wedge \varphi^3_0 + \sigma^3 \wedge (\varphi^1_3 + \varphi^2_4) \end{aligned}$$

and likewise for the differentials of the other forms. Lastly, one can easily check that the space generated by these forms is well defined, i.e., it is independent of the choice of the section ξ.

PROPOSITION 8.1. *The fibers of* $\check{\pi} : Q_2(Q_4) \to Q_4$ *are integral submanifolds of the (invariantly defined) Pfaffian system*

$$\begin{cases} \sigma^{-1} = 0 \\ \sigma^3 = 0. \end{cases} \tag{94}$$

Proof. Since $Q_2(Q_4) \subset Q_4 \times Q_2(\mathbb{R}^6)$, for $([\eta], [s_1, s_2]) \in Q_2(Q_4)$, we have

$$T_{([\eta],[s_1,s_2])} Q_2(Q_4) \subset T_{[\eta]} Q_4 \times T_{[s_1,s_2]} Q_2\left(\mathbb{R}^6\right).$$

Thus, we can regard a tangent vector of $Q_2(Q_4)$ as a pair (X, V) with $X \in T_{[\eta]} Q_4$ and $V \in T_{[s_1,s_2]} Q_2(\mathbb{R}^6)$. Now $\check{\pi}$ is the projection on the first component, so

$$\check{\pi}_{*([\eta],[s_1,s_2])}(X, V) = X$$

and

$$\ker \check{\pi}_{*([\eta],[s_1,s_2])} = \left\{(0, V) \in T_{([\eta],[s_1,s_2])} Q_2(Q_4)\right\}$$

We want to prove that

$$\begin{cases} \sigma^{-1}_{([\eta],[s_1,s_2])}(0, V) = 0 \\ \sigma^3_{([\eta],[s_1,s_2])}(0, V) = 0, \end{cases}$$

or equivalently that, if ξ is a local section of $\bar{\pi}:\mathrm{Mob}(4)\to Q_2(Q_4)$, then

$$\xi^*\Phi^A_{0\,([\eta],[s_1,s_2])}(0,V)=0.$$

To this end we set $g=\xi([\eta],[s_1,s_2])$ and compute

$$\begin{aligned}\xi^*\Phi^A_{0\,([\eta],[s_1,s_2])}(0,V)&=\Phi^A_{0\,g}(\xi_{*([\eta],[s_1,s_2])}(0,V))=\big(\Phi_g(\xi_{*([\eta],[s_1,s_2])}(0,V))\big)^A_0=\\&=(g^{-1})^A_b\big(\xi_{*([\eta],[s_1,s_2])}(0,V)\big)^b_0,\end{aligned}$$

where in the last equality we used the definition of the Maurer-Cartan form for classical groups (see [1] or [3] for details):

$$\Phi_P(X)=P^{-1}X.$$

Now take $([\tilde{\eta}],[\tilde{s}_1,\tilde{s}_2])$ in the domain of ξ, set $\tilde{g}=\xi([\tilde{\eta}],[\tilde{s}_1,\tilde{s}_2])$ and observe that

$$\bar{\pi}(\xi([\tilde{\eta}],[\tilde{s}_1,\tilde{s}_2]))=\bar{\pi}(\tilde{g})=([\tilde{g}\eta_0],[\tilde{g}\varepsilon_3,\tilde{g}\varepsilon_4])$$

and, since $\bar{\pi}\circ\xi=id$,

$$([\tilde{\eta}],[\tilde{s}_1,\tilde{s}_2])=(\bar{\pi}\circ\xi)([\tilde{\eta}],[\tilde{s}_1,\tilde{s}_2])=([\tilde{g}\eta_0],[\tilde{g}\varepsilon_3,\tilde{g}\varepsilon_4]).$$

In particular we have that $[\tilde{\eta}]=[\tilde{g}\eta_0]$ and

$$[\tilde{g}\eta_0]=[\tilde{g}_0]=[(\xi([\tilde{\eta}],[\tilde{s}_1,\tilde{s}_2]))_0]=[\xi_0([\tilde{\eta}],[\tilde{s}_1,\tilde{s}_2])],$$

that is, the projective class of the vector $\xi_0([\tilde{\eta}],[\tilde{s}_1,\tilde{s}_2])$ coincides with that of $\tilde{\eta}$. In other words, calling

$$p:\mathbb{R}^6\backslash\{0\}\to\mathbb{P}^5_{\mathbb{R}}$$

the canonical projection, we find that $p(\xi_0([\tilde{\eta}],[\tilde{s}_1,\tilde{s}_2]))=p(\tilde{\eta})$. Hence $p\circ\xi_0=\check{\pi}$ and

$$(p\circ\xi_0)_{*([\tilde{\eta}],[\tilde{s}_1,\tilde{s}_2])}(0,V)=\check{\pi}_{*([\tilde{\eta}],[\tilde{s}_1,\tilde{s}_2])}(0,V)=0,$$

that is

$$p_{*\xi_0([\tilde{\eta}],[\tilde{s}_1,\tilde{s}_2])}\xi_{0*([\tilde{\eta}],[\tilde{s}_1,\tilde{s}_2])}(0,V)=0.$$

Thus $\xi_{0*([\tilde{\eta}],[\tilde{s}_1,\tilde{s}_2])}(0,V)\in\ker p_{*\xi_0([\tilde{\eta}],[\tilde{s}_1,\tilde{s}_2])}$, implying

$$\xi_{0*([\tilde{\eta}],[\tilde{s}_1,\tilde{s}_2])}(0,V)=\lambda\xi_0([\tilde{\eta}],[\tilde{s}_1,\tilde{s}_2])$$

for some $\lambda\in\mathbb{R}$. Therefore

$$\big(\xi_{*([\eta],[s_1,s_2])}(0,V)\big)^b_0=\lambda(\xi([\eta],[s_1,s_2]))^b_0=\lambda g^b_0.$$

So eventually,

$$\xi^*\Phi^A_{0\,([\eta],[s_1,s_2])}(0,V)=\lambda(g^{-1})^A_b g^b_0=\lambda\delta^A_0=0.$$

□

Let us consider the canonical projection $c : \mathcal{Q}_2(Q_4) \to \mathcal{Q}_2\left(\mathbb{R}^6\right)$ defined by

$$c([\eta],[s_1,s_2]) = [s_1,s_2],$$

which makes the following diagram commutative

$$\begin{tikzcd}
 & \mathrm{Mob}(4) \arrow[dl, "\bar{\pi}"'] \arrow[dr, "\hat{\pi}"] & \\
\mathcal{Q}_2(Q_4) \arrow[rr, "c"] & & \mathcal{Q}_2\left(\mathbb{R}^6\right)
\end{tikzcd}$$

that is, $\hat{\pi} = c \circ \bar{\pi}$.

PROPOSITION 8.2. *The map $c : \mathcal{Q}_2(Q_4) \to \mathcal{Q}_2\left(\mathbb{R}^6\right)$ defined above is holomorphic.*

Proof. Fix $p_0 = ([\eta],[s_1,s_2]) \in \mathcal{Q}_2(Q_4)$ and consider ξ a local section of the bundle $\bar{\pi} : \mathrm{Mob}(4) \to \mathcal{Q}_2(Q_4)$, defined on a neighborhood of p_0 and ς a local section of the bundle $\hat{\pi} : \mathrm{Mob}(4) \to \mathcal{Q}_2\left(\mathbb{R}^6\right)$ defined on a neighborhood of $[s_1,s_2]$. We have to show that $c^*\zeta^0$, $c^*\zeta^k$ and $c^*\zeta^3$, defined as in (37), are forms of type $(1,0)$.

Set $g_0 = \xi(p_0)$. As in the proof of Theorem 6.1, we can assume that the section ς satisfies $\varsigma(\hat{\pi}(g_0)) = g_0$, and

$$(\varsigma \circ \hat{\pi})^*(\Phi^0_\alpha)_{g_0} = (\Phi^0_\alpha)_{g_0}.$$

Then, observing that $c = \hat{\pi} \circ \xi$, we have that

$$\left(c^*\zeta^0\right)_{p_0} = \left(\xi^*\hat{\pi}^*\zeta^0\right)_{p_0} = \xi^*\left(\hat{\pi}^*\varsigma^*(\Phi^0_3 + i\Phi^0_4)\right)_{g_0} = \xi^*\Phi^0_{3g_0} + i\xi^*\Phi^0_{4g_0} = \sigma^0_{p_0},$$

and analogously for $c^*\zeta^k$ and $c^*\zeta^3$. □

DEFINITION 8.2. *Let $f : M \to Q_4$ be an immersed oriented surface. The* ***conformal Gauss lift*** *$\Gamma_f : M \to \mathcal{Q}_2(Q_4)$ is defined as*

$$\Gamma_f = f \times \gamma_f,$$

that is, given $p \in M$ and e any Darboux frame along f, defined on a neighborhood of p,

$$\Gamma_f = \bar{\pi} \circ e;$$

in other words,

$$\Gamma_f : p \mapsto ([e_0]_p, [e_3,e_4]_p).$$

We are now ready to state the generalization of Theorem 8.1.

THEOREM 8.2. *Let M be a Riemann surface. There is a bijective correspondence between – isotropic, weakly conformal branched immersions $f : M \to Q_4$ whose* *conformal Gauss map can be continuously extended at the branch points, and holomorphic maps* $\Gamma : M \to Q_2(Q_4)$*, solutions of the Pfaffian system*

$$\begin{cases} \sigma^3 = 0 \\ \sigma^2 - i\sigma^1 = 0 \end{cases}$$

but not of $\sigma^{-1} = 0$*. The bijection is realized via the conformal Gauss lift* Γ_f*.*

Proof. Let $f : M \to Q_4$ be as in the statement of the theorem. Then, in order to show that the conformal Gauss lift Γ_f is holomorphic, we proceed as for the conformal Gauss map γ_f in the proof of Theorem 6.1. Let us fix $p_0 \in M$ such that it is not a branch point for f and choose a Darboux frame e along f defined on a neighborhood U of p_0 and a section ξ of the bundle $\bar{\pi} : \mathrm{M\ddot{o}b}(4) \to Q_2(Q_4)$ defined in a neighborhood of $\Gamma_f(p_0)$. We set $e(p_0) = g_0$; then since $\bar{\pi} \circ (\xi \circ \bar{\pi}) = \bar{\pi}$, there must exist a function $K : \bar{\pi}^{-1}(U) \to \mathrm{M\ddot{o}b}(n)_D$ such that, for every $g \in \bar{\pi}^{-1}(U)$

$$\xi(\bar{\pi}(g)) = gK(g)$$

and

$$(\xi \circ \bar{\pi})^* \Phi_g = K(g)^{-1} g^{-1} dg K(g) + K(g)^{-1} dK_g$$

In particular we have

$$\begin{aligned}
(\xi \circ \bar{\pi})^* \Phi^k_{0g} &= \big(K(g)^{-1} g^{-1} dg K(g)\big)^k_0 \\
(\xi \circ \bar{\pi})^* \Phi^0_{\alpha g} &= \big(K(g)^{-1} g^{-1} dg K(g)\big)^0_\alpha \\
(\xi \circ \bar{\pi})^* \Phi^k_{\alpha g} &= \big(K(g)^{-1} g^{-1} dg K(g)\big)^k_\alpha \\
(\xi \circ \bar{\pi})^* \Phi^\alpha_{0g} &= \big(K(g)^{-1} g^{-1} dg K(g)\big)^\alpha_0,
\end{aligned}$$

because $K^{-1}dK$ is valued in the Lie algebra of the group $\mathrm{M\ddot{o}b}(4)_D$. Replacing, if necessary, the section ξ with $\xi K(g_0)^{-1}$, we can assume that

$$\xi(\bar{\pi}(g_0)) = g_0$$

and hence

$$\begin{aligned}
(\xi \circ \bar{\pi})^* \Phi^k_{0g_0} &= \Phi^k_{0g_0} \\
(\xi \circ \bar{\pi})^* \Phi^0_{\alpha g_0} &= \Phi^0_{\alpha g_0} \\
(\xi \circ \bar{\pi})^* \Phi^k_{\alpha g_0} &= \Phi^k_{\alpha g_0} \\
(\xi \circ \bar{\pi})^* \Phi^\alpha_{0g_0} &= \Phi^\alpha_{0g_0}.
\end{aligned}$$

Therefore we can compute

$$\left(\Gamma_f^* \sigma^{-1}\right)_{p_0} = \big((\xi \circ \bar{\pi} \circ e)^* (\Phi^1_0 + i\Phi^2_0)\big)_{p_0} = \big(e^*(\Phi^1_0 + i\Phi^2_0)\big)_{p_0} = \varphi_{p_0} \tag{95}$$

and likewise for σ^k and σ^3. This proves the holomorphicity of Γ_f outside the set of branch points of f. But since f is continuous and by assumption γ_f can be continuously extended to the branch points, then $\Gamma_f = f \times \gamma_f$ is continuous on M, and therefore holomorphic.

The same computation also proves that Γ_f is a solution of the Pfaffian system $\sigma^3 = 0$, $\sigma^2 - i\sigma^1 = 0$, since it is easily verified that

$$\begin{aligned}
\Gamma_f^*\sigma^3 &= 0,\\
\Gamma_f^*\sigma^1 &= -\frac{1}{2}\left(L^3 + iL^4\right)\varphi\\
\Gamma_f^*\sigma^2 &= -\frac{i}{2}\left(L^3 + iL^4\right)\varphi.
\end{aligned}$$

Moreover, (95) assures that

$$\Gamma_f^*\sigma^{-1} \neq 0.$$

On the contrary, assume $\Gamma : M \to Q_2(Q_4)$ is a holomorphic map such that $\Gamma^*\sigma^3 = 0$, $\Gamma^*\sigma^2 = i\Gamma^*\sigma^1$ and $\Gamma^*\sigma^{-1} \neq 0$ and define $f_\Gamma = \check{\pi} \circ \Gamma$. For any local section ξ of $\bar{\pi}$, the map $e = \xi \circ \Gamma$ is a local frame along f_Γ, since

$$\pi \circ e = \pi \circ \xi \circ \Gamma = \check{\pi} \circ \bar{\pi} \circ \xi \circ \Gamma = \check{\pi} \circ \Gamma = f_\Gamma.$$

Moreover, let φ be a local $(1,0)$-form defining the complex structure on M; then, since Γ is holomorphic, there must exist a smooth function $\mu \not\equiv 0$ such that

$$e^*(\Phi_0^1 + i\Phi_0^2) = \Gamma^*\sigma^{-1} = \mu\varphi.$$

As usual, we set $\phi = e^*\Phi$, so that the previous equality becomes $\phi_0^1 + i\phi_0^2 = \mu\varphi$. Differentiating this last equality and using the structure equation we can deduce that

$$d\mu = -i\mu\left(\omega - \phi_2^1 + i\phi_0^0\right) \qquad \operatorname{mod} \varphi,$$

where ω is such that $d\varphi = i\omega \wedge \varphi$. Hence μ is of analytic type, and its zeros must be isolated and of finite order, proving that f_Γ is a weakly conformal branched immersion. In addition, since by assumption $\Gamma^*\sigma^3 = 0$, we know that e is a first order frame along f_Γ. We can prove that e is actually a Darboux frame along f_Γ using

$$\Gamma^*\sigma^2 = i\Gamma^*\sigma^1. \tag{96}$$

Indeed, setting as usual $\phi_i^\alpha = h_{ij}^\alpha \phi_0^j$, $h_{ij}^\alpha = h_{ji}^\alpha$,

$$\Gamma^*\sigma^k = e^*(\Phi_3^k + i\Phi_4^k) = -\phi_k^3 - i\phi_k^4 = -(h_{kj}^3 + ih_{kj}^4)\phi_0^j$$

and equation (96) becomes

$$\begin{cases} h_{1j}^3 = h_{2j}^4 \\ h_{2j}^3 = -h_{1j}^4 \end{cases}$$

which gives

$$h_{11}^3 = h_{21}^4 = -h_{22}^3, \qquad h_{11}^4 = -h_{21}^3 = -h_{22}^4.$$

Now since $e = \xi \circ \Gamma$ is a Darboux frame along f_Γ, it makes sense to consider its conformal Gauss map, defined as usual as

$$\gamma_{f_\Gamma} = [e_3, e_4] = \hat{\pi} \circ e$$

outside the branch points of f_Γ. We want to prove that γ_{f_Γ} can be continuously extended at the branch points, and that the extension is holomorphic. To this end, we define $\gamma : M \to Q_2\big(\mathbb{R}^6\big)$ as follows

$$\gamma = c \circ \Gamma \tag{97}$$

and observe that Proposition 8.2 implies that γ is holomorphic. By the commutativity of the following diagram

$$\begin{tikzcd}
 & & \mathrm{Mob}(4) \arrow[dl, "\bar{\pi}"'] \arrow[dr, "\hat{\pi}"] & \\
 & Q_2(Q_4) \arrow[rr, "c"] \arrow[dl, "\check{\pi}"'] & & Q_2\big(\mathbb{R}^6\big) \\
Q_4 & & M \arrow[ll, "f_\Gamma"] \arrow[ul, "\Gamma"'] \arrow[ur, "\gamma"'] &
\end{tikzcd}$$

we have that, on the open set where γ_{f_Γ} is defined,

$$\gamma_{f_\Gamma} = \hat{\pi} \circ e = \hat{\pi} \circ \xi \circ \Gamma = c \circ \bar{\pi} \circ \xi \circ \Gamma = c \circ \Gamma = \gamma.$$

Therefore γ_{f_Γ} is holomorphic, hence f_Γ is $-$ isotropic. Lastly, we obviously have

$$\Gamma_{f_\Gamma} = \bar{\pi} \circ e = \bar{\pi} \circ \xi \circ \Gamma = \Gamma$$

and

$$f_{\Gamma_f} = \check{\pi} \circ \Gamma_f = \check{\pi} \circ \bar{\pi} \circ e = \pi \circ e = f,$$

so the claim is proved. □

9. Appendix: Euclidean vs conformal description of Willmore surfaces

We shall now compare the Riemannian structure of an isometric immersion $F : M^m \to \mathbb{R}^n$ with its conformal counterpart. Let $\langle\ ,\ \rangle$ denote the metric induced on M via F. The Euclidean space $\mathbb{R}^n$ can be given the homogeneous structure $\mathrm{E}(n)/\mathrm{E}(n)_0$, where

$$\begin{aligned}
\mathrm{E}(n) &= \left\{ \begin{pmatrix} 1 & 0 \\ z & A \end{pmatrix} : \begin{array}{c} z \in \mathbb{R}^n, \\ A \in \mathrm{SO}(n) \end{array} \right\}, \\
\mathrm{E}(n)_0 &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} : A \in \mathrm{SO}(n) \right\} \simeq \mathrm{SO}(n).
\end{aligned}$$

Via the Dirac-Weyl chart χ in (1), $\mathbb{R}^n$ is isometric to $\chi(\mathbb{R}^n) \subset Q_n$ endowed with its homogeneous space structure

$$\chi(\mathbb{R}^n) = \frac{\mathbb{E}(n)}{\mathbb{E}(n)_0} \subseteq \frac{\mathrm{Mob}(n)}{\mathrm{Mob}(n)_0} = Q_n,$$

with

$$\begin{aligned}
\mathbb{E}(n) &= \left\{ \begin{pmatrix} 1 & 0 & 0 \\ z & A & 0 \\ \frac{1}{2}|z|^2 & {}^t zA & 1 \end{pmatrix} : \begin{array}{l} z \in \mathbb{R}^n, \\ A \in \mathrm{SO}(n) \end{array} \right\} \leqslant \mathrm{Mob}(n), \\
\mathbb{E}_0(n) &= \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{pmatrix} : A \in \mathrm{SO}(n) \right\} \leqslant \mathrm{Mob}(n)_0, \quad \mathrm{Mob}(n)_0 \text{ as in (3).}
\end{aligned}$$

We associate to F the immersion $f = \chi \circ F : M \to Q_n$. Therefore, we have the following commutative diagram:

$$\begin{tikzcd}
 & & \mathbb{E}(n) \arrow[r, "i"] \arrow[d, "\pi_{\mathbb{E}}"] & \mathrm{Mob}(n) \arrow[d, "\pi"] \\
M \arrow[r, "F"] & \mathbb{R}^n \arrow[r, "\chi"] & \chi(\mathbb{R}^n) \arrow[r, "i"] & Q_n
\end{tikzcd}$$

where $i : \mathbb{E}(n) \hookrightarrow \mathrm{Mob}(n)$ denotes the group inclusion. We shall identify $\mathbb{R}^n$ with $\chi(\mathbb{R}^n)$ and F with $f = \chi \circ F$, when no possible confusion arises. The Maurer-Cartan form of $\mathbb{E}(n)$ is $i^*\Phi$, where Φ is that of $\mathrm{Mob}(n)$.

Let $e : U \subseteq M \to \mathbb{E}(n)$ be a zeroth order frame along F. If $\widetilde{e}$ is another frame, then $\widetilde{e} = eK$, where K has values in the subgroup $\mathbb{E}(n)_0$.

The Cartan connections $\phi = e^* i^* \Phi$ and $\widetilde{\phi} = \widetilde{e}^* i^* \Phi$ are related as follows:

$$(\widetilde{\phi}_0^A) = {}^t A(\phi_0^A) \quad , \quad (\widetilde{\phi}_B^A) = {}^t A(\phi_B^A)A + {}^t A \mathrm{d}A,$$

the other forms being zero. Therefore, at every point there exists a frame such that $\phi_0^\alpha = 0$, and since the isotropy subgroup preserving such frames is

$$\mathbb{E}(n)_D = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : \begin{array}{l} A \in \mathrm{SO}(m) \\ B \in \mathrm{SO}(n-m) \end{array} \right\}$$

the reduction can be carried on smoothly around every point. This frames are called Euclidean Darboux frames. In a Euclidean Darboux frame, the metric $\langle\ ,\ \rangle$ is given locally by $\sum_i \phi_0^i \otimes \phi_0^i$, the volume form $\mathrm{d}V_e$ by $\phi_0^1 \wedge \ldots \wedge \phi_0^m$ and, denoting with $\{E_i\}$ the dual basis of $\{\phi_0^i\}$, the Levi-Civita connection ∇^e is given by

$$\nabla^e E_i = \phi_i^j \otimes E_j.$$

From the structure of $\mathbb{E}(n)_D$, there is a natural, well defined (Euclidean) normal bundle $N^eM \to M$ given, in a neighbourhood of every $p \in M$, by the span of $\{e_\alpha\}$. The normal connection on N^eM is defined by setting $\nabla^\perp e_\alpha = \phi_\alpha^\beta \otimes e_\beta$.

Differentiating $\phi_0^\alpha = 0$ and using the structure equations and Cartan's lemma we obtain that there exist $b_{ij}^\alpha = b_{ji}^\alpha$ such that $\phi_i^\alpha = b_{ij}^\alpha \phi_0^j$. The tensor

$$II = b_{ij}^\alpha \phi_0^j \otimes \phi_0^i \otimes e_\alpha$$

is the Riemannian second fundamental form.

Now we view the frames as sections of the larger bundle $\mathrm{M\ddot{o}b}(n) \to Q_n$, that is, for every $e : M \to \mathbb{E}(n)$ we consider $i \circ e : M \to \mathrm{M\ddot{o}b}(n)$. In this setting, since

$$(i \circ e)^*\Phi = e^*(i^*\Phi) = \phi,$$

a Euclidean Darboux frame e gives rise to a first order conformal frame $i \circ e$. For simplicity, we still denote $i \circ e$ with e. By (11), for every conformal first order frame, differentiating $\phi_0^\alpha = 0$ gives $\phi_i^\alpha = h_{ij}^\alpha \phi_0^j$, so that we deduce $h_{ij}^\alpha \equiv b_{ij}^\alpha$. To have a conformal Darboux frame, we set

$$K = \begin{pmatrix} 1 & 0 & {}^tH & \frac{|H|^2}{2} \\ 0 & I & 0 & 0 \\ 0 & 0 & I & H \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \text{where} \quad H^\alpha = \frac{1}{m} h_{kk}^\alpha = \frac{1}{m} b_{kk}^\alpha$$

is the α-th component of the mean curvature vector H. Then, by (13), $\tilde{e} = eK$ is Darboux and

$$\tilde{h}_{ij}^\alpha = b_{ij}^\alpha - \frac{1}{m} b_{kk}^\alpha \delta_{ij} = b_{ij}^\alpha - H^\alpha \delta_{ij}, \tag{98}$$

so that the definition we gave in Section 2 of umbilic points coincides with the standard definition in the Riemannian setting. The Willmore functional (26) can be written as

$$W_\Omega(f) = \frac{1}{m} \int_\Omega |II - \langle\ ,\ \rangle \otimes H|^m \mathrm{d}V_e.$$

In particular, for surfaces in $\mathbb{R}^n$, via Gauss Equations $2K = 4|H|^2 - |II|^2$ (K the Gaussian curvature of M) and Gauss-Bonnet theorem the Willmore functional for compact surfaces M without boundary has the familiar expression

$$W(f) = \int_M |H|^2 \mathrm{d}V_e - \chi(M).$$

Under $\tilde{e} = eK$, the whole set of forms changes as follows:

$$\begin{aligned} &\tilde{\phi}_i^0 = H^\alpha(\tfrac{1}{2}H^\alpha\delta_{ij} - b_{ij}^\alpha)\phi_0^j = -H^\alpha(\phi_i^\alpha - \tfrac{1}{2}H^\alpha\phi_0^i), && \tilde{\phi}_\alpha^0 = \mathrm{d}H^\alpha - H^\beta\phi_\alpha^\beta, \\ &\tilde{\phi}_i^\alpha = (b_{ij}^\alpha - H^\alpha\delta_{ij})\phi_0^j, \qquad \tilde{\phi}_0^0 = \phi_0^0 = 0, && \tilde{\phi}_B^A = \phi_B^A. \end{aligned} \tag{99}$$

Since $\tilde{e}_\alpha = e_\alpha$ and $\tilde{\phi}^\alpha_\beta = \phi^\alpha_\beta$, there is a natural isomorphism between the Euclidean normal bundle N^eM with its normal connection $\nabla^\perp$ and the conformal normal bundle N locally spanned by $\{\tilde{e}_\alpha\}$, with the connection ∇ given by (23). Substituting (98) and (99) in (17) and simplifying we get

$$h^\alpha_{ijk}\phi^k_0 = \mathrm{d}b^\alpha_{ij} - b^\alpha_{ik}\phi^k_j - b^\alpha_{kj}\phi^k_i + b^\beta_{ij}\phi^\alpha_\beta = (\nabla^e II)^\alpha_{ijk}\phi^k_0,$$

showing that $\{h^\alpha_{ijk}\}$ coincide with the coefficients of the covariant derivative of II with respect to the natural connection (still called ∇^e) on $T^*M \otimes T^*M \otimes N^eM$. It follows that

$$p^\alpha_k = \frac{1}{m}h^\alpha_{iik} = \frac{1}{m}(\nabla^e II)^\alpha_{iik} = (\nabla^\perp H)^\alpha_k,$$

where the last equality follows from the linearity of ∇^e and the fact that, by the very definition of $\nabla^e II$, the covariant derivative on the N^eM part of $T^*M \otimes T^*M \otimes N^eM$ is computed with respect to the normal connection $\nabla^\perp$. Substituting in (59) we obtain

$$\begin{aligned} p^\alpha_{ik}\phi^k_0 &= \mathrm{d}[(\nabla^\perp H)^\alpha_i] - (\nabla^\perp H)^\alpha_k\phi^k_i + (\nabla^\perp H)^\beta_i\phi^\alpha_\beta + \\ &\quad -(b^\alpha_{ji} - H^\alpha\delta_{ji})(\tfrac{1}{2}H^\beta H^\beta\delta_{jk} - H^\beta b^\beta_{jk})\phi^k_0 \\ p^\alpha_{ik} &= (\nabla^\perp\nabla^\perp H)^\alpha_{ik} + -\tfrac{1}{2}|H|^2 b^\alpha_{ik} + \tfrac{1}{2}|H|^2 H^\alpha\delta_{ik} + H^\beta b^\beta_{jk} b^\alpha_{ji} - H^\alpha H^\beta b^\beta_{ik}, \end{aligned} \tag{100}$$

where the norm of H is meant with respect to the metric on N^eM. Taking traces we deduce

$$p^\alpha_{kk} = (\Delta^\perp H)^\alpha + H^\beta b^\beta_{ik} b^\alpha_{ik} - m|H|^2 H^\alpha,$$

$\Delta^\perp$ being the Laplacian on N^eM. In particular, when M is a surface of $\mathbb{R}^n$, by Theorem 4.2 the Euler-Lagrange equations of the Willmore functional are

$$(\Delta^\perp H)^\alpha + H^\beta b^\beta_{ik} b^\alpha_{ik} - 2|H|^2 H^\alpha = 0, \qquad 3 \leqslant \alpha \leqslant 4.$$

which reduces, for $n = 2$, to

$$\Delta H + 2(H^2 - K)H = 0.$$

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