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D.V. Millionshchikov*

COMPLEX STRUCTURES ON NILPOTENT LIE ALGEBRAS AND DESCENDING CENTRAL SERIES

This article is dedicated to the memory of Sergio Console, my colleague and friend

Abstract. We study the algebraic constraints on the structure of nilpotent Lie algebra $\mathfrak{g}$, which arise because of the presence of an integrable complex structure J. Particular attention is paid to non-abelian complex structures. Constructed various examples of positive graded Lie algebras with complex structures, in particular, we construct an infinite family $\mathfrak{D}(n)$ of such algebras that we have for their nil-index $s(\mathfrak{D}(n))$:

$$s(\mathfrak{D}(n)) = \left[\frac{2}{3}\dim\mathfrak{D}(n)\right].$$

Introduction

The Newlander-Nirenberg theorem [14] implies that a left-invariant complex structure on a real simply connected Lie group G can be defined as an almost-complex structure J on the tangent Lie algebra $\mathfrak{g}$ of G (J is a linear endomorphism of $\mathfrak{g}$ such that $J^2 = -1$) satisfying *the integrability condition* (vanishing of the Nijenhuis tensor):

$$[JX, JY] = [X,Y] + J[JX,Y] + J[X,JY],\ \forall X,Y \in \mathfrak{g}.$$

Extending an almost complex structure J on the complexification $\mathfrak{g}^{\mathbb{C}}$ it is easy to see that the integrability condition is equivalent to the following one: the eigen-spaces $\mathfrak{g}^{\mathbb{C}}_{\pm i}$ of J corresponding to the eigen-values $\pm i$ are (complex) subalgebras of $\mathfrak{g}^{\mathbb{C}}$. If they are abelian subalgebras then the complex structure J is called *abelian*. It was proved in [8] that a Lie group G admitting a left-invariant abelian complex structure has to be solvable. On the another hand an abelian complex structure is nilpotent [15]. The study of nilpotent complex structures on nilmanifolds (nilpotent Lie algebras) was the subject in [6]. The properties of nilmanifolds with abelian complex structures is much more studied than the general case. For instance, the Dolbeault cohomology of a nilmanifold with an integrable abelian complex structure can be expressed in terms of the corresponding Lie algebra cohomology ([5], [6]).

Existing finite list of all real 6-dimensional nilpotent Lie algebras up to isomorphism [13] allowed S.Salamon in [15] distinguish among them algebras admitting integrable complex structures, spend their classification from this point of view. In his cassification there are examples of Lie algebras that admit only non-abelian complex structures (as well as examples of Lie algebras that does not admit any complex structure). This approach does not work in the following even dimension 8, where, on the

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one hand, such a classification does not exist, on the other hand - there are infinite families of pairwise nonisomorphic nilpotent algebras.

Another way is to find a priori algebraic constraints expressed in terms of the nil-index, the dimensions of the ideals of descending central series, the first Betti number, etc., which narrow the range of possible candidates to possess an integrable complex (or a hypercomplex) structure. This approach has been implemented in [7], [8], where the authors managed to classify 8-dimensional real nilpotent algebras admitting hypercomplex structure, despite the lack of a general classification.

As an example of an algebraic constraints, which we discussed above, we can cite the following general result by Goze and Remm [11] that a filiform Lie algebra $\mathfrak{g}$ (i.e. a nilpotent Lie algebra with the value of nil-index $s(\mathfrak{g}) = \dim\mathfrak{g} - 1$) does not admit any integrable complex structure. Later this result was extended to the class of so-called quasi-filiform Lie algebras, i.e. nilpotent Lie algebras with $s(\mathfrak{g}) = \dim\mathfrak{g} - 2$ [9]. An important role in the last proof was played by the classification of all naturally graded quasi-filifrom Lie algebras [10].

In this article we prove a general estimate (18):

$$\mathrm{codim}\mathfrak{g}^4 = \dim\mathfrak{g} - \dim[\mathfrak{g},[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]]] \geqslant 5.$$

From this estimate one can deduce the main result of [11] as a corollary.

The main purpose of this article is to study nilpotent algebras admitting (non-abelian) complex structures in high (arbitrary) dimensions. This requires a stock of examples of such algebras. As suitable examples we propose to study positively graded Lie algebras. On the one hand it is more easy to study them (cohomology computations) on the other hand they have quite interesting properties. In addition, any nilpotent algebra can be obtained as a special deformation of a positively graded Lie algebra.

The Theorem 6.1 claims that for the maximal value $s(2n)$ of nil-index $s(\mathfrak{g})$ of $2n$-dimensional nilpotent Lie algebras $\mathfrak{g}$ admitting a complex structure we have the following estimates (25):

$$\left[\frac{4n}{3}\right] \leqslant s(2n) \leqslant 2n-2.$$

It follows from [15] that $s(6)=4$. It appears possible to prove that $s(8)=5$ and improve the estimates (25) for higher dimensions.

1. Nilpotent Lie algebras

The sequence of ideals of a Lie algebra $\mathfrak{g}$

$$\mathfrak{g}^1 = \mathfrak{g} \supset \mathfrak{g}^2 = [\mathfrak{g},\mathfrak{g}] \supset \ldots \supset \mathfrak{g}^k = [\mathfrak{g},\mathfrak{g}^{k-1}] \supset \ldots$$

is called the descending central sequence of $\mathfrak{g}$.

A Lie algebra $\mathfrak{g}$ is called nilpotent if there exists a natural number $s(\mathfrak{g})$ such that:

$$\mathfrak{g}^{s(\mathfrak{g})+1} = [\mathfrak{g},\mathfrak{g}^{s(\mathfrak{g})}] = 0, \quad \mathfrak{g}^{s(\mathfrak{g})} \neq 0.$$

$s(\mathfrak{g})$ is called the nil-index of the nilpotent Lie algebra $\mathfrak{g}$ and $\mathfrak{g}$ is called $s(\mathfrak{g})$-step nilpotent Lie algebra. Thus one can regard an abelian Lie algebra as 1-step nilpotent.

EXAMPLE 1. The Heisenberg algebra $\mathfrak{h}_{2k+1}$ is defined by its basis $x_1, y_1, \dots, x_k, y_k, z$ and the commutating relations:

$$[x_i, y_i] = z, \ i = 1, \dots, k.$$

REMARK 1.1. In the sequel we will omit trivial relations $[e_i, e_j] = 0$ in the definitions of Lie algebras.

The Heisenberg Lie algebra $\mathfrak{h}_{2k+1}$ is 2-step nilpotent.

Let consider the sequence of positive integers

$$a(\mathfrak{g}) = \left(a_1(\mathfrak{g}), \dots, a_{s(\mathfrak{g})}(\mathfrak{g})\right),$$

where

$$a_i(\mathfrak{g}) = \dim(\mathfrak{g}^i/\mathfrak{g}^{i+1}), i = 1, \dots, s(\mathfrak{g}).$$

We have the following estimates on $a_i(\mathfrak{g})$:

$$a_1(\mathfrak{g}) = \dim(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]) \geqslant 2, \quad a_i(\mathfrak{g}) \geqslant 1, \ i = 2, \dots, s(\mathfrak{g}).$$

DEFINITION 1.1. *Let $(m_1, m_2, \dots, m_s)$ be a sequence of positive integers. We will call a nilpotent Lie algebra $\mathfrak{g}$ of type $(m_1, m_2, \dots, m_s)$ if $s(\mathfrak{g}) = s$ and*

$$a_i(\mathfrak{g}) = \dim \mathfrak{g}^i/\mathfrak{g}^{i+1} = m_i, i = 1, \dots, s(\mathfrak{g}).$$

REMARK 1.2. Obviously

$$a_1(\mathfrak{g}) + \dots + a_k(\mathfrak{g}) = \dim(\mathfrak{g}/\mathfrak{g}^{k+1}), k = 1, \dots, s(\mathfrak{g}).$$

It immediately follows, that we have the following estimate:

$$\dim \mathfrak{g} = a_1(\mathfrak{g}) + \dots + a_{s(\mathfrak{g})} \geqslant s(\mathfrak{g}) + 1.$$

DEFINITION 1.2. *A finite dimensional nilpotent Lie algebra $\mathfrak{g}$ is called filiform if $s(\mathfrak{g}) = \dim \mathfrak{g} - 1$.*

REMARK 1.3. A Lie algebra $\mathfrak{g}$ is filiform if and only if it is of the type $(2, 1, 1, \dots, 1)$.

EXAMPLE 2. The Lie algebra $\mathfrak{m}_0(n)$ that is defined by its basis $e_1, e_2, \dots, e_n$ and the commutating relations:

$$[e_1, e_i] = e_{i+1}, \ i = 2, \dots, n-1,$$

is obviously an example of a filiform Lie algebra.

DEFINITION 1.3. *A finite dimensional nilpotent Lie algebra $\mathfrak{g}$ is called quasi-filiform if $s(\mathfrak{g}) = \dim \mathfrak{g} - 2$.*

Obviously a quasi-filiform Lie algebra $\mathfrak{g}$ is either of type $(3,1,\dots,1)$ or of type $(2,1,\dots,1,2,1,\dots,1)$ (the last case means that $a_1(\mathfrak{g}) = a_r(\mathfrak{g}) = 2$ for some positive integer $r, 3 \leqslant r \leqslant 2n{-}2$, and $a_i(\mathfrak{g}) = 1, i \neq 1, r$). It was proved in [10] that r must to be odd (apriori it is not evident). We will prove this assertion later as an elementary corollary of Vergne's calculation [17] of the two-cohomology $H^2(m_0(n),\mathbb{R})$.

The direct sum of Lie algebras $m_0(n)\oplus\mathbb{R}$ provides us with a simplest possible example of a quasi-filiform $(n{+}1)$-dimensional Lie algebra of the type $(3,1,\dots,1)$.

The ideals $\mathfrak{g}^k$ of the descending central sequence define a decreasing filtration of the Lie algebra $\mathfrak{g}$

$$[\mathfrak{g}^k, \mathfrak{g}^l] \subset \mathfrak{g}^{k+l}.$$

DEFINITION 1.4. *The graded Lie algebra*

$$gr\mathfrak{g} = \bigoplus_{k=1} (gr\mathfrak{g})_k, \;\; (gr\mathfrak{g})_k = \mathfrak{g}^k/\mathfrak{g}^{k+1},$$

is called the naturally graded Lie algebra associated with $\mathfrak{g}$.

DEFINITION 1.5. *A graded Lie algebra*

$$\mathfrak{g} = \bigoplus_{k=1} \mathfrak{g}_k, \;\; [\mathfrak{g}_k, \mathfrak{g}_l] \subset \mathfrak{g}_{k+l}.$$

is called naturally graded if there is an isomorphism $\varphi : \mathfrak{g} \to gr\mathfrak{g}$ and $\varphi(\mathfrak{g}_k) = (gr\mathfrak{g})_k$.

REMARK 1.4. Let $\mathfrak{g} = \oplus_{k=1}\mathfrak{g}_k$ be a finite dimensional naturally graded Lie algebra then it is nilpotent and

$$\dim \mathfrak{g}_k = a_k(\mathfrak{g}), \;\; k = 1,\dots,s(\mathfrak{g}), \;\; \mathfrak{g}_k = 0, \; k \geqslant s(\mathfrak{g}){+}1.$$

THEOREM 1.1 (M. Vergne [17]). *Let $\mathfrak{g} = \oplus_{k=1}\mathfrak{g}_k$ be a naturally graded n-dimensional filiform Lie algebra then*

1) if $n = 2k+1$, then $\mathfrak{g}$ is isomorphic to $\mathfrak{m}_0(2k+1)$;

2) if $n = 2k$, then $\mathfrak{g}$ is isomorphic either to $\mathfrak{m}_0(2k)$ or to the Lie algebra $\mathfrak{m}_1(2k)$, defined by its basis $e_1,\dots,e_{2k}$ and commutating relations:

$$[e_1, e_i] = e_{i+1}, \; i = 2,\dots,2k{-}1; \qquad [e_j, e_{2k+1-j}] = (-1)^{j+1} e_{2k}, \quad j = 2,\dots,k.$$

REMARK 1.5. In the settings of the Theorem 1.1 the gradings of the algebras $\mathfrak{m}_0(n)$, $\mathfrak{m}_1(n)$ are defined as $\mathfrak{g}_1 = Span(e_1, e_2)$, $\mathfrak{g}_i = Span(e_{i+1})$, $i = 2,\dots,n{-}1$.

The first classification of naturally graded quasi-filiform Lie algebras was obtained in [10] but there were some omissions in the classification list that were corrected in [9].

2. Lie algebra cohomology and central extensions

Let us consider the cochain complex of a Lie algebra $\mathfrak{g}$ with $\dim \mathfrak{g} = n$:

$$\mathbb{K} \xrightarrow{d_0} \mathfrak{g}^* \xrightarrow{d_1} \Lambda^2(\mathfrak{g}^*) \xrightarrow{d_2} \dots \xrightarrow{d_{n-1}} \Lambda^n(\mathfrak{g}^*) \to 0$$

where $d_1 : \mathfrak{g}^* \to \Lambda^2(\mathfrak{g}^*)$ is the dual mapping to the Lie bracket $[\,,] : \Lambda^2\mathfrak{g} \to \mathfrak{g}$, and the differential d (that is a collection of d_p) is a derivation of the exterior algebra $\Lambda^*(\mathfrak{g}^*)$ that continues d_1:

$$d(\rho \wedge \eta) = d\rho \wedge \eta + (-1)^{deg\rho}\rho \wedge d\eta,\ \forall \rho, \eta \in \Lambda^*(\mathfrak{g}^*).$$

The condition $d^2 = 0$ is equivalent to the Jacobi identity in $\mathfrak{g}$.

The cohomology of $(\Lambda^*(\mathfrak{g}^*), d)$ is called the cohomology (with trivial coefficients) of the Lie algebra $\mathfrak{g}$ and is denoted by $H^*(\mathfrak{g})$. One can easily remark that $H^1(\mathfrak{g})$ is isomorphic to $(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])^*$.

Let us define a family $\{V_l\mathfrak{g}^*\}$ of subspaces in $\mathfrak{g}^*$:

1) $V_0\mathfrak{g}^* = \{0\}$,

2) $V_1\mathfrak{g}^* = \mathrm{Ker} d_1$, $d_1 f(X,Y) = f([X,Y])$,

3) $V_l\mathfrak{g}^* = \{f \in \mathfrak{g}^* : d_1 f \in \Lambda^2(V_{l-1}\mathfrak{g}^*)\}$, $l \geqslant 2$.

$$\{0\} \subset V_1\mathfrak{g}^* \subset \cdots \subset V_l\mathfrak{g}^* \subset V_{l+1}\mathfrak{g}^* \subset \dots$$

The first subspace $V_1\mathfrak{g}^*$ is the annihilator of $\mathfrak{g}^2$=$[\mathfrak{g},\mathfrak{g}]$ and it is isomorphic to the first cohomology $H^1(\mathfrak{g})$. Supposing by induction that $V_{l-1}\mathfrak{g}^*$ annihilates $\mathfrak{g}^l$ one can remark that $d_1 f \in \Lambda^2(V_l\mathfrak{g}^*)$ iff $d_1 f(X,Y) = f([X,Y])$ vanishes for all $X \in \mathfrak{g}$ and $Y \in \mathfrak{g}^l$ (f annihilates the subspace $\mathfrak{g}^l$). Hence $V_l\mathfrak{g}^*$ is the annihilator of $\mathfrak{g}^{l+1}$. Also we have

$$a_l(\mathfrak{g}) = \dim \mathfrak{g}^l/\mathfrak{g}^{l+1} = \dim V_{l+1}\mathfrak{g}^*/V_l\mathfrak{g}^*.$$

Now the nilpotency condition for a Lie algebra $\mathfrak{g}$ can be interpreted in a following way: a non-abelian $\mathfrak{g}$ is s-step nilpotent iff there exists a positive integer s such that $V_s\mathfrak{g}^* = \mathfrak{g}^*$ and $V_{s-1}\mathfrak{g}^* \neq \mathfrak{g}^*$.

Recall that an one-dimensional central extension of a Lie algebra $\mathfrak{g}$ is an exact sequence

$$0 \to \mathbb{K} \to \widetilde{\mathfrak{g}} \to \mathfrak{g} \to 0 \tag{1}$$

of Lie algebras and their homomorphisms, in which the image of the homomorphism $\mathbb{K} \to \widetilde{\mathfrak{g}}$ is contained in the centre of the Lie algebra $\mathfrak{g}$. An extension

$$0 \to \mathbb{K} \to \mathbb{K}\oplus\mathfrak{g} \to \mathfrak{g} \to 0$$

corresponds to the cocycle $c \in \Lambda^2(\mathfrak{g}^*)$, where the Lie bracket in $\mathbb{K}\oplus\mathfrak{g}$ is defined by the formula

$$[(\lambda, g), (\mu, h)] = (c(g,h), [g,h]).$$

It can be checked directly that the Jacobi identity for this Lie bracket is equivalent to c being cocycle and that to cohomologous cocycles correspond equivalent (in a obvious sense) extensions.

If $\mathfrak{g}$ is a Lie algebra of the finite dimension $\dim \mathfrak{g} = n$ defined by its basis $e_1, \dots, e_n$ and the structure relations

$$[e_i, e_j] = \sum_{k=1}^{n} c_{ij}^k e_k,$$

then the algebra $\tilde{\mathfrak{g}}$ from (1) can be defined by its basis $\tilde{e}_1, \dots, \tilde{e}_n, \tilde{e}_{n+1}$ and the structure relations

$$\begin{aligned} [\tilde{e}_i, \tilde{e}_j] &= \sum_{k=1}^{n} c_{ij}^k \tilde{e}_k + c(\tilde{e}_i, \tilde{e}_j)\tilde{e}_{n+1},\ 1 \leqslant i < j \leqslant n; \\ [\tilde{e}_i, \tilde{e}_{n+1}] &= 0,\ i = 1, \dots, n. \end{aligned} \tag{2}$$

Vergne calculated $H^2(\mathfrak{m}_0(n), \mathbb{R})$ [17], it is spanned by the following homogeneous basic cocycles:

$$e^1 \wedge e^n,\ \left\{ \omega_{2k+1} = \sum_{l=0}^{k-2} (-1)^l e^{2+l} \wedge e^{2k-1-l}, 3 \leqslant 2k-1 \leqslant n \right\}.$$

For instance

$$\omega_5 = e^2 \wedge e^3, \omega_7 = e^2 \wedge e^5 - e^3 \wedge e^4, \omega_9 = e^2 \wedge e^7 - e^3 \wedge e^6 + e^4 \wedge e^5.$$

The total number of basic cocycles are equal to the number of positive odd integers that are less or equal to n.

$$\dim H^2(\mathfrak{m}_0(n), \mathbb{R}) = \left[\frac{n+1}{2} \right].$$

One can consider a one-dimensional central extension of $\mathfrak{m}_0(n)$ that corresponds to cocycle ω_{2k+1}. We will denote it by $\mathfrak{m}_0^{2k-1}(n)$.

PROPOSITION 2.1. *The Lie algebra* $\mathfrak{m}_0^{2k-1}(n)$ *is a naturally graded quasi-filiform Lie algebra of the type* $(2, 1, \dots, 1, 2, 1, \dots, 1)$ *with* $a_{2k-1}(\mathfrak{m}_0^{2k-1}(n)) = 2$.

Proof. We have the following structure relations of $\mathfrak{m}_0^{2k-1}(n)$:

$$\begin{aligned} [\tilde{e}_1, \tilde{e}_i] &= \tilde{e}_{i+1}, i = 2, \dots, n-1, \\ [\tilde{e}_2, \tilde{e}_{2k-1}] = -[\tilde{e}_3, \tilde{e}_{2k-2}] = \dots &= (-1)^k [\tilde{e}_k, \tilde{e}_{k+1}] = \tilde{e}_{n+1}, \end{aligned} \tag{3}$$

$\tilde{e}_{2k-1} \in \mathfrak{m}_0^{2k-1}(n)^{2k-2}$ and $\tilde{e}_2 \in \mathfrak{m}_0^{2k-1}(n)^1$. Hence $\tilde{e}_{n+1} \in \mathfrak{m}_0^{2k-1}(n)^{2k-1}$. Also $\tilde{e}_{2k} \in \mathfrak{m}_0^{2k-1}(n)^{2k-1}$. On the another hand $\tilde{e}_{n+1}$ commutes with all other elements, hence $\mathfrak{m}_0^{2k-1}(n)^i = \mathfrak{m}_0(n)^i, i = 2k+2, \dots, n-2$. □

3. Integrability condition

DEFINITION 3.1. *An almost-complex structure J on a Lie algebra $\mathfrak{g}$ (i.e. J is a linear endomorphism of $\mathfrak{g}$ such that $J^2 = -1$) satisfying the integrability condition*

$$[JX, JY] = [X,Y] + J[JX,Y] + J[X,JY],\ \forall X,Y \in \mathfrak{g} \tag{4}$$

is called a complex structure on $\mathfrak{g}$.

EXAMPLE 3. Let us consider the direct sum $\mathfrak{h}_{2k+1} \oplus \mathbb{R}$, where the one-dimensional abelian $\mathbb{R}$ is spanned by w. One can define an operator J on $\mathfrak{h}_{2k+1} \oplus \mathbb{R}$:

$$Jx_i = y_i, Jy_i = -x_i\ i = 1, \ldots, k;\ \ Jz = w,\ Jw = -z.$$

$J^2 = -1$ and J satisfies the Nijenhuis condition. In fact J satisfies to an identity even stronger than (4).

DEFINITION 3.2. *An almost complex structure J on a Lie algebra $\mathfrak{g}$ is said to be an abelian complex structure iff*

$$[JX, JY] = [X,Y],\ \forall X,Y \in \mathfrak{g}. \tag{5}$$

Obviously if an almost complex structure satisfies (5) it satisfies the Nijenhuis condition (4). J from Example 1 is an abelian complex structure. It was proved in [8] that a real Lie algebra admitting an abelian complex structure has to be solvable.

Extending an almost complex structure J on the complexification $\mathfrak{g}^{\mathbb{C}}$ we have a splitting

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}^{\mathbb{C}}_{-i} \oplus \mathfrak{g}^{\mathbb{C}}_{i},$$

where $\mathfrak{g}^{\mathbb{C}}_{\pm i} = \{x - \pm iJx : x \in \mathfrak{g}\}$ are the eigen-space of the complexification of J corresponding to the eigen-values $\pm i$. It is easy to see that:

1) J is integrable iff both $\mathfrak{g}^{\mathbb{C}}_{\pm i}$ are (complex) subalgebras of $\mathfrak{g}^{\mathbb{C}}$;

2) J is an abelian complex structure iff $\mathfrak{g}^{\mathbb{C}}_{\pm i}$ are abelian subalgebras of $\mathfrak{g}^{\mathbb{C}}$.

One can point out another one important case:

3) the eigen-spaces $\mathfrak{g}^{\mathbb{C}}_{\pm i}$ of J are ideals of $\mathfrak{g}^{\mathbb{C}}$.

The last condition is equivalent to the following one:

$$[JX, Y] = J[X,Y],\ \forall X,Y \in \mathfrak{g}. \tag{6}$$

And it is the definition of a complex Lie algebra structure, i.e. $(\mathfrak{g}, J)$ can be regarded as a complex Lie algebra.

EXAMPLE 4. Let us consider a Lie algebra $\mathfrak{m}_0(n)^{\mathbb{R}}, n \geqslant 2$ defined by its basis $x_1, y_1, x_2, y_2, \ldots, x_n, y_n$ and the structure relations:

$$[x_1, x_i] = [y_i, y_1] = x_{i+1},\ [x_1, y_i] = [y_1, x_i] = y_{i+1},\ \ i = 2, \ldots, n{-}1.$$

$2n$-dimensional Lie algebra $\mathfrak{m}_0(n)^{\mathbb{R}}$ is $(n-1)$-step nilpotent.

An almost complex structure J on $\mathfrak{m}_0(n)^{\mathbb{R}}$ that is defined by $Jy_i = x_i\ i = 1, \dots, n$ satisfies (6) and $(\mathfrak{m}_0(n)^{\mathbb{R}}, J)$ is isomorphic to the complex filiform Lie algebra $\mathfrak{m}_0(n)$.

Now we are going to start study of complex structures on nilpotent Lie algebras. Let $\mathfrak{g}$ a nilpotent Lie algebra with integrable complex structure J and $\{\mathfrak{g}^j\}$ its descending central sequence.

An ideal $\mathfrak{g}^l$ is not in general an invariant subspace with respect to J. One can consider $\mathfrak{g}^l(J) = \mathfrak{g}^l + J\mathfrak{g}^l$ – the smallest J-invariant subspace of $\mathfrak{g}$ containing $\mathfrak{g}^l$. We have a decreasing sequence of J-invariant subspaces

$$\mathfrak{g}^1(J) = \mathfrak{g} \supset \mathfrak{g}^2(J) = [\mathfrak{g},\mathfrak{g}] + J[\mathfrak{g},\mathfrak{g}] \supset \dots \supset \mathfrak{g}^{s(\mathfrak{g})}(J) \supset \{0\}.$$

PROPOSITION 3.1 (S.Salamon, [15]).

$$[\mathfrak{g}^l(J), \mathfrak{g}^l(J)] \subset \mathfrak{g}^{l+1}(J).$$

Proof. One can take arbitrary $X_1, X_2, Y_1, Y_2 \in \mathfrak{g}^l$. Then

$$[X_1 + JY_1, X_2 + JY_2] = [X_1, X_2] + [JY_1, X_2] + [X_1, JY_2] + [JY_1, JY_2].$$

The first three summands on the right part of the equality are obviously in $\mathfrak{g}^{l+1}$. It follows from the integrability condition (4) that $[JY_1, JY_2]$ is in $\mathfrak{g}^l + J\mathfrak{g}^l$ as all commutators $[Y_1, Y_2], [JY_1, Y_2], [Y_1, JY_2] \in \mathfrak{g}^{l+1}$. □

COROLLARY 3.1. *A subspace $\mathfrak{g}^l(J)$ is a subalgebra in $\mathfrak{g}(J)$ and an ideal in $\mathfrak{g}^{l-1}(J)$ for all l.*

PROPOSITION 3.2 (S. Salamon, [15]).

$$\mathfrak{g}^2(J) = [\mathfrak{g},\mathfrak{g}] + J[\mathfrak{g},\mathfrak{g}] \neq \mathfrak{g} = \mathfrak{g}^1(J).$$

Proof. Let assume that $\mathfrak{g}^2(J) = \mathfrak{g}$, then exists $2 \leqslant j_0 \leqslant s(\mathfrak{g})$ such that

$$\mathfrak{g}^{j_0}(J) = \mathfrak{g}, \quad \mathfrak{g}^{j_0+1}(J) \neq \mathfrak{g}.$$

It follows that $[\mathfrak{g},\mathfrak{g}] = [\mathfrak{g}^{j_0}(J), \mathfrak{g}^{j_0}(J)] \subset \mathfrak{g}^{j_0+1}(J) \neq \mathfrak{g}$, the subspace $\mathfrak{g}^{j_0+1}(J)$ is J-invariant, hence $J[\mathfrak{g},\mathfrak{g}] \subset \mathfrak{g}^{j_0+1}(J)$ also. Combining these results we have an inclusion

$$\mathfrak{g}^2(J) = [\mathfrak{g},\mathfrak{g}] + J[\mathfrak{g},\mathfrak{g}] \subset \mathfrak{g}^{j_0+1}(J) \neq \mathfrak{g}$$

that contradicts to our initial assumption. □

4. Minimal models and complex structures

Given a Lie algebra $\mathfrak{g}$ with an integrable complex structure J. One can consider its conjugate J (we will keep the same notation for it) acting on $\mathfrak{g}^*$.

$$Jf(X) = f(JX), f \in \mathfrak{g}^*, X \in \mathfrak{g}.$$

Proceeding to the complexification of J we have a splitting

$$(\mathfrak{g}^*)^{\mathbb{C}} = \Lambda^{1,0} \oplus \Lambda^{0,1}, \tag{7}$$

where $\Lambda^{1,0} = \{f - iJf : f \in \mathfrak{g}^*\}$ and $\Lambda^{0,1} = \{f + iJf : f \in \mathfrak{g}^*\}$ are the eigen-spaces of the complexification of J that correspond to the eigen-values $\pm i$ respectively. Also we have $\Lambda^{1,0} = (\mathfrak{g}_i^{\mathbb{C}})^*$ and $\Lambda^{0,1} = (\mathfrak{g}_{-i}^{\mathbb{C}})^*$.

The splitting (7) induces a decomposition

$$\Lambda^k((\mathfrak{g}^{\mathbb{C}})^*) = \bigoplus_{p+q=k} \Lambda^{p,q},$$

where $\Lambda^{p,q} = \Lambda^p((\mathfrak{g}_i^{\mathbb{C}})^*) \otimes \Lambda^q((\mathfrak{g}_{-i}^{\mathbb{C}})^*)$ is the subspace of (p,q)-forms relative to J.

For a given subspace $\mathfrak{a} \subset \mathfrak{g}$ let us denote by $\mathfrak{a}^{ann}$ its annihilator in $\mathfrak{g}^*$:

$$\mathfrak{a}^{ann} = \{f \in \mathfrak{g}^* | f(X) = 0, \forall X \in \mathfrak{a}\}.$$

Now one can consider an obvious lemma:

LEMMA 4.1. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two subspaces of $\mathfrak{g}$ then*

$$d\mathfrak{b}^{ann} \subset \mathfrak{a}^{ann} \wedge \mathfrak{g}^*$$

if and only if $[\mathfrak{a}, \mathfrak{a}] \subset \mathfrak{b}$.

Proof. Let $f \in \mathfrak{b}^{ann}$. Then $df \in \mathfrak{a}^{ann} \wedge \mathfrak{g}^*$, or

$$df(X,Y) = f([X,Y]) = 0, \forall X,Y \in \mathfrak{a},$$

if and only if, $[X,Y] \in \mathfrak{b}, \forall X,Y \in \mathfrak{a}$. □

COROLLARY 4.1. *Let $\mathfrak{a}$ be a subspace of a Lie algebra $\mathfrak{g}$. The ideal $I(\mathfrak{a}^{ann})$ generated by the annihilator $\mathfrak{a}^{ann}$ in the exterior algebra $\Lambda(\mathfrak{g}^*)$ is d-closed*

$$d\mathfrak{a}^{ann} \subset \mathfrak{a}^{ann} \wedge \mathfrak{g}^*$$

if and only if $\mathfrak{a}$ is a Lie subalgebra of $\mathfrak{g}$.

Now we can view $\Lambda^{1,0}$ as the annihilator of the Lie subalgebra $\mathfrak{g}_{-i}^{\mathbb{C}}$ and applying Corollary 4.1 rewrite again the integrability conditions:

1) (4) holds for an almost complex structure J iff

$$d\Lambda^{1,0} \subset \Lambda^{2,0} \oplus \Lambda^{1,1};$$

2) the abelian property (5) holds for J iff

$$d\Lambda^{1,0} \subset \Lambda^{1,1};$$

3) J is a complex Lie algebra structure (6) iff

$$d\Lambda^{1,0} \subset \Lambda^{2,0}.$$

Let us consider an increasing sequence of complex subspaces in $\Lambda^{1,0}$:

$$V_0^{1,0} = \{0\} \subset V_1^{1,0} \subset \cdots \subset V_{s(\mathfrak{g})}^{1,0} = \Lambda^{1,0},$$

where

$$V_l^{1,0} = (V_l\mathfrak{g}^*)^{\mathbb{C}} \cap \Lambda^{1,0},\ l = 0, 1, \ldots, s(\mathfrak{g}).$$

REMARK 4.1. $V_1^{1,0}$ is the subspace of closed holomorphic 1-forms.

PROPOSITION 4.1. *$V_l^{1,0}$ is the annihilator of $\tilde{\mathfrak{g}}^{l+1} = \mathfrak{g}^{l+1}(J)^{\mathbb{C}} + \mathfrak{g}^{\mathbb{C}}_{-i}$.*

Proof.

$$\left((\mathfrak{g}^{l+1})^{\mathbb{C}} + \mathfrak{g}^{\mathbb{C}}_{-i}\right)^{ann} = (V_l\mathfrak{g}^*)^{\mathbb{C}} \cap (\mathfrak{g}^{\mathbb{C}}_{-i})^{ann} = V_l^{1,0}.$$

But in the same time

$$\mathfrak{g}^{l+1}(J)^{\mathbb{C}} + \mathfrak{g}^{\mathbb{C}}_{-i} = (\mathfrak{g}^{l+1})^{\mathbb{C}} + \mathfrak{g}^{\mathbb{C}}_{-i},$$

because $X + iJX \in \mathfrak{g}^{\mathbb{C}}_{-i},\ \forall X \in (\mathfrak{g}^{l+1})^{\mathbb{C}}$. □

PROPOSITION 4.2. *$\tilde{\mathfrak{g}}^l = (\mathfrak{g}^l)^{\mathbb{C}} + \mathfrak{g}^{\mathbb{C}}_{-i} = \mathfrak{g}^l(J)^{\mathbb{C}} + \mathfrak{g}^{\mathbb{C}}_{-i}$ is a Lie subalgebra of $\mathfrak{g}^{\mathbb{C}}$, moreover*

$$[\tilde{\mathfrak{g}}^l, \tilde{\mathfrak{g}}^l] \subset \tilde{\mathfrak{g}}^{l+1}.$$

Applying Lemma 4.1 and the previous two propositions we obtain

COROLLARY 4.2 ([15]).

$$dV_l^{1,0} \subset V_{l-1}^{1,0} \wedge (\mathfrak{g}^{\mathbb{C}})^*.$$

However we need to precise this statement

LEMMA 4.2.

$$dV_l^{1,0} \subset V_{l-1}^{1,0} \wedge (V_{l-1}\mathfrak{g}^*)^{\mathbb{C}}. \tag{8}$$

Proof. $V_l^{1,0} = (V_l\mathfrak{g}^*)^{\mathbb{C}} \cap \Lambda^{1,0}$ is a subspace in $(V_l\mathfrak{g}^*)^{\mathbb{C}}$ and hence

$$dV_l^{1,0} \subset d(V_l\mathfrak{g}^*)^{\mathbb{C}} \subset (V_{l-1}\mathfrak{g}^*)^{\mathbb{C}} \wedge (V_{l-1}\mathfrak{g}^*)^{\mathbb{C}}.$$

The intersection of two subspaces $V_{l-1}^{1,0} \wedge (\mathfrak{g}^{\mathbb{C}})^*$ and $(V_{l-1}\mathfrak{g}^*)^{\mathbb{C}} \wedge (V_{l-1}\mathfrak{g}^*)^{\mathbb{C}}$ gives the answer. □

THEOREM 4.1 ([15]). *A real nilpotent $2n$-dimensional Lie algebra $\mathfrak{g}$ admits an integrable complex structure if and only if $(\mathfrak{g}^{\mathbb{C}})^*$ has a basis $\{\omega^1,\dots\omega^n,\bar\omega^1,\dots\bar\omega^n\}$ such that*

$$d\omega^{l+1} \in I(\omega^1,\dots\omega^l),\ l = 0,\dots,n-1,$$

where $I(\omega^1,\dots\omega^l)$ is an ideal in $\Lambda^*((\mathfrak{g}^{\mathbb{C}})^*)$ generated by $\omega^1,\dots\omega^l$.

COROLLARY 4.3 ([15]). *A nilpotent Lie algebra $\mathfrak{g}$ admits an abelian complex structure if and only if $(\mathfrak{g}^*)^{\mathbb{C}}$ has a basis $\{\omega^1,\dots\omega^n,\bar\omega^1,\dots\bar\omega^n\}$ such that*

$$d\omega^{i+1} \in \Lambda^{1,1}(\omega^1,\dots\omega^i,\bar\omega^1,\dots\bar\omega^i),\ i = 0,\dots,n-1,$$

EXAMPLE 5. Define a graded Lie algebra $\mathfrak{B}(n) = \oplus_{l=1}^n \mathfrak{B}_l$ such that:

$$\dim \mathfrak{B}_l = 2,\ l = 1,2,\dots,n.$$

Let x_l, y_l denote basic elements in $\mathfrak{B}_l$ for $l = 1,\dots,n$. Then a Lie algebra structure of $\mathfrak{B}(n)$ is defined by:

$$[x_1,y_1] = y_2,\ [x_1,x_l] = [y_1,y_l] = x_{l+1},\ [x_1,y_l] = [x_l,y_1] = y_{l+1},\ \ l = 2,3,\dots,n-1.$$

One can define an abelian complex structure J:

$$Jx_l = -y_l,\ Jy_l = x_l,\ l = 1,2,\dots,n.$$

Taking 1-forms $\omega^l = x^l + iy^l$, $l = 1,2,\dots$ we have

$$d\omega^1 = 0,\ d\omega^2 = \frac{1}{2}\bar\omega^1\wedge\omega^1, d\omega^3 = \bar\omega^1\wedge\omega^2,\dots,d\omega^n = \bar\omega^1\wedge\omega^{n-1}. \tag{9}$$

Obviously $\mathfrak{B}(n)$ is n-step nilpotent Lie algebra and real 1-forms x^1, y^1, x^2 are closed. Moreover $\dim H^1(\mathfrak{B}(n)) = b_1(\mathfrak{B}(n)) = 3$.

REMARK 4.2. A complex structure J on a nilpotent Lie algebra $\mathfrak{g}$ corresponding to a basis $\{\omega^1,\dots\omega^n,\bar\omega^1,\dots\bar\omega^n\}$ of $\mathfrak{g}^*$ such that

$$d\omega^{i+1} \in \Lambda^2(\omega^1,\dots\omega^i,\bar\omega^1,\dots\bar\omega^i),\ i = 0,\dots,n-1,$$

was called in [6] a nilpotent complex structure.

One can give an invariant definition of nilpotent complex structure.

DEFINITION 4.1 ([15]). *An almost complex structure J on a Lie algebra $\mathfrak{g}$ is called (integrable) nilpotent complex structure if for all $l = 1,\dots,s(\mathfrak{g})$,*

$$dV_l^{1,0} \subset V_{l-1}^{1,0} \wedge (V_{l-1}^{1,0} \oplus V_{l-1}^{0,1}).$$

PROPOSITION 4.3. *A nilpotent $2n$-dimensional Lie algebra $\mathfrak{g}$ admits a complex Lie algebra structure, i.e. $\mathfrak{g}$ can be regarded as **a n-dimensional Lie algebra over** $\mathbb{C}$, if and only if $(\mathfrak{g}^*)^{\mathbb{C}}$ has a basis $\{\omega^1,\dots\omega^n,\bar\omega^1,\dots\bar\omega^n\}$ such that*

$$d\omega^{i+1} \in \Lambda^2(\omega^1,\dots\omega^i),\ i = 0,\dots,n-1.$$

Obviously an abelian complex structure and a complex Lie algebra structure are examples of nilpotent complex structures. However there are examples of nilpotent Lie algebras that admit only non-nilpotent complex structures.

EXAMPLE 6 (S. Salamon, [15]). Let us consider the 6-dimensional Lie algebra $\mathfrak{g}_{6,8}$ defined by its basis $e_1,\dots,e_6$ and structure relations:

$$[e_1,e_2]=e_3,\ [e_1,e_3]=e_4,\ [e_2,e_3]=e_5,\ [e_1,e_4]=[e_2,e_5]=e_6.$$

One can verify that an almost complex structure J on $\mathfrak{g}_{6,8}$ defined by

$$Je_1=-e_2, Je_2=e_1,\ Je_4=-e_5, Je_5=e_4,\ Je_3=e_6, Je_6=-e_3,$$

is integrable and non-abelian.

Proceeding to the dual picture (taking the dual basis $e^1,e^2,\dots,e^6$ of $\mathfrak{g}^*_{6,8}$) we have the following relations for the differentials of basic forms:

$$de^1=0,\ de^2=0,\ de^3=e^1\wedge e^2,\ de^4=e^1\wedge e^3,\ de^5=e^2\wedge e^3,\ de^6=e^1\wedge e^4+e^2\wedge e^5.$$

Now let us consider the complexification $(\mathfrak{g}^{\mathbb{C}}_{6,8})^*$ and complex forms

$$\omega^1=e^1+ie^2,\ \omega^2=e^4+ie^5,\ \omega^3=e^6+ie^3.$$

They are a basis of holomorphic forms $\Lambda^{1,0}(\mathfrak{g}^{\mathbb{C}}_{6,8})^*$.

Now one can verify the formulas

$$d\omega^1=0,\ d\omega^2=\omega^1\wedge\frac{i}{2}(\bar\omega^3-\omega^3),\ d\omega^3=\frac{1}{2}(\omega^1\wedge\bar\omega^2+\bar\omega^1\wedge\omega^2)-\frac{1}{2}\omega^1\wedge\bar\omega^1.$$

One can generalize this example for an arbitrary even dimension $\dim\mathfrak{g}\geqslant 6$.

EXAMPLE 7. A Lie algebra $\mathfrak{C}(n+1), n\geqslant 3$,: let $w_2, w_{n+1}, x_l, y_l, l=1,3,4,\dots,n$, be its basis. Structure constants are defined by:

$$\begin{gathered}[x_1,y_1]=z_2, [x_1,w_2]=x_3, [y_1,w_2]=y_3, [x_1,x_n]=[y_1,y_n]=w_{n+1},\\ [x_1,x_l]=[y_1,y_l]=x_{l+1}, [y_1,x_l]=[x_1,y_l]=y_{l+1},\ l=3,\dots,n-1.\end{gathered} \tag{10}$$

We define a complex structure J:

$$Jw_2=w_{n+1}, Jw_{n+1}=-w_2, Jx_l=-y_l,\ Jy_l=x_l,\ l=1,3,4,\dots,n.$$

Taking 1-forms $\omega^2=w^{n+1}+iw^2, \omega^l=x^l+iy^l,\ l=1,3,4,\dots,n$, we have

$$\begin{gathered}d\omega^1=0,\ d\omega^2=\omega^1\wedge\frac{i}{2}\left(\bar\omega^{n+1}-\omega^{n+1}\right), d\omega^3=\bar\omega^1\wedge\omega^2,\dots,\\ d\omega^n=\bar\omega^1\wedge\omega^{n-1},\ d\omega^{n+1}=\frac{1}{2}\left(\omega^1\wedge\bar\omega^n+\bar\omega^1\wedge\omega^n\right)-\frac{1}{2}\omega^1\wedge\bar\omega^1.\end{gathered} \tag{11}$$

5. Integrable complex structures and algebraic constraints

Consider again the decreasing sequence of Lie subalgebras $\mathfrak{g}^k(J)$:

$$\mathfrak{g} = \mathfrak{g}^1(J) \supset \mathfrak{g}^2(J) \supset \cdots \supset \mathfrak{g}^{s(\mathfrak{g})}(J) \supset \{0\}. \tag{12}$$

We have already noted that the first inclusion in this sequence is strict, but this is not necessarily so for the other inclusions.

Recall the example 6 of the 6-dimensional nilpotent Lie algebra $\mathfrak{g}_{6,8}$ endowed with the non-abelian complex structure J. $\mathfrak{g}_{6,8}$ is 4-step nilpotent and one can easily remark that

$$\mathfrak{g}_{6,8} \supset \mathfrak{g}^2_{6,8}(J) = \mathfrak{g}^3_{6,8}(J) \supset \mathfrak{g}^4_{6,8}(J) \supset \{0\},$$

where

$$\mathfrak{g}^2_{6,8}(J) = Span(e_3,e_4,e_5,e_6) = \mathfrak{g}^3_{6,8}(J), \quad \mathfrak{g}^4_{6,8}(J) = Span(e_5,e_6).$$

Let E denotes the total number of equalities in the sequence (12).

Obviously we have the following estimate:

$$2\,(s(\mathfrak{g}) - E) \leqslant \dim \mathfrak{g}, \tag{13}$$

it follows from the fact that the dimension $\dim \mathfrak{g}^k(J)$ decreases at each strict inclusion at least by two.

PROPOSITION 5.1. *Let $\mathfrak{g}$ be a Lie algebra endowed by nilpotent complex structure. Then we have the following estimate on its nil-index:*

$$s(\mathfrak{g}) \leqslant \frac{1}{2} \dim \mathfrak{g}. \tag{14}$$

Proof. It follows that for a nilpotent complex structure J in the decreasing sequence (12) all inclusions are strict, i.e. $E = 0$. The Example 5 shows that our estimate on nil-index of nilpotent Lie algebras with nilpotent or abelian complex structures is sharp.

The estimate (14) of nil-index have not been discussed in [6], although it easily follows from the arguments there. □

REMARK 5.1 ([6]). Another restriction imposed on the algebraic structure of $\mathfrak{g}$ by the existence of a nilpotent complex structure was the following estimate [6]:

$$b_1(\mathfrak{g}) = \dim \mathfrak{g} - \dim [\mathfrak{g},\mathfrak{g}] \geqslant 3.$$

This estimate follows from the properties of the canonical basis $\omega^1, \bar{\omega}^1, \dots, \omega^n, \bar{\omega}^n$ in $(\mathfrak{g}^*)^{\mathbb{C}}$. Real and imaginary parts of ω^1 are linearly independent closed forms. If $d\omega^2 = 0$ then it evidently follows that $b_1(\mathfrak{g}) \geqslant 4$. If $d\omega^2 \neq 0$ one can choose ω_2 such that $d\omega^2 = \omega^1 \wedge \bar{\omega}^1$, but the 2-form $\omega^1 \wedge \bar{\omega}^1$ is pure imaginary and hence the real part of ω^2 must to be closed, that gives us at least three linearily independent closed real 1-forms.

The complex Lie algebra structure J can be regarded as abelian complex structure as we have seen. But in order to have sharp estimates we have to precise our estimates.

PROPOSITION 5.2. *If a real nilpotent Lie algebra $\mathfrak{g}$ admits a complex Lie algebra structure, then*

$$s(\mathfrak{g}) \leqslant \frac{1}{2}\dim\mathfrak{g} - 1, \quad b_1(\mathfrak{g}) = \dim\mathfrak{g} - \dim[\mathfrak{g},\mathfrak{g}] \geqslant 4. \tag{15}$$

Proof. The $2n$-dimensional real Lie algebra $\mathfrak{g}$ can be regarded as a complex n-dimensional Lie algebra and hence

$$s(\mathfrak{g}) \leqslant n-1 = \frac{1}{2}\dim\mathfrak{g} - 1.$$

Its first cohomology groupe $H^1(\mathfrak{g})$ as a complex space has dimension at least 2. Hence $\dim\mathfrak{g} - \dim[\mathfrak{g},\mathfrak{g}] = b_1(\mathfrak{g}) \geqslant 4$. The Lie algebra $\mathfrak{m}_0^{\mathbb{R}}(n)$ from Example 4 shows that these estimates are sharp. □

As we have already noticed, the nil-index $s(\mathfrak{g})$ of even-dimesional real nilpotent Lie algebra $\mathfrak{g}$ can exceed the value $\frac{1}{2}\dim\mathfrak{g}$ if we have positive number $E > 0$ of equalities in the sequence (12).

LEMMA 5.1. *Let $k \geqslant 2$ and*

$$\mathfrak{g}^{k-1}(J) \supset \mathfrak{g}^k(J) = \mathfrak{g}^{k+1}(J) = \dots = \mathfrak{g}^{k+p}(J) \supset \mathfrak{g}^{k+p+1}(J),\ p \geqslant 1, \tag{16}$$

where the first and the last inclusions are strict. Then

$$\dim\mathfrak{g}^{k+p+1} - \dim\mathfrak{g}^{k+p+2} \geqslant 2.$$

Proof. The condition (16) is equivalent to the dual one

$$V_{k-1}^{1,0} \subset V_k^{1,0} = V_{k+1}^{1,0} = \dots = V_{k+p}^{1,0} \subset V_{k+p+1}^{1,0},\ k \geqslant 2.$$

Let us consider $\omega \in V_{k+p+1}^{1,0}, \omega \notin V_{k+p}^{1,0}$. Fix a base $\omega_1^1, \dots, \omega_1^{j_1}$ of $V_1^{1,0}$. Add thereto new elements $\omega_2^1, \dots, \omega_2^{j_2}$ (if necessary) to obtain a basis of $V_2^{1,0}$. Continue this process sequentially. In the last step we add 1-forms $\omega_k^1, \dots, \omega_k^{j_k} \in V_k^{1,0}$ such that the whole set

$$\omega_1^1, \dots, \omega_1^{j_1}, \omega_2^1, \dots, \omega_2^{j_2}, \dots, \omega_k^1, \dots, \omega_k^{j_k},$$

would form a basis of $V_k^{1,0}$. It follows from (4.2) that

$$\begin{gathered} d\omega = \Omega = \omega_1^1 \wedge \xi_1^1 + \dots + \omega_1^{j_1} \wedge \xi_1^{j_1} + \sum_{2 \leqslant m \leqslant k, 1 \leqslant l \leqslant j_m} \omega_m^l \wedge \xi_m^l, \\ \omega_m^l \in V_m^{1,0},\ \xi_m^l \in V_{k+p-m}(\mathfrak{g}^*)^{\mathbb{C}},\ 1 \leqslant m \leqslant k,\ 1 \leqslant l \leqslant j_m. \end{gathered} \tag{17}$$

Consider two inclusions $\mathfrak{g}^2 \subset \mathfrak{g}^2(J) \subset \mathfrak{g}$. We recall that the second one is strict. We choose a basis in the annihilator $\mathfrak{g}^2(J)^{ann}$ and complete it (if necessary, i.e. $\mathfrak{g}^2 \neq \mathfrak{g}^2(J)$) to a whole basic of $V_1\mathfrak{g}$. We denote the elements that we add by

$$e_1^1, \dots, e_1^r.$$

Remark that

$$\omega_1^1, \dots, \omega_1^{j_1}, \bar{\omega}_1^1, \dots, \bar{\omega}_1^{j_1}, e_1^1, \dots, e_1^r,$$

is a basis of subspace $V_1(\mathfrak{g}^*)^{\mathbb{C}}$. We denote by

$$e_k^1, e_k^2, \dots, e_k^{j_k},\ 2 \leqslant k \leqslant s(\mathfrak{g}),$$

linear independent 1-forms such that

$$V_k\mathfrak{g} = Span(e_k^1, e_k^2, \dots, e_k^{j_k}) \oplus V_{k-1}\mathfrak{g},\ 2 \leqslant k \leqslant s(\mathfrak{g}).$$

It is obvious that

$$V_k(\mathfrak{g})^{\mathbb{C}} = Span^{\mathbb{C}}(e_k^1, e_k^2, \dots, e_k^{j_k}) \oplus V_{k-1}(\mathfrak{g})^{\mathbb{C}},\ 2 \leqslant k \leqslant s(\mathfrak{g}).$$

PROPOSITION 5.3. *Cohomology classes* $[\Omega]$ *and* $[\bar{\Omega}]$ *are linearly independent in* $\ker \varphi_{k+p} \subset H^2(\Lambda^2(V_{k+p}(\mathfrak{g}^*)^{\mathbb{C}}))$*, where the mapping in the cohomology*

$$\varphi_{k+p} : H^2(\Lambda^2(V_{k+p}(\mathfrak{g}^*)^{\mathbb{C}})) \to H^2((\mathfrak{g}^*)^{\mathbb{C}}),$$

is induced by the inclusion of exterior d-algebras

$$\Lambda^*(V_{k+p}(\mathfrak{g}^*)^{\mathbb{C}}) \to \Lambda^*((\mathfrak{g}^*)^{\mathbb{C}}).$$

Proof. Among $\xi_1^1, \dots, \xi_1^{j_1}$ there is at least one $\xi_1^{k_0}$ that belongs to $V_{k+p}(\mathfrak{g}^*)^{\mathbb{C}}$ and does not belong to $V_{k+p-1}(\mathfrak{g}^*)^{\mathbb{C}}$ (otherwise $\Omega \in \Lambda^2(V_{k+p-1}(\mathfrak{g}^*)^{\mathbb{C}})$ and hence $\omega \in V_{k+p}(\mathfrak{g}^*)^{\mathbb{C}}$ which contradicts to our choice of ω).

We decompose ξ_{k_0} into a linear combination of basis vectors:

$$\xi_1^{k_0} = \sum_{j=1}^{m_{k+p}} A_{j,k+p} e_{k+p}^j + \sum_{1 \leqslant r \leqslant m_q, 1 \leqslant q \leqslant k+p-1} A_{r,q} e_q^r + \sum_{t=1}^{j_1} B_t \omega_1^t + \sum_{s=1}^{j_1} C_s \bar{\omega}_1^s.$$

There is $l_0, 1 \leqslant l_0 \leqslant j_m$, such that $A_{l_0,k+p} \neq 0$. Hence

$$\Omega = A_{l_0,k+p} \omega_1^{k_0} \wedge e_{k+p}^{l_0} + \Omega_1,\ A_{l_0,k+p} \neq 0.$$

The key observation is that in the expansion of Ω there is no term proportional to $\bar{\omega}_1^{k_0} \wedge e_{k+p}^{l_0}$. On the other hand, the expansion of an arbitrary coboundary $d\rho, \rho \in V_{k+p}(\mathfrak{g}^*)^{\mathbb{C}}$ can not contain terms $\omega_1^{k_0} \wedge e_{k+p}^{l_0}$ and $\bar{\omega}_1^{k_0} \wedge e_{k+p}^{l_0}$, because

$$d\rho \in \Lambda^2(V_{k+p-1}(\mathfrak{g}^*)^{\mathbb{C}}),\ \rho \in V_{k+p}(\mathfrak{g}^*)^{\mathbb{C}}.$$

Thus the cohomology classes $[\Omega]$ and $[\bar{\Omega}]$ are linearly independent in $\ker \varphi_{k+p}$. □

EXAMPLE 8. Consider again Example 6. In the first step we chose $\omega^1, \bar{\omega}^1$ as a basis of $V_1(\mathfrak{g}_{6,8})^{\mathbb{C}}$. The kernel of the cohomology map induced by the inclusion

$$\Lambda^2(\omega^1, \bar{\omega}^1) \to \Lambda^2((\mathfrak{g}_{6,8}^*)^{\mathbb{C}}),$$

is one-dimensional and it is spanned by $[\omega_1 \wedge \bar{\omega}_1] = -2i[e^1 \wedge e^2]$. In the second step $V_1^{1,0} = V_2^{1,0}$ and we added a new generator e^3 in order to kill the kernel $\ker \varphi_1$:

$$de^3 = e^1 \wedge e^2 = \frac{i}{2}\omega^1 \wedge \bar{\omega}^1.$$

The third step. We have a strict inclusion $V_2^{1,0} \subset V_3^{1,0}$ and and we took ω^2, which together with ω^1 constitutes the basis of the subspace $V_3^{1,0}$. Thus the following equality holds

$$d\omega^2 = \Omega_2 = \omega^1 \wedge e^3.$$

The cocycles $\Omega_2 = \omega^1 \wedge e^3$ and $\bar{\Omega}_2 = \bar{\omega}^1 \wedge e^3$ are obviously linearly independent in $\Lambda^2(\omega^1, \bar{\omega}^1, e^3)$. An arbitrary 2-coboundary has the form $\alpha\omega^1 \wedge \bar{\omega}^1, \alpha \in \mathbb{C}$. Hence cohomology classes $[\Omega_2]$ and $[\bar{\Omega}_2]$ are linearly independent in $\ker \varphi_2$ in 2-cohomology and they span this kernel. So we add new generators $\omega^2, \bar{\omega}^2$ and kill $\ker \varphi_2$.

The last step. Again we have a strict inclusion $V_3^{1,0} \subset V_4^{1,0}$. We add ω_3 and get a basis $\omega^1, \omega^2, \omega^3$ of $\Lambda^{1,0}$:

$$d\omega^3 = \Omega_3 = \frac{1}{2}\left(\omega^1 \wedge \bar{\omega}^2 + \bar{\omega}^1 \wedge \omega^2\right) - \frac{1}{2}\omega^1 \wedge \bar{\omega}^1.$$

The kernel of the cohomology map induced by the inclusion

$$\Lambda^2(\omega^1, \bar{\omega}^1, e^3, \omega^2, \bar{\omega}^2) \to \Lambda^2((\mathfrak{g}_{6,8}^*)^{\mathbb{C}}),$$

is one-dimensional and it is spanned by $[\Omega_3] = [\omega^1 \wedge \bar{\omega}^2 + \bar{\omega}^1 \wedge \omega^2] = 2[e^1 \wedge e^4 + e^2 \wedge e^5]$.

We see that

$$\Omega_3 - \bar{\Omega}_3 = \omega^1 \wedge \bar{\omega}^1 = d(-2ie^3).$$

So in this case 2-classes Ω_3 and $\bar{\Omega}_3$ are linearly dependent in $\ker \varphi_3$. There is no contradiction with Lemma 5.1, because we have the case of two consecutive strict inclusions:

$$V_2^{1,0} \subset V_3^{1,0} \subset V_4^{1,0}.$$

□

PROPOSITION 5.4. *Let* $\mathfrak{g}$ *be a nilpotent Lie algebra endowed with an integrable complex structure and* $\dim \mathfrak{g} \geqslant 6$. *Then we have the following estimate:*

$$\dim \mathfrak{g} - \dim \mathfrak{g}^4 = \dim \mathfrak{g} - \dim [\mathfrak{g}, [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]]] \geqslant 5. \quad (18)$$

Proof. Our assertion is necessary to prove only for Lie algebras $\mathfrak{g}$ with the first Betti number $b_1(\mathfrak{g}) = 2$ (for all other Lie algebras our inequality holds automatically). In this case forms $\omega^1, \bar{\omega}^1$ span the space $V_1(\mathfrak{g}^*)^{\mathbb{C}}$ and

$$d\omega^2 = \omega^1 \wedge \xi^{(0)}, \xi^{(0)} \in V_k(\mathfrak{g}^*)^{\mathbb{C}}, k \geqslant 2, \xi^{(0)} \notin V_{k-1}(\mathfrak{g}^*)^{\mathbb{C}}.$$

If $\xi^{(0)} \in V_1(\mathfrak{g}^*)^{\mathbb{C}}$ then either ω^2 is closed or $\omega^1 \wedge \xi^{(0)}$ is proportional to $\omega^1 \wedge \bar{\omega}^1$ which is pure imaginary form and hence in this case the real part of ω^2 has to be closed and it condradicts to the assumption that $b_1(\mathfrak{g}) = 2$.

We see that $d\xi^{(0)} = \omega^1 \wedge \xi^{(1)}, \xi^{(1)} \in V_{k-1}(\mathfrak{g}^*)^{\mathbb{C}}$.

Continuing step-by-step process, we get at the end two linearly independent 2-cocycles

$$\omega^1 \wedge \xi^{(k-2)} = \omega^1 \wedge (\alpha e^3 + \beta\omega^1 + \gamma\bar{\omega}^1), \;\; \bar{\omega}^1 \wedge \bar{\xi}^{(k-2)} = \bar{\omega}^1 \wedge (\bar{\alpha} e^3 + \bar{\beta}\bar{\omega}^1 + \bar{\gamma}\omega^1),$$

where $de^3 = e^1 \wedge e^2$, $\omega^1 = e^1 + ie^2$ and $\alpha \neq 0$. Hence

$$\dim V_3(\mathfrak{g}^*)^{\mathbb{C}} - \dim V_2(\mathfrak{g}^*)^{\mathbb{C}} \geqslant 2,$$

which means that $\dim \mathfrak{g}^3 - \dim \mathfrak{g}^4 \geqslant 2$, Finally we have

$$\left(\dim \mathfrak{g} - \dim \mathfrak{g}^2\right) + \left(\dim \mathfrak{g}^2 - \dim \mathfrak{g}^3\right) + \left(\dim \mathfrak{g}^3 - \dim \mathfrak{g}^4\right) \geqslant 5.$$

□

As an elementary corollary we get the result obtained by Goze and Remm [11]:

COROLLARY 5.1. *A filiform Lie algebra* $\mathfrak{g}$ *does not admit any integrable complex structure.*

Proof. It follows from the definition that for an arbitrary filiform Lie algebra $\mathfrak{g}$ the following equality holds on:

$$\dim \mathfrak{g} - \dim \mathfrak{g}^4 = 2 + 1 + 1 = 4.$$

□

6. Main example

EXAMPLE 9. Define a positively graded Lie algebra $\mathfrak{D}(n) = \oplus_{l=1}^{n} \mathfrak{D}_l$ such that:

$$\dim \mathfrak{D}_l(n) = \begin{cases} 1, & l = 2k, l \leqslant n; \\ 2, & l = 2k-1, l \leqslant n. \end{cases}$$

Let v_{2k-1}, u_{2k-1} denote basic elements in $\mathfrak{D}_{2k-1}(n)$ and w_{2k} in $\mathfrak{D}_{2k}(n)$ respectively. Then a Lie algebra structure of $\mathfrak{D}(n)$ is defined by:

(19)

$$[v_i, w_j] = \begin{cases} u_{i+j}, i+j \leqslant n; \\ 0, \;\; i+j > n. \end{cases}, [w_j, u_l] = \begin{cases} v_{j+l}, j+l \leqslant n; \\ 0, \;\; j+l > n. \end{cases}, [u_l, v_i] = \begin{cases} w_{l+i}, l+i \leqslant n; \\ 0, \;\; l+i > n. \end{cases}$$

We recall that indexes i, l (index j) in (19) are taking odd (even) positive integer values.

REMARK 6.1. $\mathfrak{D}(4)$ is isomorphic to the algebra $\mathfrak{g}_{6,8}$ from the Example 6.

PROPOSITION 6.1. *$\mathfrak{D}(n)$ is naturally graded nilpotent Lie algebra and*

$$\dim \mathfrak{D}(n) = \begin{cases} 6m, & n = 4m, \\ 6m+2, & n = 4m+1, \\ 6m+3, & n = 4m+2, \\ 6m+5, & n = 4m+3, \end{cases} , \; s(\mathfrak{D}(n)) = n.$$

Proof. It is easy to see that

$$\mathfrak{D}(n)^2 = [\mathfrak{D}(n), \mathfrak{D}(n)] = \oplus_{l=2}^{n} \mathfrak{D}_l.$$

Continue this process step-by-step we have that

$$\mathfrak{D}(n)^m = \left[\mathfrak{D}(n), \mathfrak{D}(n)^{m-1}\right] = \oplus_{l=m}^{n} \mathfrak{D}_l, \; m = 2, \dots, n.$$

Hence $\mathfrak{D}(n)^m / \mathfrak{D}(n)^{m+1} = \mathfrak{D}_m(n)$ that proves that $\mathfrak{D}(n)$ is naturally graded. The formulas for the dimension and nil-index can be easily verified. □

PROPOSITION 6.2. *The positively graded (nilpotent) Lie algebras $\mathfrak{D}(4m)$ and $\mathfrak{D}(4m+1)$ admit complex structures.*

Proof. Define an almost complex structure J on $\mathfrak{D}(4m)$ by

(20) $$Jv_{2l+1} = u_{2l+1}, l = 0, 1, \dots, 2m-1, \; Jw_{4k+2} = w_{4k+4}, \; k = 0, 1, \dots, m-1.$$

The proof consists of verifying the integrability condition (4) for basic elements u_j, v_k, w_m.

To define a complex structure on $\mathfrak{D}(4m+1)$ one should add to (20) one relation:

(21) $$Jv_{4m+1} = u_{4m+1}, \; Ju_{4m+1} = -v_{4m+1}.$$

Taking the duals u^r, v^p, w^l one can define the following 1-forms for $k = 1, 2, \dots, m-1$:

(22) $$\begin{aligned} \omega_{3k} &= w^{4k-2} - iw^{4k}, \\ \omega_{3k+1} &= u^{4k+1} + iv^{4k+1}, \\ \omega_{3k+2} &= u^{4k+3} + iv^{4k+3}, \end{aligned}$$

It follows from (19) that

(23) $$du^r = \sum_{i+j=r} v^i \wedge w^j, \; dv^p = \sum_{i+j=p} w^i \wedge u^j, \; dw^l = \sum_{i+j=l} u^i \wedge v^j.$$

One can easily write out the formulae for the differentials $d\omega_i$:

(24)

$$d\omega_1 = 0, \; d\omega_2 = -\omega_1 \wedge iw^2, \; d\omega_3 = \frac{1}{2}(\omega_1 \wedge \bar{\omega}_2 - \bar{\omega}_1 \wedge \omega_2) + \frac{i}{2}\omega_1 \wedge \bar{\omega}_1,$$
$$d\omega_4 = -\omega_1 \wedge iw^4 - \omega_2 \wedge iw^2, \; d\omega_5 = -\omega_1 \wedge iw^6 - \omega_2 \wedge iw^4 - \omega_4 \wedge iw^2,$$
$$d\omega_6 = \frac{1}{2}(\omega_1 \wedge \bar{\omega}_5 - \bar{\omega}_1 \wedge \omega_5) + \frac{1}{2}(\omega_2 \wedge \bar{\omega}_4 - \bar{\omega}_2 \wedge \omega_4) + \frac{i}{2}(\omega_1 \wedge \bar{\omega}_4 - \bar{\omega}_1 \wedge \omega_4) + \frac{i}{2}\omega_2 \wedge \bar{\omega}_2,$$
$$\dots, \qquad \dots$$

□

Remark that the algebras $\mathfrak{D}(4m+2)$ and $\mathfrak{D}(4m+3)$ are odd-dimensional ones.

PROPOSITION 6.3. *Graded Lie algebras $\mathfrak{D}(4m+2\oplus\mathbb{R}$ and $\mathfrak{D}(4m+3\oplus\mathbb{R}$ are $(6m+4)$-dimensional and $(6m+6)$-dimensional respectively and they admit complex structures defined by (20) and (21) with the additional relation*

$$Jw_{4k+2} = t,\ Jt = -w_{4k+2}.$$

Obviously $s(\mathfrak{D}(4m+2)\oplus\mathbb{R}) = 4m+2$ and $s(\mathfrak{D}(4m+3)\oplus\mathbb{R}) = 4m+3$ because the new generator t belongs to the centre of our algebra.

We proved the Theorem

THEOREM 6.1. *Let $s(2n)$ denotes the maximal value of nil-index $s(\mathfrak{g})$ of $2n$-dimensional nilpotent Lie algebra $\mathfrak{g}$ that admits a complex structure. Then we have the following estimates:*

$$\left[\frac{4n}{3}\right] \leqslant s(2n) \leqslant 2n-2. \tag{25}$$

REMARK 6.2. It follows from [15] that $s(6) = 4$. But it appears possible to improve the estimates (25) for dimensions $2n \geqslant 8$.

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Dmitry MILLIONSHCHIKOV,
Steklov Mathematical Institute of Russian Academy of Sciences,
Moscow, Russia
e-mail: `million@mech.math.msu.su`

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