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THE X-RAY TRANSFORM ON 2-STEP NILPOTENT LIE GROUPS OF HIGHER RANK

Abstract. We prove injectivity and a support theorem for the X-ray transform on 2-step nilpotent Lie groups with many totally geodesic 2-dimensional flats. The result follows from a general reduction principle for manifolds with uniformly escaping geodesics.

Dedicated to the memory of Sergio Console

1. Background

The X-ray transform of a sufficiently rapidly decreasing continuous function f on the Euclidean plane $\mathbb{R}^2$ is a function $\mathcal{X}f$ defined on the set of all straight lines via integration along these lines. More precisely, if ξ is a straight line, given by a point $x \in \xi$ and a unit vector $\theta \in \mathbb{R}^2$ such that $\xi = x + \mathbb{R}\theta$, then

$$\mathcal{X}f(\xi) = \mathcal{X}f(x,\theta) = \int_{-\infty}^{\infty} f(x+s\theta)\,ds.$$

It is natural to ask about injectivity of this transform and, if yes, for an explicit inversion formula. If $f(x) = O(|x|^{-(2+\varepsilon)})$ for some $\varepsilon > 0$, the function f can be recovered via the following inversion formula, going back to J. Radon [18] in 1917:

$$f(x) = -\frac{1}{\pi}\int_0^{\infty} \frac{F_x'(t)}{t}\,dt, \tag{1}$$

where $F_x(t)$ is the mean value of $\mathcal{X}f(\xi)$ over all lines ξ at distance t from x:

$$F_x(t) = \frac{1}{2\pi}\int_{S^1} \mathcal{X}f(x+t\theta^{\perp},\theta)\,d\theta,$$

where $(x,y)^{\perp} = (y,-x)$. Zalcman [29] gave an example of a non-trivial function $f \in C^{\infty}(\mathbb{R}^2)$ with $f(x) = O(|x|^{-2})$ and $\mathcal{X}f(\xi) = 0$ for all lines $\xi \subset \mathbb{R}^2$ and, therefore, the decay condition for the inversion formula is optimal.

Under stronger decay conditions, it is possible to prove the following support theorem (see [5, Thm. 2.1] or [7, Thm. I.2.6]):

THEOREM 1.1 (Support Theorem). *Let $R > 0$ and $f \in C(\mathbb{R}^2)$ with $f(x) = O(|x|^{-k})$ for all $k \in \mathbb{N}$. Assume that $\mathcal{X}f(\xi) = 0$ for all lines ξ with $d(\xi,0) > R$. Then we have $f(x) = 0$ for all $|x| > R$.*

Again, the stronger decay condition is needed here by a counterexample of D.J. Newman given in Weiss [26] (see also [7, Rmk. I.2.9]). The Euclidean X-ray transform plays a prominent role in medical imaging techniques like the CT and PET (see, e.g., [12]).

The X-ray transform can naturally be generalized to other complete, simply connected Riemannian manifolds, by replacing straight lines by complete geodesics. Radon mentioned in [18] that there is an analogous inversion formula in the (real) hyperbolic plane $\mathbb{H}^2$, where the denominator in the integral of (1) has to be replaced by $\sinh(t)$ (see also [7, Thm. III.1.12(ii)]). There is also an analogue of the support theorem for the hyperbolic space (see [7, Thm. III.1.6]), valid for functions f satisfying $f(x) = O(e^{-kd(x_0,x)})$ for all $k \in \mathbb{N}$ and $x_0 \in \mathbb{H}^n$.

In the case of a continuous function f on a *closed* Riemannian manifold X, the domain of $\mathcal{X}f$ is the set of all *closed* geodesics. Continuous functions f can only be recovered from their X-ray transform $\mathcal{X}f$ if the union of all closed geodesics is dense in X. But this condition is not sufficient as the following simple example of the two-sphere $\mathbb{S}^2$ shows. Every *even* continuous function f on $\mathbb{S}^2$ (i.e., $f(-x) = f(x)$) can be recovered by its integrals over all great circles. This fact and a solution similar to (1) goes back to Minkowski 1911 and Funk 1913 (see [7, Section II.4.A] and the references therein). But, on the other hand, it is easy to see that $\mathcal{X}f$ vanishes for all *odd* functions, so the restriction to even functions is essential. For injectivity and support theorems of the X-ray transform on compact symmetric spaces X other than $\mathbb{S}^n$ see, e.g., [7, Section IV.1]. Injectivity properties of the extended X-ray transform for symmetric k-tensors on closed manifolds (with respect to the solenoidal part) play an important role in connection with *spectral rigidity* (see [4]) and were proved for closed manifolds with Anosov geodesic flows (see [3, Thms 1.1 and 1.3] for $k = 0, 1$) or strictly negative curvature (see [2] for arbitrary $k \in \mathbb{N}$).

Another class of manifolds for which the X-ray transform and its extension to symmetric k-tensors has been studied are *simple manifolds*, i.e., manifolds X with strictly convex boundary and without conjugate points (see [23]). An application is the *boundary rigidity problem*, i.e., whether it is possible to reconstruct the metric of X (modulo isometries fixing the boundary) from the knowledge of the distance function between points on the boundary ∂X. Solenoidal injectivity is known for $k = 0, 1$ for all simple manifolds (see [13] and [1]), and for all $k \in \mathbb{N}$ for surfaces [16] and for negatively curved manifolds [15]. There are also support type theorem for the X-ray transform on simple manifolds (see [10, 11] and [25] and the references therein). A very recommendable survey with a list of open problems is [17].

2. A reduction principle for manifolds with uniformly escaping geodesics

In this article, we will only consider complete Riemannian manifolds X whose geodesics escape in the sense of e.g. [27], [28], [9], in a uniform way. Simply connected manifolds without conjugate points have this property, but we like to stress that the main examples in this article will be *manifolds with conjugate points*. Geodesics will always be parametrized by arc length.

DEFINITION 2.1. *A Riemannian manifold X has* uniformly escaping geodesics *if for each* $r \in \mathbb{R}_0^+$ *there is* $P(r) \in \mathbb{R}_0^+$ *such that for every geodesic* $\gamma\colon \mathbb{R} \to X$ *and every* $t > P(r)$*, we have* $d(\gamma(t), \gamma(0)) > r$*. We call* P *an* escape function *of* X*.*

The smallest such function P,

$$P(r) := \sup\{t \geqslant 0 \mid \exists \text{ geodesic } \gamma\colon \mathbb{R} \to X, d(\gamma(0), \gamma(t)) \leqslant r\}$$

is thus required to be finite for all r. After time $P(r)$ every geodesic has left a closed ball $\mathrm{B}_r(p)$ of radius $r \in \mathbb{R}_0^+$ around its center $p \in X$. The function P increases and satisfies $P(r) \geqslant r$. Note that P may not be continuous.

Manifolds with this property must be simply connected and non-compact. As mentioned earlier, simply connected Riemannian manifolds without conjugate points have this property with escape function $P(r) = r$.

The class of compactly supported continuous functions on such a manifold is preserved under restriction to totally geodesic immersed submanifolds. Thus if f is a compactly supported continuous function on X, say $\operatorname{supp}(f) \subset \mathrm{B}_r(p)$ for some $p \in X$ and $r > 0$, and $\phi\colon Y \to X$ a totally geodesic isometric immersion, then f has compact support on Y and $\operatorname{supp}(f \circ \phi) \subset \mathrm{B}^Y_{P(r)}(p)$. In particular, this holds for geodesics (as 1-dimensional immersions) and the integral of f over any geodesic in X is thus defined.

Before we formulate the reduction principle, let us first fix some notation. The unit tangent bundle of X is denoted by SX. For a Riemannian manifold X let $C_c(X)$ be the space of all continuous functions $f\colon X \to \mathbb{C}$ with compact support. By $G(X)$ we denote the set of (unparametrized oriented) geodesics, i.e.

$$G(X) = \{\gamma(\mathbb{R}) \mid \gamma\colon \mathbb{R} \to X \text{ geodesic }\}$$

The X-ray transform of $f \in C_c(X)$ is the function $\mathcal{X}f\colon G(X) \to \mathbb{C}$ with

$$\mathcal{X}f(L) = \int_L f = \int_{-\infty}^{+\infty} f(\gamma(t))dt$$

if $L = \gamma(\mathbb{R})$ and γ a unit speed geodesic.

DEFINITION 2.2. *Let* $r_0 \geqslant 0$ *and* $\sigma\colon [r_0, \infty) \to \mathbb{R}_0^+$ *be a function. We say that* the σ-support theorem holds on X *if for* $p \in X$ *and* $f \in C_c(X)$*,* $r \in [r_0, \infty)$ *we have that* $\mathcal{X}f|_{G(X \setminus \mathrm{B}_{\sigma(r)}(p))} = 0$ *implies* $f|_{X \setminus \mathrm{B}_r(p)} = 0$*. We say that* X has a support theorem *if this holds for a function* σ *with* $\lim_{r\to\infty} \sigma(r) = \infty$*.*

REMARK 2.1. If X has a σ-support theorem, then X has a support theorem for all smaller functions as well. Moreover, we can always modify $\sigma : [r_0, \infty) \to \mathbb{R}_0^+$ to be monotone non-decreasing. If $r_0 = 0$, i.e., $\sigma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, the σ-support theorem implies injectivity of the X-ray transform.

Then we have the following reduction principle.

THEOREM 2.1. *Let* X *be a complete, Riemannian manifold which has uniformly escaping geodesics with escape function* P*.*

(i) *Assume there exists, for every $x \in X$, a closed totally geodesic immersed submanifold $Y \subset X$ through x such that the X-ray transform on Y is injective. Then the X-ray transform on X is also injective.*

(ii) *Let $\mu\colon [r_0,\infty) \to \mathbb{R}_0^+$ be a function with $\mu \geqslant P(0)$. Assume there exists, for every $v \in SX$, a closed totally geodesic immersed submanifold $Y \subset X$ with $v \in SY$ such that the μ-support theorem holds on Y. Then a σ-support theorem holds on X for any function $\sigma\colon [r_0,\infty) \to \mathbb{R}_0^+$ with $P(\sigma(r)) \leqslant \mu(r)$ for all $r \geqslant r_0$. In particular, we can choose σ to be unbounded if μ is unbounded.*

Proof. (i) is obviously true by restriction since all geodesics in Y are also geodesics in X.

For (ii), let $f \in C_c(X)$ and $r \geqslant r_0$. We fix a point $p \in X$ and let $\mathcal{Y}_p$ be a set of closed totally geodesic immersed submanifolds Y with μ-support theorem and so that each geodesic through p lies in one of the $Y \in \mathcal{Y}_p$.

We then have

$$f|_{X \setminus \mathrm{B}_r^X(p)} = 0$$

if

$$\forall Y \in \mathcal{Y}_p : f|_{Y \setminus \mathrm{B}_r^Y(p)} = 0,$$

since, by assumption, each geodesic in X is contained in some Y. Now, by the μ-support theorem in $Y \in \mathcal{Y}_p$, we have

$$f|_{Y \setminus \mathrm{B}_r^Y(p)} = 0$$

if

$$\mathcal{X} f|_{G(Y \setminus \mathrm{B}_{\mu(r)}^Y(p))} = 0.$$

Since X has uniformly escaping geodesics property, this is guaranteed if

$$\mathcal{X} f|_{G(X \setminus \mathrm{B}_s^X(p))} = 0$$

for any $s \geqslant 0$ with $P(s) \leqslant \mu(r)$. Thus X has a σ-support theorem for any function $\sigma\colon [r_0,\infty) \to \mathbb{R}_0^+$ satisfying $P(\sigma(r)) \leqslant \mu(r)$. If the escape function $P\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is left-continuous, i.e. $\lim_{s \nearrow r} P(s) = P(r)$, we can choose $\sigma(r) = \sup\{s \geqslant 0 \mid P(s) \leqslant \mu(r)\}$. □

3. Applications of the reduction principle

In this section we demonstrate that many interesting examples can be derived by the reduction principle from $\mathbb{R}^2$ and $\mathbb{H}^2$. The X-ray transform on the euclidean and on the hyperbolic plane is injective and both have a μ-support theorem with $\mu(r) = r$. This follows directly from the euclidean or hyperbolic version of Radon's classical inversion formula (1), or Theorem 1.1.

If $X = X_1 \times X_2$ is the product of two Riemannian manifolds of positive dimension with uniformly escaping geodesics, with escape functions P_1 and P_2 respectively, then X has uniformly escaping geodesics with function P satisfying

$$\max\{P_1(r), P_2(r)\} \leqslant P(r) = \sup\left\{\sqrt{P_1(r_1)^2 + P_2(r_2)^2} \mid r_1^2 + r_2^2 \leqslant r^2\right\} \leqslant P_1(r) + P_2(r).$$

Each vector $v \in S(X_1 \times X_2)$ lies in a 2-flat $F \subset X_1 \times X_2$, i.e. a totally geodesic immersed flat submanifold. By the reduction principle, the σ-support theorem holds on $X_1 \times X_2$ for any function σ with $P(\sigma(r)) \leqslant r$ for all $r \in [P(0), \infty)$.

The reduction principle can also be applied to symmetric spaces of noncompact type. These spaces have no conjugate points and each of their geodesics is contained in a flat of dimension at least 2 if their rank is at least 2. In non-compact rank-1 symmetric spaces each geodesic is contained in a real hyperbolic plane. Therefore, the reduction principle yields injectivity of the X-ray transform and a support theorem with $\sigma(r) = r$ ([6], also [7, Cor. IV.2.1]).

Another interesting family are noncompact harmonic manifolds, which do not have conjugate points. Prominent examples in this family are Damek-Ricci spaces. In [21], Rouviere used the fact that each geodesic of a Damek-Ricci space is contained in a totally geodesic complex hyperbolic plane $\mathbb{CH}^2$ to obtain a support theorem with $\sigma(r) = r$ for Damek Ricci spaces.

The main result in this article is about injectivity of the X-ray transform and a support theorem for a certain class of 2-step nilpotent Lie groups with a left invariant metric and higher rank introduced in [22]. By[14] these spaces have conjugate points. Therefore, the methods of [10] do not immediately apply to these spaces. The spaces in [22] differ also significantly from Heisenberg-type groups which do not even infinitesimally have higher rank.

3.1. 2-step nilpotent Lie groups have uniformly escaping geodesics.

The Lie algebra of a 2-step nilpotent Lie algebra $\mathfrak{n}$ splits orthogonally as $\mathfrak{n} = \mathfrak{h} \oplus \mathfrak{z}$, $\mathfrak{z} = [\mathfrak{n}, \mathfrak{n}]$ the commutator and $\mathfrak{h} = \mathfrak{z}^\perp$ its orthogonal complement. We can thus view $\mathfrak{z} \subset \mathfrak{so}(\mathfrak{h})$ as a vectorspace of skew symmetric endomorphisms of $\mathfrak{h}$. We have

$$\langle [h,k] \mid z \rangle = \langle zh \mid k \rangle$$

for $h, k \in \mathfrak{h}$, $z \in \mathfrak{z}$. We show that 2-step nilpotent Lie groups have uniformly escaping geodesics, hence the X-ray transform for all functions with compact support is defined.

THEOREM 3.1. *Let N be a simply connected 2-step nilpotent Lie group with Lie algebra $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{h}$, $\mathfrak{z} \subset \mathfrak{so}(h)$. Then N has uniformly escaping geodesics with a continuous escape function P.*

Proof. We will prove that for each $r \in \mathbb{R}_0^+$ there is $P(r) \in \mathbb{R}^+$ such that every geodesic γ with $\gamma(0) = e$ (the neutral element of N) we have that $d(\gamma(t), e) \leqslant r$ implies $t \leqslant P(r)$.

We denote by $\exp^{\mathfrak{n}} : \mathfrak{n} \to N$ the exponential map of the Lie group. Since N is simply connected nilpotent this is a diffeomorphism. In particular, $(\exp^{\mathfrak{n}})^{-1}(\mathrm{B}_r(e)) \subset \mathrm{B}^{\mathfrak{n}}_{\rho(r)}(0)$ for some increasing continuous function $\rho(r) : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $\rho(0) = 0$. We will show that there is $P(r)$ such that for every geodesic γ in N with $\gamma(0) = e$, the curve $(\exp^{\mathfrak{n}})^{-1} \circ \gamma$ has left $\mathrm{B}^{\mathfrak{n}}_{\rho(r)}(0)$ after time $P(r)$.

From [8] for a geodesic $\gamma(t) = \exp^{\mathfrak{n}}(z(t) + h(t))$ with $z(t) \in \mathfrak{z}$, $h(t) \in \mathfrak{h}$, $\gamma'(0) = z_0 + h_0$, we have

$$h''(t) = z_0 h'(t),$$

$$z'(t) = z_0 + \frac{1}{2}[h(t), h'(t)],$$

which we need to solve subject to the initial conditions

$$\gamma(0) = \exp^{\mathfrak{n}}(z(0) + h(0)) = e \text{ hence } z(0) = 0 = h(0),$$

$$\gamma'(0) = z_0 + h_0 = z'(0) + h'(0),$$

so that $\|z_0\|^2 + \|h_0\|^2 = 1$. The solution to the first equation is

$$h(t) = \left((e^{tz_0} - 1)z_0^{-1}\right) h_0.$$

Note that this is well defined even if z_0 is not invertible. Inserting this into the second equation gives

$$z'(t) = z_0 + \frac{1}{2}\left[\left((e^{tz_0} - 1)z_0^{-1}\right) h_0, e^{tz_0} h_0\right].$$

Taking the scalar product of this with z_0 gives

$$\begin{aligned} \langle z'(t) \mid z_0 \rangle &= \|z_0\|^2 + \frac{1}{2}\langle z_0 \mid \left[\left((e^{tz_0} - 1)z_0^{-1}\right) h_0, e^{tz_0} h_0\right]\rangle \\ &= \|z_0\|^2 + \frac{1}{2}\langle z_0(e^{tz_0} - 1)z_0^{-1} h_0 \mid e^{tz_0} h_0\rangle \\ &= \|z_0\|^2 + \frac{1}{2}\|h_0\|^2 - \frac{1}{2}\langle h_0 \mid e^{tz_0} h_0\rangle, \end{aligned}$$

since e^{tz_0} is orthogonal. In order to compute $\langle z(t) \mid z_0 \rangle$, we integrate,

$$\langle z(t) \mid z_0 \rangle = t\|z_0\|^2 + \frac{t}{2}\|h_0\|^2 + \frac{1}{2}\langle h_0 \mid (1 - e^{tz_0})z_0^{-1} h_0\rangle.$$

It follows that

$$z(t) = tz_0 + \frac{t\|h_0\|^2 + \langle h_0 \mid (1 - e^{tz_0})z_0^{-1} h_0\rangle}{2\|z_0\|^2} z_0 + w(t)$$

with $w(t) \in \mathfrak{z}$ perpendicular to z_0. Hence, in the norm $\|\cdot\|$ of $\mathfrak{n}$, we can estimate

$$\|z(t)+h(t)\|^2 \geqslant \left\| \left((e^{tz_0}-1)z_0^{-1}\right) h_0\right\|^2 + \frac{1}{4\|z_0\|^2}\left(2\|z_0\|^2 t + t\|h_0\|^2 + \langle h_0 \mid (1-e^{tz_0})z_0^{-1}h_0\rangle\right)^2.$$

We split $\mathfrak{h} = \bigoplus_{\lambda\in\mathbb{R}} E(z_0, i\lambda)$ into the eigenspaces of z_0 and let $h_{\max} \in E(z_0, i\lambda)$ be the largest component of h_0, $i\lambda$ the corresponding eigenvalue. Thus $|h_{\max}|^2 \geqslant \frac{1}{\dim \mathfrak{h}}\|h_0\|^2$. Disregarding all other components, we estimate

$$\begin{aligned}
\|z(t)+h(t)\|^2 &\geqslant \left|\frac{e^{it\lambda}-1}{i\lambda}\right|^2 |h_{\max}|^2 + \frac{1}{4\|z_0\|^2}\left(2\|z_0\|^2 t + t\|h_{\max}\|^2 + \mathrm{Re}\left(\frac{1-e^{it\lambda}}{i\lambda}\right)|h_{\max}|^2\right)^2 \\
&= \frac{2-2\cos(\lambda t)}{\lambda^2}|h_{\max}|^2 + \frac{1}{4\|z_0\|^2}\left(2\|z_0\|^2 t + \left(t - \frac{\sin(t\lambda)}{\lambda}\right)|h_{\max}|^2\right)^2 \\
&= \|z_0\|^2 t^2 + |h_{\max}|^2\left(\frac{2-2\cos(\lambda t)}{\lambda^2} + t\left(t-\frac{\sin(\lambda t)}{\lambda}\right) + \frac{|h_{\max}|^2}{4\|z_0\|^2}\left(t-\frac{\sin(\lambda t)}{\lambda}\right)^2\right).
\end{aligned}$$

We now consider the cases:

$\|z_0\|^2 \geqslant \frac{1}{2}$: Then $\|z(t)+h(t)\|^2 \geqslant \frac{1}{2}t^2$.

If $\|z_0\|^2 \leqslant \frac{1}{2}$, then $\|h_0\|^2 = 1 - \|z_0\|^2 \geqslant \frac{1}{2}$, hence $|h_{\max}|^2 \geqslant \frac{1}{2\dim \mathfrak{h}}$. We can therefore estimate

$$\|z(t)+h(t)\|^2 \geqslant \frac{1}{2\dim \mathfrak{h}}\left(\frac{2-2\cos(\lambda t)}{\lambda^2} + t\left(t-\frac{\sin(\lambda t)}{\lambda}\right) + \frac{1}{4\dim \mathfrak{h}}\left(t-\frac{\sin(\lambda t)}{\lambda}\right)^2\right).$$

If $\lambda = 0$ the bracket evaluates to t^2, hence $\|z(t)+h(t)\|^2 \geqslant \frac{1}{2\dim \mathfrak{h}}t^2$.

If $0 \leqslant t \leqslant \frac{\pi}{2\lambda}$ then $\cos(\lambda t) \leqslant 1 - \frac{1}{\pi}(\lambda t)^2$. The other two summands are always nonnegative. Hence in this case,

$$\|z(t)+h(t)\|^2 \geqslant \frac{t^2}{\pi \dim \mathfrak{h}}.$$

If $t > \frac{\pi}{2\lambda}$ then $t - \frac{\sin(\lambda t)}{\lambda} \geqslant \left(1-\frac{2}{\pi}\right)t$. Observing that the rightmost and the leftmost summand are nonnegative, we get in this case that

$$\|z(t)+h(t)\|^2 \geqslant \frac{(\pi-2)t^2}{2\pi \dim \mathfrak{h}}.$$

Thus we have shown that

$$\|z(t)+h(t)\|^2 \geqslant t^2 \min\left\{\frac{1}{2}, \frac{1}{\pi\dim\mathfrak{h}}, \frac{\pi-2}{2\pi\dim\mathfrak{h}}\right\} = t^2\frac{\pi-2}{2\pi\dim\mathfrak{h}}.$$

Thus the curve $(\exp^{\mathfrak{n}})^{-1}(\gamma(t)) = z(t)+h(t)$ has left $\mathrm{B}^{\mathfrak{n}}_{\rho(r)}(0)$ after time $t = P(r) := \rho(r)\sqrt{\frac{2\pi\dim\mathfrak{h}}{\pi-2}}$. □

3.2. X-ray transform on certain 2-step nilpotent Lie groups

Let $\mathfrak{h} = \mathbb{R}^{2q} = \mathbb{C}^q$ and $\mathfrak{z} = \mathfrak{t}_{q-1} \subset \mathfrak{su}(q) \subset \mathfrak{so}(2q)$ be the Lie algebra of the maximal torus of $SU(q)$ and consider the 2-step nilpotent Lie group N_q with Lie algebra $\mathfrak{n}_q = \mathfrak{z} \oplus \mathfrak{h} = \mathfrak{t}_q \oplus \mathbb{R}^{2q}$ endowed with a left invariant metric. In [22], it was shown that for every $q \in \mathbb{N}$, $q \geqslant 3$, the Lie group N_q has the property that each geodesic is contained in a totally geodesic immersed 2-dimensional flat submanifold. The reduction principle, Theorem 3.1, and the continuity of P immediately yield

THEOREM 3.2. *The X-ray transform on N_q is injective and has a support theorem.*

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