

H. Tadano*

REMARK ON HARNACK INEQUALITIES FOR THE POROUS MEDIUM EQUATION ON RIEMANNIAN MANIFOLDS

Dedicated to the memory of Professor Sergio Console

Abstract. In this paper, we shall derive some new local gradient estimates of Li–Yau and Hamilton types for positive solutions $u = u(x, t)$ to the porous medium equation

$$\frac{\partial u}{\partial t} = \Delta u^m, \quad m > 1$$

on a complete Riemannian manifold (X, g) with Ricci curvature bounded from below. As applications, we shall give some parabolic Harnack inequalities for positive solutions to the porous medium equation on a complete non-compact Riemannian manifold with Ricci curvature bounded from below. In particular, our results improve the recent work by Huang, Huang, and Li (J. Geom. Anal. 23, 1851–1875, 2013).

1. Introduction

Differential Harnack inequalities for parabolic equations are powerful tools in geometric analysis and are originated with the celebrated work [6] by Li and Yau, where they studied positive solutions to the *heat equation*

$$(1) \quad \frac{\partial u}{\partial t} = \Delta u$$

on a complete Riemannian manifold (X, g) with Ricci curvature bounded from below and derived the following local gradient estimate for such positive solutions:

THEOREM 1.1 (Li and Yau [6]). *Let (X, g) be an n -dimensional complete Riemannian manifold with $\text{Ric}_g(B_p(2R)) \geq -K$ for $K \geq 0$. Suppose that $u = u(x, t)$ is a positive solution to (1) on $B_p(2R) \times [0, T]$. Then, on $B_p(R) \times (0, T]$,*

$$(2) \quad \frac{|\nabla u|^2}{u^2} - \alpha \frac{u_t}{u} \leq \frac{C(n)\alpha^2}{R^2} \left(\frac{\alpha^2}{\alpha - 1} + 1 + \sqrt{KR} \right) + \frac{n\alpha^2 K}{\sqrt{2}(\alpha - 1)} + \frac{n\alpha^2}{2t},$$

where $\alpha > 1$ is a constant and $C(n)$ is a constant depending only on n .

By taking $R \rightarrow +\infty$ in (2), we have the following global gradient estimate for positive solutions to the heat equation on a complete non-compact Riemannian manifold:

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COROLLARY 1.1 (Li and Yau [6]). *Let (X, g) be an n -dimensional complete non-compact Riemannian manifold with $\text{Ric}_g \geq -K$ for $K \geq 0$. Suppose that $u = u(x, t)$ is a positive solution to (1) on $X \times [0, T]$. Then, on $X \times (0, T]$,*

$$\frac{|\nabla u|^2}{u^2} - \alpha \frac{u_t}{u} \leq \frac{n\alpha^2 K}{\sqrt{2}(\alpha - 1)} + \frac{n\alpha^2}{2t},$$

where $\alpha > 1$ is a constant.

On the other hand, Hamilton [3] proved the following global gradient estimate for positive solutions to the heat equation which may be compared with Corollary 1.1:

THEOREM 1.2 (Hamilton [3]). *Let (X, g) be an n -dimensional complete non-compact Riemannian manifold with $\text{Ric}_g \geq -K$ for $K \geq 0$. Suppose that $u = u(x, t)$ is a positive solution to (1) on $X \times [0, T]$. Then, on $X \times (0, T]$,*

$$\frac{|\nabla u|^2}{u^2} - e^{2Kt} \frac{u_t}{u} \leq e^{4Kt} \frac{n}{2t}.$$

To give sharp gradient estimates for positive solutions to the heat equation (1) on a Riemannian manifold is an important topic in geometric analysis. We refer the reader to the work [5] by Li and Xu for recent improvement on this topic.

Recently, some non-linear extensions of the heat equation (1) have been studied. A porous medium equation is defined as

$$(3) \quad \frac{\partial u}{\partial t} = \Delta u^m, \quad m > 1.$$

The porous medium equation (3) is of great interest due to their importance in mathematics and physics [1, 8]. For various values of $m > 1$, it arises in different applications to model diffusive phenomena such as groundwater infiltration ($m = 2$), flow of gas in porous media ($m \geq 2$), heat radiation in plasmas ($m > 4$), and liquid thin films moving under gravity ($m = 4$). We refer the reader to the book [8] for basic facts on the porous medium equation.

Aronson and Bénylan [1] obtained a second order differential inequality for positive solutions to the porous medium equation on the Euclidean space. On the other hand, Lu, Ni, Vázquez, and Villani [7] first established the following local gradient estimate of Li–Yau type for the porous medium equation on a complete Riemannian manifold:

THEOREM 1.3 (Lu, Ni, Vázquez, and Villani [7]). *Let (X, g) be an n -dimensional complete Riemannian manifold with $\text{Ric}_g(B_p(2R)) \geq -K$ for $K \geq 0$. Suppose that $u = u(x, t)$ is a positive solution to (3) on $B_p(2R) \times [0, T]$. Put*

$$v = \frac{m}{m-1} u^{m-1}, \quad L = (m-1) \max_{B_p(2R) \times [0, T]} v, \quad \text{and} \quad a = \frac{n(m-1)}{n(m-1)+2}.$$

Then, on $B_p(R) \times (0, T]$,

$$(4) \quad \frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} \leq \frac{C(n)L\alpha^2}{R^2} \left\{ \frac{\alpha\alpha^2 m^2}{(\alpha-1)(m-1)} + 1 + \sqrt{KR} \right\} + \frac{\alpha\alpha^2 KL}{\alpha-1} + \frac{\alpha\alpha^2}{t},$$

where $\alpha > 1$ is a constant and $C(n)$ is a constant depending only on n .

By taking $R \rightarrow +\infty$ in (4), we have the following global gradient estimate for the porous medium equation on a complete non-compact Riemannian manifold:

COROLLARY 1.2 (Lu, Ni, Vázquez, and Villani [7]). *Let (X, g) be an n -dimensional complete non-compact Riemannian manifold with $\text{Ric}_g \geq -K$ for $K \geq 0$. Suppose that $u = u(x, t)$ is a positive solution to (3) on $X \times [0, T]$. Put*

$$v = \frac{m}{m-1} u^{m-1}, \quad L = (m-1) \max_{X \times [0, T]} v, \quad \text{and} \quad a = \frac{n(m-1)}{n(m-1)+2}.$$

Then, on $X \times (0, T]$,

$$\frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} \leq \frac{\alpha\alpha^2 KL}{\alpha-1} + \frac{\alpha\alpha^2}{t},$$

where $\alpha > 1$ is a constant.

Recently, Huang, Huang, and Li [4] proved the following local gradient estimate of Li–Yau type for the porous medium equation:

THEOREM 1.4 (Huang, Huang, and Li [4]). *Let (X, g) be an n -dimensional complete Riemannian manifold with $\text{Ric}_g(B_p(2R)) \geq -K$ for $K \geq 0$. Suppose that $u = u(x, t)$ is a positive solution to (3) on $B_p(2R) \times [0, T]$. Put*

$$v = \frac{m}{m-1} u^{m-1}, \quad L = (m-1) \max_{B_p(2R) \times [0, T]} v, \quad \text{and} \quad a = \frac{n(m-1)}{n(m-1)+2}.$$

Then, on $B_p(R) \times (0, T]$,

$$(5) \quad \frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} \leq \alpha\alpha^2 \left\{ \left(\frac{C(n)L}{R^2} \cdot \frac{\alpha\alpha^2 m^2}{(\alpha-1)(m-1)} \right)^{\frac{1}{2}} + \left[\frac{KL}{2(\alpha-1)} + \frac{1}{t} + \frac{C(n)L}{R^2} \left(1 + \sqrt{KR} \coth \left(\sqrt{\frac{K}{n-1}} R \right) \right) \right]^{\frac{1}{2}} \right\}^2,$$

where $\alpha > 1$ is a constant and $C(n)$ is a constant depending only on n .

By taking $R \rightarrow +\infty$ in (5), we have the following global gradient estimate which improves Corollary 1.2:

COROLLARY 1.3 (Huang, Huang, and Li [4]). *Let (X, g) be an n -dimensional complete non-compact Riemannian manifold with $\text{Ric}_g \geq -K$ for $K \geq 0$. Suppose that $u = u(x, t)$ is a positive solution to (3) on $X \times [0, T]$. Put*

$$v = \frac{m}{m-1}u^{m-1}, \quad L = (m-1) \max_{X \times [0, T]} v, \quad \text{and} \quad a = \frac{n(m-1)}{n(m-1)+2}.$$

Then, on $X \times (0, T]$,

$$(6) \quad \frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} \leq \frac{\alpha^2 KL}{2(\alpha-1)} + \frac{\alpha^2}{t},$$

where $\alpha > 1$ is a constant.

Moreover, Huang, Huang, and Li [4] established the following local gradient estimate of Hamilton type for the porous medium equation:

THEOREM 1.5 (Huang, Huang, and Li [4]). *Let (X, g) be an n -dimensional complete Riemannian manifold with $\text{Ric}_g(B_p(2R)) \geq -K$ for $K \geq 0$. Suppose that $u = u(x, t)$ is a positive solution to (3) on $B_p(2R) \times [0, T]$. Put*

$$v = \frac{m}{m-1}u^{m-1}, \quad L = (m-1) \max_{B_p(2R) \times [0, T]} v, \quad \text{and} \quad a = \frac{n(m-1)}{n(m-1)+2}.$$

Then, on $B_p(R) \times (0, T]$,

$$(7) \quad \frac{|\nabla v|^2}{v} - \alpha(t) \frac{v_t}{v} \leq \frac{C(n)L\alpha^2(t)}{R^2} \left\{ \frac{\alpha^2(t)m^2}{2(\alpha(t)-1)(m-1)} + 1 \right. \\ \left. + \sqrt{KR} \coth \left(\sqrt{\frac{K}{n-1}} R \right) \right\} + \frac{\alpha^2(t)}{t},$$

where $\alpha(t) = e^{2KLt}$ and $C(n)$ is a constant depending only on n .

By taking $R \rightarrow +\infty$ in (7), we have the following global gradient estimate which may be compared with Corollary 1.3:

COROLLARY 1.4 (Huang, Huang, and Li [4]). *Let (X, g) be an n -dimensional complete non-compact Riemannian manifold with $\text{Ric}_g \geq -K$ for $K \geq 0$. Suppose that $u = u(x, t)$ is a positive solution to (3) on $X \times [0, T]$. Put*

$$v = \frac{m}{m-1}u^{m-1}, \quad L = (m-1) \max_{X \times [0, T]} v, \quad \text{and} \quad a = \frac{n(m-1)}{n(m-1)+2}.$$

Then, on $X \times (0, T]$,

$$(8) \quad \frac{|\nabla v|^2}{v} - \alpha(t) \frac{v_t}{v} \leq \frac{a\alpha^2(t)}{t},$$

where $\alpha(t) = e^{2KLt}$.

In this paper, we shall further study gradient estimates for positive solutions to the porous medium equation (3) on a complete Riemannian manifold with Ricci curvature bounded from below. Our first result in this paper is the following:

THEOREM 1.6. *Let (X, g) be an n -dimensional complete Riemannian manifold with $\text{Ric}_g(B_p(2R)) \geq -K$ for $K \geq 0$. Suppose that $u = u(x, t)$ is a positive solution to (3) on $B_p(2R) \times [0, T]$. Put*

$$v = \frac{m}{m-1}u^{m-1}, \quad L = (m-1) \max_{B_p(2R) \times [0, T]} v, \quad \text{and} \quad a = \frac{n(m-1)}{n(m-1)+2}.$$

Then, on $B_p(R) \times (0, T]$,

$$(9) \quad \frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} \leq a\alpha^2 \left\{ \left(\frac{C(n)L}{R^2} \cdot \frac{a\alpha^2 m^2}{(\alpha-1)(m-1)} \right)^{\frac{1}{2}} + \left[\frac{KL}{2(\alpha-1)} + \frac{1}{1+a(\alpha-1)} \cdot \frac{1}{t} + \frac{C(n)L}{R^2} \left(1 + \sqrt{KR} \coth \left(\sqrt{\frac{K}{n-1}} R \right) \right) \right]^{\frac{1}{2}} \right\}^2,$$

where $\alpha > 1$ is a constant and $C(n)$ is a constant depending only on n .

By taking $R \rightarrow +\infty$ in (9), we have the following global gradient estimate which improves Corollary 1.3:

COROLLARY 1.5. *Let (X, g) be an n -dimensional complete non-compact Riemannian manifold with $\text{Ric}_g \geq -K$ for $K \geq 0$. Suppose that $u = u(x, t)$ is a positive solution to (3) on $X \times [0, T]$. Put*

$$v = \frac{m}{m-1}u^{m-1}, \quad L = (m-1) \max_{X \times [0, T]} v, \quad \text{and} \quad a = \frac{n(m-1)}{n(m-1)+2}.$$

Then, on $X \times (0, T]$,

$$(10) \quad \frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} \leq \frac{a\alpha^2 KL}{2(\alpha-1)} + \frac{1}{1+a(\alpha-1)} \cdot \frac{a\alpha^2}{t},$$

where $\alpha > 1$ is a constant.

By integrating the global gradient estimate (10) along a space-time path, we may obtain the following Harnack inequality for the porous medium equation:

COROLLARY 1.6. *Let (X, g) be an n -dimensional complete non-compact Riemannian manifold with $\text{Ric}_g \geq -K$ for $K \geq 0$. Suppose that $u = u(x, t)$ is a positive solution to (3) on $X \times [0, T]$. Put*

$$v = \frac{m}{m-1}u^{m-1}, \quad L = (m-1) \sup_{X \times [0, T]} v, \quad M := \inf_{X \times [0, T]} v, \quad \text{and} \quad a = \frac{n(m-1)}{n(m-1)+2}.$$

Then, for any $x_1, x_2 \in X$ and $0 < t_1 < t_2 < T$,

$$(11) \quad v(x_1, t_1) \leq v(x_2, t_2) \left(\frac{t_2}{t_1}\right)^{\frac{\alpha\alpha}{1+\alpha(\alpha-1)}} \exp\left\{\frac{\alpha \operatorname{dist}^2(x_1, x_2)}{4M(t_2 - t_1)} + \frac{a\alpha KL}{2(\alpha - 1)}(t_2 - t_1)\right\},$$

where $\alpha > 1$ is a constant and $\operatorname{dist}(x_1, x_2)$ is the distance between x_1 and x_2 .

Moreover, we shall derive the following local gradient estimate of Hamilton type for the porous medium equation:

THEOREM 1.7. *Let (X, g) be an n -dimensional complete Riemannian manifold with $\operatorname{Ric}_g(B_p(2R)) \geq -K$ for $K \geq 0$. Suppose that $u = u(x, t)$ is a positive solution to (3) on $B_p(2R) \times [0, T]$. Put*

$$v = \frac{m}{m-1} u^{m-1}, \quad L = (m-1) \max_{B_p(2R) \times [0, T]} v, \quad \text{and} \quad a = \frac{n(m-1)}{n(m-1) + 2}.$$

Then, on $B_p(R) \times (0, T]$,

$$(12) \quad \frac{|\nabla v|^2}{v} - \alpha(t) \frac{v_t}{v} \leq a\alpha^2(t) \left\{ \left(\frac{C(n)L}{R^2} \cdot \frac{a\alpha^2(t)m^2}{(\alpha(t)-1)(m-1)} \right)^{\frac{1}{2}} + \left[\frac{1}{1+a(\alpha(t)-1)} \cdot \frac{1}{t} + \frac{C(n)L}{R^2} \left(1 + \sqrt{KR} \coth \left(\sqrt{\frac{K}{n-1}} R \right) \right) \right]^{\frac{1}{2}} \right\}^2,$$

where $\alpha(t) = e^{2KLt}$.

By taking $R \rightarrow +\infty$ in (12), we have the following global gradient estimate which improves Corollary 1.4:

COROLLARY 1.7. *Let (X, g) be an n -dimensional complete non-compact Riemannian manifold with $\operatorname{Ric}_g \geq -K$ for $K \geq 0$. Suppose that $u = u(x, t)$ is a positive solution to (3) on $X \times [0, T]$. Put*

$$v = \frac{m}{m-1} u^{m-1}, \quad L = (m-1) \max_{X \times [0, T]} v, \quad \text{and} \quad a = \frac{n(m-1)}{n(m-1) + 2}.$$

Then, on $X \times (0, T]$,

$$(13) \quad \frac{|\nabla v|^2}{v} - \alpha(t) \frac{v_t}{v} \leq \frac{1}{1+a(\alpha(t)-1)} \cdot \frac{a\alpha^2(t)}{t},$$

where $\alpha(t) = e^{2KLt}$.

By integrating the global gradient estimate (13) along a space-time path, we may obtain the following Harnack inequality for the porous medium equation:

COROLLARY 1.8. *Let (X, g) be an n -dimensional complete non-compact Riemannian manifold with $\text{Ric}_g \geq -K$ for $K \geq 0$. Suppose that $u = u(x, t)$ is a positive solution to (3) on $X \times [0, T]$. Put*

$$v = \frac{m}{m-1} u^{m-1}, \quad L = (m-1) \sup_{X \times [0, T]} v, \quad M := \inf_{X \times [0, T]} v, \quad \text{and} \quad a = \frac{n(m-1)}{n(m-1)+2}.$$

Then, for any $x_1, x_2 \in X$ and $0 < t_1 < t_2 < T$,

$$(14) \quad v(x_1, t_1) \leq v(x_2, t_2) \exp \left\{ \frac{\alpha(t_2) - \alpha(t_1)}{2KL} \left(\frac{\text{dist}^2(x_1, x_2)}{4M(t_2 - t_1)^2} + \frac{1}{1 + a(\alpha(t_1) - 1)} \cdot \frac{a}{t_1} \right) \right\},$$

where $\alpha(t) = e^{2KLt}$ and $\text{dist}(x_1, x_2)$ is the distance between x_1 and x_2 .

This paper is organized as follows: In Section 2, we shall give a lemma which shall play important roles in proving Theorem 1.6 and 1.7. Ending with Section 3, we shall prove Theorem 1.6, 1.7 and Corollary 1.6, 1.8.

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2. Prerequisites

Throughout this paper, (X, g) is assumed to be an n -dimensional complete Riemannian manifold with Ricci curvature bounded from below. We shall denote by r, ∇ , and Δ the distance function, the Riemannian connection, and the Laplace–Beltrami operator on (X, g) , respectively. We shall also denote by $B_p(R)$ the geodesic ball of radius $R > 0$ centered at $p \in X$. Suppose that $u = u(x, t)$ is a positive solution to the porous medium equation (3) on $B_p(2R) \times [0, T]$. Let $v = \frac{m}{m-1} u^{m-1}$. Note that v is positive. By simple calculations, (3) may be written as

$$(15) \quad \frac{v_t}{v} = (m-1)\Delta v + \frac{|\nabla v|^2}{v}.$$

Following [7], we introduce the linear differential operator

$$\mathcal{L} := \frac{\partial}{\partial t} - (m-1)v\Delta.$$

We put

$$F := \frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v}$$

and consider $F = F(x, t)$ on $B_p(2R) \times [0, T]$, where $\alpha = \alpha(t)$ is a real-valued function depending only on t .

The following lemma plays important roles in proving Theorem 1.6 and 1.7:

LEMMA 2.1 (Huang, Huang, and Li [4]). *Under the notation above, we have*

$$(16) \quad \begin{aligned} \mathcal{L}(F) = & -2(m-1)v_{ij}^2 - 2(m-1)R^{ij}v_i v_j \\ & + 2m \langle \nabla v, \nabla F \rangle - ((m-1)\Delta v)^2 - (\alpha-1) \left(\frac{v_t}{v}\right)^2 - \alpha' \frac{v_t}{v}. \end{aligned}$$

Proof. To make this paper self-contained, we shall recall the proof of Lemma 2.1. First, by direct calculations, for any two smooth functions φ, ψ on (X, g) , we have

$$(17) \quad \mathcal{L}\left(\frac{\varphi}{\psi}\right) = \frac{1}{\psi} \mathcal{L}(\varphi) - \frac{\varphi}{\psi^2} \mathcal{L}(\psi) + 2(m-1)v \left\langle \nabla\left(\frac{\varphi}{\psi}\right), \nabla \log \psi \right\rangle.$$

By using (17), we obtain

$$(18) \quad \begin{aligned} \mathcal{L}\left(\frac{|\nabla v|^2}{v}\right) &= 2(m-1)\Delta v \frac{|\nabla v|^2}{v} + \frac{2}{v} \langle \nabla v, \nabla |\nabla v|^2 \rangle - 2(m-1)v_{ij}^2 \\ &\quad - 2(m-1)R^{ij}v_i v_j - \frac{|\nabla v|^4}{v^2} + 2(m-1)v \left\langle \nabla\left(\frac{|\nabla v|^2}{v}\right), \nabla \log v \right\rangle, \\ \mathcal{L}\left(\frac{v_t}{v}\right) &= (m-1)\Delta v \frac{v_t}{v} + \frac{2}{v} \langle \nabla v, \nabla v_t \rangle \\ &\quad - \frac{v_t}{v} \cdot \frac{|\nabla v|^2}{v} + 2(m-1)v \left\langle \nabla\left(\frac{v_t}{v}\right), \nabla \log v \right\rangle. \end{aligned}$$

It follows from (18) that

$$(19) \quad \begin{aligned} \mathcal{L}(F) &= \mathcal{L}\left(\frac{|\nabla v|^2}{v}\right) - \alpha \mathcal{L}\left(\frac{v_t}{v}\right) - \alpha' \frac{v_t}{v} \\ &= 2(m-1)\Delta v \frac{|\nabla v|^2}{v} + \frac{2}{v} \langle \nabla v, \nabla |\nabla v|^2 \rangle - 2(m-1)v_{ij}^2 \\ &\quad - 2(m-1)R^{ij}v_i v_j - \frac{|\nabla v|^4}{v^2} + 2(m-1)v \left\langle \nabla\left(\frac{|\nabla v|^2}{v}\right), \nabla \log v \right\rangle \\ &\quad - \alpha(m-1)\Delta v \frac{v_t}{v} - \alpha \frac{2}{v} \langle \nabla v, \nabla v_t \rangle \\ &\quad + \alpha \frac{v_t}{v} \cdot \frac{|\nabla v|^2}{v} - 2\alpha(m-1)v \left\langle \nabla\left(\frac{v_t}{v}\right), \nabla \log v \right\rangle - \alpha' \frac{v_t}{v}. \end{aligned}$$

By using (15), we have

$$(20) \quad \begin{aligned} &2(m-1)\Delta v \frac{|\nabla v|^2}{v} - \frac{|\nabla v|^4}{v^2} - \alpha(m-1)\Delta v \frac{v_t}{v} + \alpha \frac{v_t}{v} \cdot \frac{|\nabla v|^2}{v} \\ &= 2 \frac{|\nabla v|^2}{v} \left(\frac{v_t}{v} - \frac{|\nabla v|^2}{v}\right) - \frac{|\nabla v|^4}{v^2} - \alpha \frac{v_t}{v} \left(\frac{v_t}{v} - \frac{|\nabla v|^2}{v}\right) + \alpha \frac{v_t}{v} \cdot \frac{|\nabla v|^2}{v} \\ &= (2\alpha+2) \frac{v_t}{v} \cdot \frac{|\nabla v|^2}{v} - 3 \frac{|\nabla v|^4}{v^2} - \alpha \left(\frac{v_t}{v}\right)^2. \end{aligned}$$

On the other hand, it follows that

$$\begin{aligned}
 & \frac{2}{v} \langle \nabla v, \nabla |\nabla v|^2 \rangle - \alpha \frac{2}{v} \langle \nabla v, \nabla v_t \rangle \\
 (21) \quad &= \frac{2}{v} \langle \nabla v, \nabla (Fv) \rangle = 2F \frac{|\nabla v|^2}{v} + 2 \langle \nabla v, \nabla F \rangle, \\
 & 2(m-1)v \left\langle \nabla \left(\frac{|\nabla v|^2}{v} \right), \nabla \log v \right\rangle - 2\alpha(m-1)v \left\langle \nabla \left(\frac{v_t}{v} \right), \nabla \log v \right\rangle \\
 &= 2(m-1) \langle \nabla v, \nabla F \rangle.
 \end{aligned}$$

Hence, by combining (19), (20), and (21), we obtain

$$\begin{aligned}
 \mathcal{L}(F) &= (2\alpha + 2) \frac{v_t}{v} \cdot \frac{|\nabla v|^2}{v} - 3 \frac{|\nabla v|^4}{v^2} - \alpha \left(\frac{v_t}{v} \right)^2 \\
 &\quad + 2F \frac{|\nabla v|^2}{v} + 2 \langle \nabla v, \nabla F \rangle + 2(m-1) \langle \nabla v, \nabla F \rangle \\
 &\quad - 2(m-1)v_{ij}^2 - 2(m-1)R^{ij}v_iv_j - \alpha' \frac{v_t}{v},
 \end{aligned}$$

from where we obtain (16). The proof of Lemma 2.1 is completed. \square

3. Proofs of Theorem 1.6, 1.7 and Corollary 1.6, 1.8

Now, we are in a position to prove Theorem 1.6.

Proof of Theorem 1.6. Our proof of Theorem 1.6 is given by a slight modification of the proof of Theorem 1.4 by Huang, Huang, and Li [4]. Define

$$F := \frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v},$$

where $\alpha > 1$ is a constant. It follows from (16) and $\text{Ric}_g(B_p(2R)) \geq -K$ that

$$\begin{aligned}
 (22) \quad \mathcal{L}(F) &\leq -2(m-1)v_{ij}^2 + 2(m-1)K|\nabla v|^2 \\
 &\quad + 2m \langle \nabla v, \nabla F \rangle - ((m-1)\Delta v)^2 - (\alpha-1) \left(\frac{v_t}{v} \right)^2.
 \end{aligned}$$

By applying the Cauchy–Schwarz inequality

$$v_{ij} \geq \frac{1}{n}(\Delta v)^2$$

to the right hand side of (22), we have

$$\begin{aligned}
 \mathcal{L}(F) &\leq -\frac{2}{n(m-1)}((m-1)\Delta v)^2 + 2(m-1)K|\nabla v|^2 \\
 &\quad + 2m\langle \nabla v, \nabla F \rangle - ((m-1)\Delta v)^2 - (\alpha-1)\left(\frac{v_t}{v}\right)^2 \\
 &= -\frac{1}{a}((m-1)\Delta v)^2 + 2(m-1)K|\nabla v|^2 \\
 (23) \quad &\quad + 2m\langle \nabla v, \nabla F \rangle - (\alpha-1)\left(\frac{v_t}{v}\right)^2 \\
 &\leq -\frac{1}{\alpha^2}\left(F + (\alpha-1)\frac{|\nabla v|^2}{v}\right)^2 + 2KL\frac{|\nabla v|^2}{v} \\
 &\quad + 2m\langle \nabla v, \nabla F \rangle - \frac{(\alpha-1)}{\alpha^2}\left(F - \frac{|\nabla v|^2}{v}\right)^2,
 \end{aligned}$$

where the last inequality follows from

$$(m-1)\Delta v = \frac{v_t}{v} - \frac{|\nabla v|^2}{v} = -\frac{1}{\alpha}\left(F + (\alpha-1)\frac{|\nabla v|^2}{v}\right).$$

Take a cut-off function $\phi = \phi(x)$ of the form

$$\phi(x) := \xi\left(\frac{r(x)}{R}\right),$$

where $\xi = \xi(t)$ is a smooth monotone function satisfying

$$(24) \quad \xi(t) \equiv 1 \text{ for } t \leq 1, \quad \xi(t) \equiv 0 \text{ for } t \geq 2, \quad \frac{(\xi')^2}{\xi} \leq C, \quad \text{and } \xi'' \geq -C$$

for some positive constant $C > 0$. Here, $r = r(x)$ is the distance between x and p . Then, it follows from (24) that

$$(25) \quad \frac{|\nabla \phi|^2}{\phi} = \frac{(\xi')^2}{\xi} \cdot \frac{|\nabla \rho|^2}{R^2} \leq \frac{C}{R^2}.$$

Since $\coth t$ is decreasing and $\xi'(t) = 0$ for $t \leq 1$, we have

$$(26) \quad \Delta \phi = \frac{\xi''|\nabla r|^2}{R^2} + \frac{\xi'\Delta r}{R} \geq -\frac{C(n)}{R^2} \left\{ 1 + \sqrt{KR} \coth\left(\sqrt{\frac{K}{n-1}}R\right) \right\},$$

where $C(n)$ is a constant depending only on n . Here, we have used the Laplacian comparison theorem which asserts that

$$\Delta r(x) \leq \begin{cases} \frac{n-1}{r(x)} & \text{if } \text{Ric}_g \geq 0, \\ \sqrt{(n-1)K} \coth\left(\sqrt{\frac{K}{n-1}}r(x)\right) & \text{if } \text{Ric}_g \geq -K \text{ for } K > 0. \end{cases}$$

For the Laplacian comparison theorem, see the book [2] for example.

Define $G := t\phi F$. We shall apply the maximum principle to G on $B_p(2R) \times [0, T]$. Assume that G attains its maximum at some point $(x_0, s) \in B_p(2R) \times [0, T]$. Without loss of generality, we may assume that $G(x_0, s) > 0$, which implies $s > 0$. Then at the point (x_0, s) , we have

$$(27) \quad \mathcal{L}(G) \geq 0 \quad \text{and} \quad \nabla F = -\frac{F}{\phi} \nabla \phi.$$

It follows from the last inequality in (23) and (27) that

$$(28) \quad \begin{aligned} 0 \leq \mathcal{L}(G) &= s\phi \mathcal{L}(F) - (m-1)vsF\Delta\phi - 2(m-1)vs\langle \nabla F, \nabla \phi \rangle + \phi F \\ &= s\phi \mathcal{L}(F) - (m-1)v\frac{\Delta\phi}{\phi}G + 2(m-1)v\frac{|\nabla\phi|^2}{\phi^2}G + \frac{G}{s} \\ &\leq s\phi \left[-\frac{1}{\alpha\alpha^2} \left(F + (\alpha-1)\frac{|\nabla v|^2}{v} \right)^2 + 2KL\frac{|\nabla v|^2}{v} \right. \\ &\quad \left. - 2m\langle \nabla v, \nabla \phi \rangle \frac{F}{\phi} - \frac{(\alpha-1)}{\alpha^2} \left(F - \frac{|\nabla v|^2}{v} \right)^2 \right] \\ &\quad - (m-1)v\frac{\Delta\phi}{\phi}G + 2(m-1)v\frac{|\nabla\phi|^2}{\phi^2}G + \frac{G}{s}. \end{aligned}$$

Let $\frac{|\nabla v|^2}{v} = \mu F$ at the point (x_0, s) . Since $G(x_0, s) > 0$, we have $\mu \geq 0$. It follows from the last inequality in (28) that

$$(29) \quad \begin{aligned} 0 &\leq -\frac{1}{\alpha\alpha^2}s\phi(1+(\alpha-1)\mu)^2F^2 + 2KL\mu s\phi F \\ &\quad - 2m\langle \nabla v, \nabla \phi \rangle sF - \frac{(\alpha-1)}{\alpha^2}s\phi(1-\mu)^2F^2 \\ &\quad - (m-1)v\frac{\Delta\phi}{\phi}G + 2(m-1)v\frac{|\nabla\phi|^2}{\phi^2}G + \frac{G}{s} \\ &\leq -\frac{1}{\alpha\alpha^2s\phi}(1+(\alpha-1)\mu)^2G^2 + 2KL\mu G \\ &\quad + 2m\frac{|\nabla\phi|}{s^{\frac{1}{2}}\phi^{\frac{3}{2}}} \cdot \frac{L^{\frac{1}{2}}\mu^{\frac{1}{2}}}{(m-1)^{\frac{1}{2}}}G^{\frac{3}{2}} - \frac{(\alpha-1)}{\alpha^2s\phi}(1-\mu)^2G^2 \\ &\quad - (m-1)v\frac{\Delta\phi}{\phi}G + 2(m-1)v\frac{|\nabla\phi|^2}{\phi^2}G + \frac{G}{s}. \end{aligned}$$

Multiplying both sides of the last inequality in (29) by $\frac{\phi}{G}$, we have

$$\begin{aligned}
 & \frac{1}{\alpha\alpha^2s} \left\{ (1 + (\alpha - 1)\mu)^2 + a(\alpha - 1)(1 - \mu)^2 \right\} G - 2m \frac{|\nabla\phi|}{s^{\frac{1}{2}}\phi^{\frac{1}{2}}} \cdot \frac{L^{\frac{1}{2}}\mu^{\frac{1}{2}}}{(m - 1)^{\frac{1}{2}}} G^{\frac{1}{2}} \\
 (30) \quad & \leq 2KL\mu\phi - (m - 1)v\Delta\phi + 2(m - 1)v \frac{|\nabla\phi|^2}{\phi} + \frac{\phi}{s} \\
 & \leq 2KL\mu - L\Delta\phi + 2L \frac{|\nabla\phi|^2}{\phi} + \frac{1}{s}.
 \end{aligned}$$

From the quadratic inequality $Ax^2 - 2Bx \leq C$, we have $x \leq \frac{2B}{A} + \left(\frac{C}{A}\right)^{\frac{1}{2}}$. By applying this inequality to the last inequality in (30) by setting $x = G^{\frac{1}{2}}$, we have

$$\begin{aligned}
 (31) \quad G^{\frac{1}{2}} & \leq \frac{2\alpha\alpha^2ms^{\frac{1}{2}}L^{\frac{1}{2}}\mu^{\frac{1}{2}}}{(1 + (\alpha - 1)\mu)^2 + a(\alpha - 1)(1 - \mu)^2} \cdot \frac{1}{(m - 1)^{\frac{1}{2}}} \cdot \frac{|\nabla\phi|}{\phi^{\frac{1}{2}}} \\
 & + \left\{ \frac{\alpha\alpha^2s}{(1 + (\alpha - 1)\mu)^2 + a(\alpha - 1)(1 - \mu)^2} \left(2KL\mu - L\Delta\phi + 2L \frac{|\nabla\phi|^2}{\phi} + \frac{1}{s} \right) \right\}^{\frac{1}{2}}.
 \end{aligned}$$

Note that

$$\begin{aligned}
 & \frac{2\alpha\alpha^2ms^{\frac{1}{2}}L^{\frac{1}{2}}\mu^{\frac{1}{2}}}{(1 + (\alpha - 1)\mu)^2 + a(\alpha - 1)(1 - \mu)^2} \leq \frac{2\alpha\alpha^2ms^{\frac{1}{2}}L^{\frac{1}{2}}\mu^{\frac{1}{2}}}{(1 + (\alpha - 1)\mu)^2} \\
 & = \frac{\alpha\alpha^2ms^{\frac{1}{2}}L^{\frac{1}{2}}}{(\alpha - 1)^{\frac{1}{2}}} \cdot \frac{2((\alpha - 1)\mu)^{\frac{1}{2}}}{(1 + (\alpha - 1)\mu)^2} \\
 (32) \quad & < \frac{\alpha\alpha^2ms^{\frac{1}{2}}L^{\frac{1}{2}}}{(\alpha - 1)^{\frac{1}{2}}},
 \end{aligned}$$

where the last inequality follows from $\frac{2\sqrt{x}}{(1+x)^2} < 1$. Moreover, we have

$$\begin{aligned}
 & \frac{\alpha\alpha^2s}{(1 + (\alpha - 1)\mu)^2 + a(\alpha - 1)(1 - \mu)^2} \cdot 2KL\mu \leq \frac{\alpha\alpha^2s}{(1 + (\alpha - 1)\mu)^2} \cdot 2KL\mu \\
 & = \frac{\alpha\alpha^2sKL}{\alpha - 1} \cdot \frac{2(\alpha - 1)\mu}{(1 + (\alpha - 1)\mu)^2} \\
 (33) \quad & \leq \frac{\alpha\alpha^2sKL}{2(\alpha - 1)}
 \end{aligned}$$

and

$$(34) \quad \frac{\alpha\alpha^2s}{(1 + (\alpha - 1)\mu)^2 + a(\alpha - 1)(1 - \mu)^2} \cdot \frac{1}{s} \leq \frac{\alpha\alpha^2}{1 + a(\alpha - 1)},$$

where the inequality (33) follows from $\frac{2x}{(1+x)^2} \leq \frac{1}{2}$. Hence, by applying (25), (26), (32), (33), and (34) to (31), we have

$$\begin{aligned} G^{\frac{1}{2}}(x, T) &\leq G^{\frac{1}{2}}(x_0, s) \\ &\leq \frac{a\alpha^2 m s^{\frac{1}{2}} L^{\frac{1}{2}}}{(\alpha-1)^{\frac{1}{2}}} \cdot \frac{1}{(m-1)^{\frac{1}{2}}} \cdot \frac{|\nabla\phi|}{\phi^{\frac{1}{2}}} \\ &\quad + \left\{ a\alpha^2 \left(\frac{sKL}{2(\alpha-1)} - sL\Delta\phi + 2sL \frac{|\nabla\phi|^2}{\phi} + \frac{1}{1+a(\alpha-1)} \right) \right\}^{\frac{1}{2}} \\ &\leq a^{\frac{1}{2}} \alpha T^{\frac{1}{2}} \left(\frac{C(n)L}{R^2} \cdot \frac{a\alpha^2 m^2}{(\alpha-1)(m-1)} \right)^{\frac{1}{2}} + a^{\frac{1}{2}} \alpha T^{\frac{1}{2}} \left\{ \frac{KL}{2(\alpha-1)} \right. \\ &\quad \left. + \frac{C(n)L}{R^2} \left(1 + \sqrt{KR} \coth \left(\sqrt{\frac{K}{n-1}} R \right) \right) + \frac{1}{1+a(\alpha-1)} \cdot \frac{1}{T} \right\}^{\frac{1}{2}}, \end{aligned}$$

from where it follows that for any $x \in B_\rho(R)$,

$$\begin{aligned} F^{\frac{1}{2}}(x, T) &\leq a^{\frac{1}{2}} \alpha \left(\frac{C(n)L}{R^2} \cdot \frac{a\alpha^2 m^2}{(\alpha-1)(m-1)} \right)^{\frac{1}{2}} + a^{\frac{1}{2}} \alpha \left\{ \frac{KL}{2(\alpha-1)} \right. \\ &\quad \left. + \frac{C(n)L}{R^2} \left(1 + \sqrt{KR} \coth \left(\sqrt{\frac{K}{n-1}} R \right) \right) + \frac{1}{1+a(\alpha-1)} \cdot \frac{1}{T} \right\}^{\frac{1}{2}}. \end{aligned}$$

Since T is arbitrary, by squaring both sides of the inequality just above, we obtain (9). The proof of Theorem 1.6 is completed. \square

REMARK 3.1. Huang, Huang, and Li [4] estimated $\mathcal{L}(F)$ from above by

$$\mathcal{L}(F) \leq -\frac{1}{a\alpha^2} \left(F + (\alpha-1) \frac{|\nabla v|^2}{v} \right)^2 + 2KL \frac{|\nabla v|^2}{v} + 2m \langle \nabla v, \nabla F \rangle$$

and derived the gradient estimate (6), while we estimated $\mathcal{L}(F)$ from above by

$$\begin{aligned} \mathcal{L}(F) &\leq -\frac{1}{a\alpha^2} \left(F + (\alpha-1) \frac{|\nabla v|^2}{v} \right)^2 + 2KL \frac{|\nabla v|^2}{v} \\ &\quad + 2m \langle \nabla v, \nabla F \rangle - \frac{(\alpha-1)}{\alpha^2} \left(F - \frac{|\nabla v|^2}{v} \right)^2 \end{aligned}$$

in the last inequality of (23) and derived the better gradient estimate (10).

Next, we shall prove Corollary 1.6.

Proof of Corollary 1.6. From a global gradient estimate

$$(35) \quad \frac{|\nabla v|^2}{v} - \alpha(t) \frac{v_t}{v} \leq \varphi(t),$$

we may establish a Harnack inequality as follows: First, (35) may be written as

$$-\frac{v_t}{v} \leq \frac{1}{\alpha(t)} \left(\varphi(t) - \frac{|\nabla v|^2}{v} \right).$$

By setting $f = \log v$, we have

$$(36) \quad -f_t \leq \frac{1}{\alpha(t)} (\varphi(t) - M|\nabla f|^2),$$

where we have used definition of M . For any points x_1 and x_2 in X , let $\gamma: [t_1, t_2] \rightarrow X$ be a shortest geodesic such that $\gamma(t_1) = x_1$ and $\gamma(t_2) = x_2$. We define a curve $\eta: [t_1, t_2] \rightarrow X \times (0, \infty)$ by $\eta(t) = (\gamma(t), t)$. Then we obtain $\eta(t_1) = (x_1, t_1), \eta(t_2) = (x_2, t_2)$ and $|\dot{\eta}| = \frac{\text{dist}(x_1, x_2)}{t_2 - t_1}$. Then it follows from (36) that

$$(37) \quad \begin{aligned} f(x_1, t_1) - f(x_2, t_2) &= \int_{t_2}^{t_1} \frac{d}{dt} f(\eta(t)) dt \\ &= \int_{t_2}^{t_1} (\langle \dot{\eta}, \nabla f \rangle + f_t) dt \\ &= \int_{t_1}^{t_2} (-\langle \dot{\eta}, \nabla f \rangle - f_t) dt \\ &\leq \int_{t_1}^{t_2} \left(|\dot{\eta}| \cdot |\nabla f| + \frac{1}{\alpha(t)} (\varphi(t) - M|\nabla f|^2) \right) dt \\ &= \int_{t_1}^{t_2} \left(-\frac{M}{\alpha(t)} |\nabla f|^2 + |\dot{\eta}| \cdot |\nabla f| \right) dt + \int_{t_1}^{t_2} \frac{\varphi(t)}{\alpha(t)} dt \\ &\leq \frac{\text{dist}^2(x_1, x_2)}{4M(t_2 - t_1)^2} \int_{t_1}^{t_2} \alpha(t) dt + \int_{t_1}^{t_2} \frac{\varphi(t)}{\alpha(t)} dt, \end{aligned}$$

where the last inequality follows from $-Ax^2 + Bx \leq \frac{B^2}{4A}$.

Now, we are in a position to prove (11). Let $\alpha > 1$ be a constant and set

$$\varphi(t) = \frac{a\alpha^2 KL}{2(\alpha - 1)} + \frac{1}{1 + a(\alpha - 1)} \cdot \frac{a\alpha^2}{t}.$$

By applying (37), we have

$$\begin{aligned} f(x_1, t_1) - f(x_2, t_2) &\leq \frac{\text{dist}^2(x_1, x_2)}{4M(t_2 - t_1)^2} \int_{t_1}^{t_2} \alpha dt + \int_{t_1}^{t_2} \left(\frac{a\alpha KL}{2(\alpha - 1)} + \frac{1}{1 + a(\alpha - 1)} \cdot \frac{a\alpha}{t} \right) dt \\ &= \frac{\alpha \text{dist}^2(x_1, x_2)}{4M(t_2 - t_1)} + \frac{a\alpha KL}{2(\alpha - 1)} (t_2 - t_1) + \frac{a\alpha}{1 + a(\alpha - 1)} \log \left(\frac{t_2}{t_1} \right), \end{aligned}$$

from where we obtain (11). The proof of Corollary 1.6 is completed. \square

Next, we shall prove Theorem 1.7.

Proof of Theorem 1.7. Define

$$F := \frac{|\nabla v|^2}{v} - \alpha(t) \frac{v_t}{v},$$

where $\alpha(t) = e^{2KLt}$. It follows from (16) and $\text{Ric}_g(B_p(2R)) \geq -K$ that

$$(38) \quad \begin{aligned} \mathcal{L}(F) \leq & -2(m-1)v_{ij}^2 + 2(m-1)K|\nabla v|^2 \\ & + 2m \langle \nabla v, \nabla F \rangle - ((m-1)\Delta v)^2 - (\alpha-1) \left(\frac{v_t}{v}\right)^2 - \alpha' \frac{v_t}{v}. \end{aligned}$$

By applying the Cauchy–Schwarz inequality

$$v_{ij} \geq \frac{1}{n}(\Delta v)^2$$

to the right hand side of (38), we have

$$(39) \quad \begin{aligned} \mathcal{L}(F) \leq & -\frac{2}{n(m-1)}((m-1)\Delta v)^2 + 2(m-1)K|\nabla v|^2 \\ & + 2m \langle \nabla v, \nabla F \rangle - ((m-1)\Delta v)^2 - (\alpha-1) \left(\frac{v_t}{v}\right)^2 - \alpha' \frac{v_t}{v} \\ = & -\frac{1}{n}((m-1)\Delta v)^2 + 2(m-1)K|\nabla v|^2 \\ & + 2m \langle \nabla v, \nabla F \rangle - (\alpha-1) \left(\frac{v_t}{v}\right)^2 - \alpha' \frac{v_t}{v} \\ \leq & -\frac{1}{\alpha\alpha^2} \left(F + (\alpha-1) \frac{|\nabla v|^2}{v}\right)^2 + 2KL \frac{|\nabla v|^2}{v} \\ & + 2m \langle \nabla v, \nabla F \rangle - \frac{(\alpha-1)}{\alpha^2} \left(F - \frac{|\nabla v|^2}{v}\right)^2 - \alpha' \frac{v_t}{v}, \end{aligned}$$

where the last inequality follows from

$$(m-1)\Delta v = \frac{v_t}{v} - \frac{|\nabla v|^2}{v} = -\frac{1}{\alpha} \left(F + (\alpha-1) \frac{|\nabla v|^2}{v}\right).$$

From the last inequality in (39), we obtain

$$\begin{aligned}
 \mathcal{L}(\alpha^{-1}F) &= (\alpha^{-1})'F + \alpha^{-1}\mathcal{L}(F) \\
 &\leq -\frac{2KL}{\alpha} \cdot \frac{|\nabla v|^2}{v} + 2KL \frac{v_t}{v} \\
 &\quad - \frac{1}{\alpha\alpha^3} \left(F + (\alpha-1) \frac{|\nabla v|^2}{v} \right)^2 + \frac{2KL}{\alpha} \cdot \frac{|\nabla v|^2}{v} \\
 (40) \quad &\quad + 2m \langle \nabla v, \nabla(\alpha^{-1}F) \rangle - \frac{(\alpha-1)}{\alpha^3} \left(F - \frac{|\nabla v|^2}{v} \right)^2 - 2KL \frac{v_t}{v} \\
 &= -\frac{1}{\alpha\alpha^3} \left(F + (\alpha-1) \frac{|\nabla v|^2}{v} \right)^2 \\
 &\quad + 2m \langle \nabla v, \nabla(\alpha^{-1}F) \rangle - \frac{(\alpha-1)}{\alpha^3} \left(F - \frac{|\nabla v|^2}{v} \right)^2.
 \end{aligned}$$

Take a cut-off function $\phi = \phi(x)$ of the form

$$\phi(x) := \xi \left(\frac{r(x)}{R} \right),$$

where $\xi = \xi(t)$ is a smooth monotone function satisfying

$$(41) \quad \xi(t) \equiv 1 \quad \text{for } t \leq 1, \quad \xi(t) \equiv 0 \quad \text{for } t \geq 2, \quad \frac{(\xi')^2}{\xi} \leq C, \quad \text{and } \xi'' \geq -C$$

for some positive constant $C > 0$. Here, $r = r(x)$ is the distance between x and p . Then, it follows from (41) that

$$(42) \quad \frac{|\nabla \phi|^2}{\phi} = \frac{(\xi')^2}{\xi} \cdot \frac{|\nabla \rho|^2}{R^2} \leq \frac{C}{R^2}.$$

Since $\coth t$ is decreasing and $\xi'(t) = 0$ for $t \leq 1$, we have

$$(43) \quad \Delta \phi = \frac{\xi'' |\nabla r|^2}{R^2} + \frac{\xi' \Delta r}{R} \geq -\frac{C(n)}{R^2} \left\{ 1 + \sqrt{KR} \coth \left(\sqrt{\frac{K}{n-1}} R \right) \right\},$$

where $C(n)$ is a constant depending only on n . Here, we have used the Laplacian comparison theorem.

Define $G := t\phi\alpha^{-1}F$. We shall apply the maximum principle to G on $B_p(2R) \times [0, T]$. Assume that G attains its maximum at some point $(x_0, s) \in B_p(2R) \times [0, T]$. Without loss of generality, we may assume that $G(x_0, s) > 0$, which implies $s > 0$. Then at the point (x_0, s) , we have

$$(44) \quad \mathcal{L}(G) \geq 0 \quad \text{and} \quad \nabla(\alpha^{-1}F) = -\frac{\alpha^{-1}F}{\phi} \nabla \phi.$$

It follows from the last equality in (40) and (44) that

$$\begin{aligned}
 (45) \quad 0 &\leq \mathcal{L}(G) = s\phi\mathcal{L}(\alpha^{-1}F) - (m-1)v\alpha^{-1}F\Delta\phi - 2(m-1)vs\langle\nabla(\alpha^{-1}F), \nabla\phi\rangle + \phi\alpha^{-1}F \\
 &= s\phi\mathcal{L}(\alpha^{-1}F) - (m-1)v\frac{\Delta\phi}{\phi}G + 2(m-1)v\frac{|\nabla\phi|^2}{\phi^2}G + \frac{G}{s} \\
 &\leq s\phi\left[-\frac{1}{\alpha\alpha^3}\left(F + (\alpha-1)\frac{|\nabla v|^2}{v}\right)^2 - 2m\langle\nabla v, \nabla\phi\rangle\frac{\alpha^{-1}F}{\phi} \right. \\
 &\quad \left. - \frac{(\alpha-1)}{\alpha^3}\left(F - \frac{|\nabla v|^2}{v}\right)^2\right] \\
 &\quad - (m-1)v\frac{\Delta\phi}{\phi}G + 2(m-1)v\frac{|\nabla\phi|^2}{\phi^2}G + \frac{G}{s}.
 \end{aligned}$$

Let $\frac{|\nabla v|^2}{v} = \mu F$ at the point (x_0, s) . Since $G(x_0, s) > 0$, we have $\mu \geq 0$. It follows from the last inequality in (45) that

$$\begin{aligned}
 (46) \quad 0 &\leq -\frac{1}{\alpha\alpha^3}s\phi(1 + (\alpha-1)\mu)^2F^2 \\
 &\quad - 2m\langle\nabla v, \nabla\phi\rangle\alpha^{-1}sF - \frac{(\alpha-1)}{\alpha^3}s\phi(1-\mu)^2F^2 \\
 &\quad - (m-1)v\frac{\Delta\phi}{\phi}G + 2(m-1)v\frac{|\nabla\phi|^2}{\phi^2}G + \frac{G}{s} \\
 &\leq -\frac{1}{\alpha\alpha s\phi}(1 + (\alpha-1)\mu)^2G^2 \\
 &\quad + 2m\frac{|\nabla\phi|}{s^{\frac{1}{2}}\phi^{\frac{3}{2}}}\cdot\frac{\alpha^{\frac{1}{2}}L^{\frac{1}{2}}\mu^{\frac{1}{2}}}{(m-1)^{\frac{1}{2}}}G^{\frac{3}{2}} - \frac{(\alpha-1)}{\alpha s\phi}(1-\mu)^2G^2 \\
 &\quad - (m-1)v\frac{\Delta\phi}{\phi}G + 2(m-1)v\frac{|\nabla\phi|^2}{\phi^2}G + \frac{G}{s}.
 \end{aligned}$$

Multiplying both sides of the last inequality in (46) by $\frac{\phi}{G}$, we have

$$\begin{aligned}
 (47) \quad &\frac{1}{\alpha\alpha s}\left\{(1 + (\alpha-1)\mu)^2 + \alpha(\alpha-1)(1-\mu)^2\right\}G - 2m\frac{|\nabla\phi|}{s^{\frac{1}{2}}\phi^{\frac{1}{2}}}\cdot\frac{\alpha^{\frac{1}{2}}L^{\frac{1}{2}}\mu^{\frac{1}{2}}}{(m-1)^{\frac{1}{2}}}G^{\frac{1}{2}} \\
 &\leq -(m-1)v\Delta\phi + 2(m-1)v\frac{|\nabla\phi|^2}{\phi} + \frac{\phi}{s} \\
 &\leq -L\Delta\phi + 2L\frac{|\nabla\phi|^2}{\phi} + \frac{1}{s}.
 \end{aligned}$$

From the quadratic inequality $Ax^2 - 2Bx \leq C$, we have $x \leq \frac{2B}{A} + \left(\frac{C}{A}\right)^{\frac{1}{2}}$. By applying

this inequality to the last inequality in (47) by setting $x = G^{\frac{1}{2}}$, we have

$$(48) \quad G^{\frac{1}{2}} \leq \frac{2a\alpha^{\frac{3}{2}}ms^{\frac{1}{2}}L^{\frac{1}{2}}\mu^{\frac{1}{2}}}{(1+(\alpha-1)\mu)^2+a(\alpha-1)(1-\mu)^2} \cdot \frac{1}{(m-1)^{\frac{1}{2}}} \cdot \frac{|\nabla\phi|}{\phi^{\frac{1}{2}}} \\ + \left\{ \frac{a\alpha s}{(1+(\alpha-1)\mu)^2+a(\alpha-1)(1-\mu)^2} \left(-L\Delta\phi + 2L\frac{|\nabla\phi|^2}{\phi} + \frac{1}{s} \right) \right\}^{\frac{1}{2}}.$$

Note that

$$(49) \quad \frac{2a\alpha^{\frac{3}{2}}ms^{\frac{1}{2}}L^{\frac{1}{2}}\mu^{\frac{1}{2}}}{(1+(\alpha-1)\mu)^2+a(\alpha-1)(1-\mu)^2} \leq \frac{2a\alpha^{\frac{3}{2}}ms^{\frac{1}{2}}L^{\frac{1}{2}}\mu^{\frac{1}{2}}}{(1+(\alpha-1)\mu)^2} \\ = \frac{a\alpha^{\frac{3}{2}}ms^{\frac{1}{2}}L^{\frac{1}{2}}}{(\alpha-1)^{\frac{1}{2}}} \cdot \frac{2((\alpha-1)\mu)^{\frac{1}{2}}}{(1+(\alpha-1)\mu)^2} \\ < \frac{a\alpha^{\frac{3}{2}}ms^{\frac{1}{2}}L^{\frac{1}{2}}}{(\alpha-1)^{\frac{1}{2}}},$$

where the last inequality follows from $\frac{2\sqrt{x}}{(1+x)^2} < 1$. Moreover, we have

$$(50) \quad \frac{a\alpha s}{(1+(\alpha-1)\mu)^2+a(\alpha-1)(1-\mu)^2} \cdot \frac{1}{s} \leq \frac{a\alpha}{1+a(\alpha-1)}.$$

Hence, by applying (42), (43), (49), and (50) to (48), we have

$$G^{\frac{1}{2}}(x, T) \leq G^{\frac{1}{2}}(x_0, s) \\ \leq \frac{a\alpha^{\frac{3}{2}}ms^{\frac{1}{2}}L^{\frac{1}{2}}}{(\alpha-1)^{\frac{1}{2}}} \cdot \frac{1}{(m-1)^{\frac{1}{2}}} \cdot \frac{|\nabla\phi|}{\phi^{\frac{1}{2}}} \\ + \left\{ a\alpha \left(-sL\Delta\phi + 2sL\frac{|\nabla\phi|^2}{\phi} + \frac{1}{1+a(\alpha-1)} \right) \right\}^{\frac{1}{2}} \\ \leq a^{\frac{1}{2}}\alpha^{\frac{1}{2}}T^{\frac{1}{2}} \left(\frac{C(n)L}{R^2} \cdot \frac{a\alpha^2m^2}{(\alpha-1)(m-1)} \right)^{\frac{1}{2}} \\ + a^{\frac{1}{2}}\alpha^{\frac{1}{2}}T^{\frac{1}{2}} \left\{ \frac{C(n)L}{R^2} \left(1 + \sqrt{KR} \coth \left(\sqrt{\frac{K}{n-1}}R \right) \right) + \frac{1}{1+a(\alpha-1)} \cdot \frac{1}{T} \right\}^{\frac{1}{2}},$$

from where it follows that for any $x \in B_p(R)$,

$$F^{\frac{1}{2}}(x, T) \leq a^{\frac{1}{2}}\alpha \left(\frac{C(n)L}{R^2} \cdot \frac{a\alpha^2m^2}{(\alpha-1)(m-1)} \right)^{\frac{1}{2}} \\ + a^{\frac{1}{2}}\alpha \left\{ \frac{C(n)L}{R^2} \left(1 + \sqrt{KR} \coth \left(\sqrt{\frac{K}{n-1}}R \right) \right) + \frac{1}{1+a(\alpha-1)} \cdot \frac{1}{T} \right\}^{\frac{1}{2}}.$$

Since T is arbitrary, by squaring both sides of the inequality just above, we obtain (12). The proof of Theorem 1.7 is completed. \square

REMARK 3.2. Huang, Huang, and Li [4] estimated $\mathcal{L}(F)$ from above by

$$\begin{aligned} \mathcal{L}(F) \leq & -\frac{1}{a\alpha^2} \left(F + (\alpha - 1) \frac{|\nabla v|^2}{v} \right)^2 + 2KL \frac{|\nabla v|^2}{v} \\ & + 2m \langle \nabla v, \nabla F \rangle - \alpha' \frac{v_t}{v} \end{aligned}$$

and derived the gradient estimate (8), while we estimated $\mathcal{L}(F)$ from above by

$$\begin{aligned} \mathcal{L}(F) \leq & -\frac{1}{a\alpha^2} \left(F + (\alpha - 1) \frac{|\nabla v|^2}{v} \right)^2 + 2KL \frac{|\nabla v|^2}{v} \\ & + 2m \langle \nabla v, \nabla F \rangle - \frac{(\alpha - 1)}{\alpha^2} \left(F - \frac{|\nabla v|^2}{v} \right)^2 - \alpha' \frac{v_t}{v} \end{aligned}$$

in the last inequality of (39) and derived the better gradient estimate (13).

Finally, we shall prove Corollary 1.8.

Proof of Corollary 1.8. Let

$$\alpha(t) = e^{2KLt} \quad \text{and} \quad \varphi(t) = \frac{1}{1 + a(\alpha(t) - 1)} \cdot \frac{a\alpha^2(t)}{t}.$$

By applying (37), we obtain

$$\begin{aligned} f(x_1, t_1) - f(x_2, t_2) & \leq \frac{\text{dist}^2(x_1, x_2)}{4M(t_2 - t_1)^2} \int_{t_1}^{t_2} \alpha(t) dt + \int_{t_1}^{t_2} \left(\frac{1}{1 + a(\alpha(t) - 1)} \cdot \frac{a\alpha(t)}{t} \right) dt \\ & \leq \frac{\text{dist}^2(x_1, x_2)}{4M(t_2 - t_1)^2} \cdot \frac{e^{2KLt_2} - e^{2KLt_1}}{2KL} \\ & \quad + \frac{1}{1 + a(\alpha(t_1) - 1)} \cdot \frac{a}{t_1} \cdot \frac{e^{2KLt_2} - e^{2KLt_1}}{2KL}, \end{aligned}$$

from where we obtain (14). The proof of Corollary 1.8 is completed. \square

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Homare TADANO,
Department of Mathematics, Graduate School of Science, Osaka University
1-1 Machikaneyama, Toyonaka, Osaka 560-0043, JAPAN
e-mail: h-tadano@cr.math.sci.osaka-u.ac.jp

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