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FOUR-DIMENSIONAL PSEUDO-RIEMANNIAN LIE GROUPS

Dedicated to the memory of Sergio Console

Abstract. We investigate four-dimensional Lie groups equipped with a left-invariant pseudo-Riemannian metric. We shall describe the general procedure to classify such pseudo-Riemannian Lie groups, and apply it to obtain an explicit classification of the Einstein examples.

1. Introduction

It is a somewhat surprizing fact that the classification of four-dimensional homogeneous pseudo-Riemannian manifolds $(G/H, g)$ proceeded faster in the case of nontrivial isotropy than for $H = 0$, that is, for pseudo-Riemannian Lie groups.

Indeed, a complete local classification of four-dimensional homogeneous pseudo-Riemannian manifolds with nontrivial isotropy was achieved in [11]. This permitted to determine all invariant metrics on these spaces satisfying some geometric conditions, like for example the solutions to the Einstein-Maxwell equation (in particular, Einstein metrics) again in [11], the Kähler metrics [3] and the Ricci soliton metrics [4].

On the other hand, up to recently, a systematic study of left-invariant metrics on four-dimensional Lie groups only concerned the Riemannian case (see for example [1]). The approach used in Riemannian settings made use of the following very useful facts:

1. each four-dimensional Lie algebra $\mathfrak{g}$ can be described in terms of a semi-direct product between $\mathbb{R}$ and a three-dimensional Lie algebra, and
2. the restriction of a positive definite inner product to any subspace of $\mathfrak{g}$ is again positive definite.

The above point 1. does not depend on the signature of the inner product. On the other hand, it is clear that point 2. fails completely in other signatures. Hence, their study requires a completely different approach.

The above cited results led in a natural way to consider the problem of studying and classifying four-dimensional pseudo-Riemannian Lie groups. Such investigation has been undertaken in [7] and [8] (see also [6] for the conformally flat examples and [5] for cyclic Lorentzian metrics). In this paper, we shall illustrate the basic ideas used to study four-dimensional pseudo-Riemannian Lie groups, and to obtain the complete classification of the Einstein examples.

In Section 2 we shall discuss the general structure of a four-dimensional Lie algebra equipped with an inner product of any signature. Then, a complete classification

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of four-dimensional simply connected Einstein Lie groups will be given in Sections 3 and 4, for the Lorentzian and the neutral signature cases respectively. We shall see that differently from the Riemannian case, there exist left-invariant Einstein metrics, Lorentzian and of neutral signature, on four-dimensional Lie groups, which are not symmetric (not even locally symmetric). We observe that the results of these Sections, together with the ones obtained in [11], lead to the complete local classification of four-dimensional pseudo-Riemannian homogeneous Einstein manifolds.

2. Four-dimensional Lie groups

Four-dimensional homogeneous *Riemannian* manifolds were classified by Bérard-Bérgery [2]. This classification yields that a simply connected four-dimensional homogeneous Riemannian manifold is either symmetric, or isometric to some Lie group equipped with a left-invariant Riemannian metric. The classification of four-dimensional simply connected Riemannian Lie groups is resumed in the following.

PROPOSITION 1. [1] *A simply connected four-dimensional Riemannian Lie group is:*

(i) either one of the unsolvable direct products $\mathbb{R} \times SU(2)$ *and* $\mathbb{R} \times \widetilde{SL}(2,\mathbb{R})$*; or*

(ii) one of the following solvable Lie groups:

- *(ii1) the non-trivial semi-direct products* $\mathbb{R} \ltimes E(2)$ *and* $\mathbb{R} \ltimes E(1,1)$*;*
- *(ii2) the non-nilpotent semi-direct products* $\mathbb{R} \ltimes H$*, where* H *denotes the Heisenberg group;*
- *(ii3) the semi-direct products* $\mathbb{R} \ltimes \mathbb{R}^3$.

Consider now an n-dimensional simply connected Lie group G and the corresponding Lie algebra $\mathfrak{g}$. Left-invariant metrics on G, of prescribed signature, are in a one-to-one correspondence with inner products of the same signature on $\mathfrak{g}$. Thus, it suffices to work at the Lie algebra level henceforth.

It is easily seen that G admits left-invariant metrics of any signature $(p, n-p)$. In fact, one only needs to choose a basis $\mathcal{B} = \{e_1, \dots, e_n\}$ of $\mathfrak{g}$ and consider the inner product g on $\mathfrak{g}$, uniquely determined by having $\mathcal{B}$ as a pseudo-orthonormal basis, with $e_1, \dots, e_p$ space-like and $e_{p+1}, \dots, e_n$ time-like vectors.

In particular, suppose now that $\dim G = 4$. Let $\bar{g}$ be a positive definite inner product on $\mathfrak{g}$. By a well-known argument of linear algebra, it exists an ortohonormal basis $\{e_1, e_2, e_3, e_4\}$ of $\mathfrak{g}$. Then, a corresponding left-invariant metric g of neutral (respectively, Lorentzian) signature on G is uniquely determined at the Lie algebra level by having $\{e_1, e_2, e_3, e_4\}$ as a pseudo-orthonormal basis of $\mathfrak{g}$, with e_1, e_2 space-like and e_3, e_4 time-like (respectivey, with e_1, e_2, e_3 space-like and e_4 time-like).

Conversely, if g is an inner product of either Lorentzian or neutral signature on $\mathfrak{g}$, it suffices to consider a pseudo-orthonormal basis $\{e_1, e_2, e_3, e_4\}$ for g, and we get a corresponding left-invariant Riemannian metric $\bar{g}$ on G, described at the Lie algebra level by having $\{e_1,e_2,e_3,e_4\}$ as an orthonormal basis. Thus, G is necessarily one of Lie groups listed in Proposition 1, and we proved the following result.

PROPOSITION 2. *Every n-dimensional simply connected Lie group G admits left-invariant metrics of any prescribed signature* $(p,n-p)$. *In particular, if G is a four-dimensional simply connected Lie group, equipped with a left-invariant metric of any signature, then G is one of Lie groups listed in Proposition 1.*

The crucial fact in dimension four is that each simply connected four-dimensional Lie group can be described in terms of a semi-direct product of $\mathbb{R}$ by a three-dimensional Lie group (also including in this description the case of direct products of $\mathbb{R}$ by one of unsolvable Lie groups $SU(2)$ or $\widetilde{SL}(2,\mathbb{R})$).

Correspondingly, the Lie algebra of G can be described as $\mathfrak{g} = \mathfrak{r} \ltimes \mathfrak{g}_3$, that is, $\mathfrak{g}$ is the direct sum of $\mathfrak{r}$ and $\mathfrak{g}_3$, where $\mathfrak{g}_3$ is a three-dimensional Lie algebra, and the generator of the one-dimensional Lie algebra $\mathfrak{r}$ acts as a derivation on $\mathfrak{g}_3$.

By Proposition 2, pseudo-Riemannian and Riemannian Lie groups coincide in any dimension. However, the study of left-invariant pseudo-Riemannian metrics on Lie groups cannot use the same techniques of the Riemannian case.

In fact, if g is a positive definite inner product over $\mathfrak{g} = \mathfrak{r} \ltimes \mathfrak{g}_3$, then its restriction to $\mathfrak{g}_3$ is still positive definite. Hence, one can use the description of three-dimensional Riemannian Lie groups given in [12] and then choose a basis of the Lie algebra adapted to the inner product. An example of this technique is given by the study of curvature properties of four-dimensional Riemannian Lie groups made in [1]. On the other hand:

- If g is Lorenzian, then the restriction of g over $\mathfrak{g}_3$ is

  (L1) either positive definite, (L2) Lorentzian, or (L3) *degenerate*.

- If g is of neutral signature $(2,2)$, then its restriction to $\mathfrak{g}_3$ is

  (N1) either of signature $(2,1)$, (N1′) of signature $(1,2)$, or (N2) *degenerate*.

We referred to the first two cases listed for g neutral as "(N1)" and "(N1′)" because they are indeed equivalent to one another, up to reversing the metric. In fact, in case (N1′), we have a neutral inner product g over a four-dimensional Lie algebra $\mathfrak{g} = \mathfrak{r} \ltimes \mathfrak{g}_3$, where a space-like vector e_4 (spanning $\mathfrak{r}$) acts as a derivation over a three-dimensional Lie algebra $\mathfrak{g}_3$, on which g has signature $(1,2)$. But *reversing the metric* [13], we get the same Lie algebra $\mathfrak{g}$, equipped with the neutral inner product $-g$, for which a time-like vector e_4 acts as a derivation over the three-dimensional Lorentzian Lie algebra $\mathfrak{g}_3$ of signature $(2,1)$, that is, case (N1). We also explicitly observe that being homothetic, the metrics g and $-g$ share the same curvature properties.

In some of the above cases (namely, (L1), (L2) and (N1)), the argument used in the Riemannian case could still be applied successfully, using the classifications of three-dimensional Riemannian [12] and Lorentzian [14] Lie groups.

However, in cases (L3) and (N2), since the restriction of g to $\mathfrak{g}_3$ is degenerate, the approach used in the Riemannian case to study these metrics fails completely. Observe that such cases explicitly occurred, for example, in the classification of left-invariant conformally flat neutral metrics on four-dimensional Lie groups [6].

To study inner products over an arbitrary four-dimensional Lie algebra $\mathfrak{g} = \mathfrak{r} \ltimes \mathfrak{g}_3$, we shall first discuss the standard forms of such inner products with respect to the semi-direct product structure of $\mathfrak{g}$, treating separately the Lorentzian and neutral cases. Then, we can impose the required curvature condition, like the metric being Einstein, together with the Jacobi identity. This gives us all the possible solutions, that is, the Lie algebras satisfying the required curvature property. At that point, we can identify *a posteriori* the corresponding simply connected Lie groups.

2.1. Lorentzian case

It is well known that any symmetric bilinear form g admits an orthogonal basis. Moreover, if g is nondegenerate of signature (p,q), then $r = \min(p,q)$ is the maximal dimension of a vector subspace W such that $g|W = 0$.

In particular, in the four-dimensional Lorentzian case, the maximal dimension of a subspace on which g vanishes is equal to one. Consequently, the nullity index of $g|_{\mathfrak{g}_3}$ is either 0 or 1. Up to reversing the metric, the possible cases in terms of the signature of $g|_{\mathfrak{g}_3}$ are then the following:

(I) $\mathrm{sgn}(g|_{\mathfrak{g}_3}) = (3,0,0)$, which leads to case (L1);

(II) $\mathrm{sgn}(g|_{\mathfrak{g}_3}) = (2,1,0)$, which yields case (L2);

(III) $\mathrm{sgn}(g|_{\mathfrak{g}_3}) = (1,1,1)$, which is incompatible with the fact that g is Lorentzian;

(IV) $\mathrm{sgn}(g|_{\mathfrak{g}_3}) = (1,0,2)$, wich leads to case (L3).

Using the above cases, the following key result was obtained ([7],[8]).

PROPOSITION 3. *Let $(\mathfrak{g}, g)$ be an arbitrary four-dimensional Lorentzian Lie algebra. Then, there exists a basis $\{e_1,e_2,e_3,e_4\}$ of $\mathfrak{g}$, such that*

- $\mathfrak{h} = \mathrm{span}(e_1,e_2,e_3)$ *is a three-dimensional Lie algebra and e_4 acts as a derivation on $\mathfrak{h}$ (that is, $\mathfrak{g} = \mathfrak{h} \rtimes \mathfrak{r}$, where $\mathfrak{r} = span(e_4)$), and*
- *with respect to $\{e_1,e_2,e_3,e_4\}$, the Lorentzian inner product takes one of the following forms:*

$$(L1)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},\ (L2)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ (L3)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

2.2. Neutral case

For a neutral four-dimensional inner product g, the maximal dimension of a vector subspace W such that $g|W = 0$ is 2, and the following result holds ([8]).

PROPOSITION 4. *Let* $\mathfrak{g}$ *denote any four-dimensional Lie algebra and* g *an inner product on* $\mathfrak{g}$*, of signature* $(2,2)$*. Then, there exists a basis* $\{e_1,e_2,e_3,e_4\}$ *of* $\mathfrak{g}$*, such that*

- $\mathfrak{g}_3 = \mathrm{Span}(e_1,e_2,e_3)$ *is a three-dimensional Lie algebra and* e_4 *acts as a derivation on* $\mathfrak{g}_3$ *(that is,* $\mathfrak{g} = \mathfrak{r} \ltimes \mathfrak{g}_3$*, where* $\mathfrak{r} = \mathrm{Span}(e_4)$*), and*
- *with respect to* $\{e_1,e_2,e_3,e_4\}$*, the neutral inner product* g *takes one of the following forms:*

$$(N1)\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}, \qquad (N2)\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}.$$

3. Lorentzian Einstein 4D Lie groups

The basic idea to classify four-dimensional Einstein Lorentzian Lie groups is the following. By Proposition 3, the Lie algebra $\mathfrak{g}$ of G is a semi-direct product $\mathfrak{r} \ltimes \mathfrak{g}_3$, where $\mathfrak{r} = \mathrm{span}(e_4)$ acts on $\mathfrak{g}_3 = \mathrm{span}(e_1,e_2,e_3)$, and the Lorentzian inner product on $\mathfrak{g}$ is described by one of conditions (l1),(l2),(l3). The general form of the semi-direct product Lie algebra $\mathfrak{g} = \mathfrak{r} \ltimes \mathfrak{g}_3$ is given by

$$\begin{array}{ll} [e_1,e_2] = a_1e_1+a_2e_2+a_3e_3, & [e_1,e_3] = b_1e_1+b_2e_2+b_3e_3, \\ [e_1,e_4] = c_1e_1+c_2e_2+c_3e_3, & [e_2,e_3] = d_1e_1+d_2e_2+d_3e_3, \\ [e_2,e_4] = p_1e_1+p_2e_2+p_3e_3, & [e_3,e_4] = q_1e_1+q_2e_2+q_3e_3, \end{array} \tag{3.1}$$

for some real constants $a_i,\dots,q_i$, which must satisfy the Jacobi identity. The following algorithm can be then applied:

1. Treat separately the cases (L1), (L2), (L3), requiring that the solutions satisfy both the Jacobi identity and the Einstein equation;
2. Discuss the possible isometries between different solutions;
3. Determine the simply connected Lie group corresponding to the remaining solutions.

In this way, the following classification result was proved in [7] for Einstein Lorentzian Lie groups in dimension four.

THEOREM 1. *Let G be a four-dimensional simply connected Lie group. If g is a left-invariant Lorentzian Einstein metric on G, then the Lie algebra $\mathfrak{g}$ of G is isometric* *to $\mathfrak{g}=\mathfrak{r}\ltimes\mathfrak{g}_3$, where $\mathfrak{g}_3=\operatorname{span}\{e_1,e_2,e_3\}$ and $\mathfrak{r}=\operatorname{span}\{e_4\}$, and one of the following cases occurs.*

(L1) $\{e_i\}_{i=1}^4$ is a pseudo-orthonormal basis, with e_3 time-like. In this case, G is isometric to one of the following semi-direct products $\mathbb{R}\ltimes G_3$:

a1) $\mathbb{R}\ltimes H$, where H is the Heisenberg group and $\mathfrak{g}$ is described by one of the following sets of conditions:

1) $[e_1,e_2]=\varepsilon Ae_1, [e_1,e_3]=Ae_1$, $[e_1,e_4]=\delta Ae_1$, $[e_3,e_4]=-2A\delta(\varepsilon e_2-e_3)$,

2) $[e_1,e_2]=\frac{\varepsilon\sqrt{A^2-B^2}}{2}e_1$, $[e_1,e_3]=-\frac{\varepsilon\delta\sqrt{A^2-B^2}}{2}e_1$, $[e_1,e_4]=\frac{\delta A+B}{2}e_1$,
$[e_2,e_4]=B(e_2+\delta e_3)$, $[e_3,e_4]=A(e_2+\delta e_3)$,

3) $[e_1,e_2]=\frac{\varepsilon A\sqrt{A^2-B^2}}{B}e_1$, $[e_1,e_3]=\varepsilon\sqrt{A^2-B^2}e_1$, $[e_2,e_4]=Be_2-Ae_3$,
$[e_3,e_4]=Ae_2-\frac{A^2}{B}e_3$,

4) $[e_1,e_2]=\varepsilon\sqrt{A^2-B^2}e_1+Be_2, \quad [e_3,e_4]=Ae_3$.

a2) $\mathbb{R}\ltimes\mathbb{R}^3$, where $\mathfrak{g}$ is described by one of the following sets of conditions:

5) $[e_1,e_4]=-(A+B)e_1, \quad [e_2,e_4]=Be_2-\varepsilon\sqrt{A^2+AB+B^2}e_3$,
$[e_3,e_4]=\varepsilon\sqrt{A^2+AB+B^2}e_2+Ae_3$,

6) $[e_1,e_4]=-2Ae_1$, $[e_2,e_4]=-5Ae_2+6\varepsilon Ae_3$, $[e_3,e_4]=Ae_3$,

7) $[e_1,e_4]=Ae_1$, $[e_2,e_4]=Ae_2+Be_3$, $[e_3,e_4]=Be_2+Ae_3$,

8) $[e_1,e_4]=\varepsilon\frac{A+B}{3}e_1$, $[e_2,e_4]=\varepsilon\frac{5B-A}{6}e_2+Be_3$, $[e_3,e_4]=Ae_2+\varepsilon\frac{5A-B}{6}e_3$,

9) $[e_1,e_4]=\frac{5A}{2}e_1+3\varepsilon Ae_3$, $[e_2,e_4]=Ae_2$, $[e_3,e_4]=-\frac{A}{2}e_3$,

10) $[e_1,e_4]=Ae_1+\varepsilon\sqrt{B^2-A^2-C^2-AC}e_2$,
$[e_2,e_4]=\varepsilon\sqrt{B^2-A^2-C^2-AC}e_1-(A+C)e_2-Be_3$, $[e_3,e_4]=Be_2+Ce_3$,

11) $[e_1,e_4]=-\frac{2\varepsilon\sqrt{2}A}{3}e_1+\delta Ae_3$, $[e_2,e_4]=\frac{\varepsilon\sqrt{2}A}{3}e_2, \quad [e_3,e_4]=Ae_2-\frac{\varepsilon\sqrt{2}A}{6}e_3$.

(L3) $\{e_i\}_{i=1}^4$ is a basis, with the inner product g on $\mathfrak{g}$ completely determined by $g(e_1,e_1)=g(e_2,e_2)=g(e_3,e_4)=g(e_4,e_3)=1$ and $g(e_i,e_j)=0$ otherwise. In this case, G is isometric to one of the following semi-direct products $\mathbb{R}\ltimes G_3$:

c1) $\mathbb{R}\ltimes H$, where $\mathfrak{g}$ is described by one of the following sets of conditions:

12) $[e_1,e_2]=\varepsilon(A+B)e_3, [e_1,e_4]=Ce_1+Ae_2+De_3, [e_2,e_4]=Be_1+Ee_3, [e_3,e_4]=Ce_3,$

13) $[e_1,e_2]=Be_3, [e_1,e_4]=\frac{(C+D)^2-B^2}{4A}e_1+De_2+Fe_3, [e_2,e_4]=Ce_1+Ae_2+Ee_3,$ $[e_3,e_4]=\frac{(C+D)^2-B^2+4A^2}{4A}e_3,$

14) $[e_1,e_2]=\varepsilon\sqrt{(A+D)^2+4B^2}e_3, \quad [e_1,e_4]=-Be_1+De_2+Ee_3,$ $[e_2,e_4]=Ae_1+Be_2+Ce_3.$

c1) $\mathbb{R}\ltimes\mathbb{R}^3$*, where* $\mathfrak{g}$ *is described by one of the following sets of conditions:*

15) $[e_1,e_4]=Ae_2+Be_3, \quad [e_2,e_4]=-Ae_1+Ce_3,$

16) $[e_1,e_4]=Ae_1+Be_2+Ce_3, [e_2,e_4]=De_1+Ee_2+Fe_3, [e_3,e_4]=\frac{(B+D)^2+2(A^2+E^2)}{2(E+A)}e_3.$

In all the cases listed above, $\varepsilon=\pm1$ *and* $\delta=\pm1$.

We explicitly remark that the above Theorem 1, together with the results obtained in [11], permits to obtain the complete local classification of all four-dimensional Lorentzian homogeneous Einstein manifolds.

Observe that in Theorem 1 we did not list solutions corresponding to "Case (L2)", that is, a time-like vector acting as a derivation on a three-dimensional Riemannian Lie algebra. Such solutions do occur. However, each of them is also isometric to one of cases listed in case L1). For example, an explicit solution for case (L2) is given by

$$[e_1,e_2]=Ae_3, \quad [e_1,e_4]=-Be_3, \quad [e_2,e_3]=Ae_1, \quad [e_3,e_4]=Be_1, \tag{3.2}$$

for some real constants A,B. Thus, $[\mathfrak{g},\mathfrak{g}]=\mathrm{span}(e_1,e_3)$, and the time-like vector e_4 acts as a derivation on the Riemannian Lie algebra $\mathfrak{g}_3=\mathrm{span}(e_1,e_2,e_3)$.

On the other hand, the above equation (3.2) yields that the space-like vector e_2 also acts as a derivation on the Lorentzian Lie algebra $\mathfrak{g}_3'=\mathrm{span}(e_1,e_3,e_4)$, which is the Lie algebra of the Heisenberg group. Therefore, this example is already included in case L1).

At this point, we can investigate the geometry of the Einstein examples classified in Theorem 1. In particular, for each of them we can determine whether they are conformally flat (and so, being Einstein, of constant sectional curvature) or locally symmetric, and specify whether they are flat or Ricci-flat. With respect to the basis $\{e_1,e_2,e_3,e_4\}$ used to describe the Lie algebra $\mathfrak{g}$, the conformal flatness condition is equivalent to the system of algebraic equations

$$W_{ijkh}=R_{ijkh}-\tfrac{1}{2}(g_{ik}\rho_{jh}+g_{jh}\rho_{ik}-g_{ih}\rho_{jk}-g_{jk}\rho_{ih})+\tfrac{r}{6}(g_{ik}g_{jh}-g_{ih}g_{jk})=0, \tag{3.3}$$

for all indices $i,j,k,h=1,\dots,4$, where W_{ijkh} denote the components of the *Weyl tensor* with respect to the basis $\{e_i\}$ and r the scalar curvature, and local symmetry condition $\nabla R=0$ is equivalent to the system of algebraic equations

$$\begin{aligned}\nabla_s R_{ijkh} = & -R(\nabla_{e_s}e_i,e_j,e_k,e_h)-R(e_i,\nabla_{e_s}e_j,e_k,e_h)\\ & -R(e_i,e_j\nabla_{e_s}e_k,e_h)-R(e_i,e_j,e_k,\nabla_{e_s}e_h)=0,\end{aligned} \tag{3.4}$$

for all indices $s,i,j,k,h=1,\dots,4$. We apply Equations (3.3) and (3.4) to examples **1)-16)** listed in Theorem 1 and obtain the following result.

THEOREM 2. *Among four-dimensional Einstein Lorentzian Lie groups, as classified in Theorem 1 up to isometries, the locally symmetric, conformally flat, flat and Ricci-flat examples are listed in the following* Table I, *where the checkmark means that the corresponding condition holds for all Lie algebras of that form.*

(G,g)	*Locally symmetric*	*Constant curvature*	*Flat*	*Ricci-flat*
1)	✓	✓	$A=0$	$A=0$
2)	✓	✓	$B=-\delta A$	$B=-\delta A$
3)	✓	$B=\pm A$	$B=\pm A$	$B=\pm A$
4)	✓	$A=0$	$A=0$	$A=0$
5)	$B=-A$	$B=-A$	$B=-A$	✓
6)	$A=0$	$A=0$	$A=0$	$A=0$
7)	✓	✓	$A=0$	$A=0$
8)	$B=\pm A$	$B=\pm A$	$B=-A$	$B=-A$
9)	$A=0$	$A=0$	$A=0$	$A=0$
10)	$B=C=0$ *or* $A=B\pm C=0$	$B=C=0$ *or* $A=B\pm C=0$	$B=C=0$ *or* $A=B\pm C=0$	✓
11)	$A=0$	$A=0$	$A=0$	$A=0$
12)	$\varepsilon-1=A=0$ *or* $\varepsilon+1=B=0$ *or* $A+B=C=0$	$\varepsilon-1=A=0$ *or* $\varepsilon+1=B=0$ *or* $A+B=C=0$	$\varepsilon-1=A=0$ *or* $\varepsilon+1=B=0$ *or* $A+B=C=0$	✓
13)	$B=C-D$	$B=C-D$	$B=C-D$	✓
14)	$A+D=B=0$ *or* $AD+B^2=0$	$A+D=B=0$ *or* $AD+B^2=0$	$A+D=B=0$ *or* $AD+B^2=0$	✓
15)	✓	✓	✓	✓
16)	$A=E,D=-B$ *or* $B=D,AE=D^2$	$A=E,D=-B$ *or* $B=D,AE=D^2$	$A=E,D=-B$ *or* $B=D,AE=D^2$	✓

Table I: Geometry of four-dimensional Einstein Lorentzian Lie groups

4. Neutral Einstein 4D Lie groups

The same argument illustrated in the previous Section also applies to neutral inner products over four-dimensional Lie algebras, described as semi-direct products $\mathbb{R} \ltimes \mathfrak{g}_3$, treating separately cases (N1) and (N2) of Proposition 4. The result is the following classification of left-invariant neutral Einstein metrics on four-dimensional Lie groups, proved in [8].

THEOREM 3. *Let G be a four-dimensional simply connected Lie group. If g is a left-invariant neutral Einstein metric on G, then the Lie algebra $\mathfrak{g}$ of G is isometric to $\mathfrak{g} = \mathfrak{r} \ltimes \mathfrak{g}_3$, where $\mathfrak{g}_3 = \mathrm{Span}\{e_1, e_2, e_3\}$ and $\mathfrak{r} = \mathrm{Span}\{e_4\}$, and one of the following cases occurs.*

(N1) $\{e_1, e_2, e_3, e_4\}$ is a pseudo-orthonormal basis, with e_3, e_4 time-like. In this case, G is isometric to one of the following semi-direct products $\mathbb{R} \ltimes G_3$:

a1) $\mathbb{R} \ltimes H$, where $\mathfrak{g}$ is described by one of the following sets of conditions:

1) $[e_1,e_2] = \frac{\varepsilon\sqrt{A^2-B^2}}{2}e_1, \quad [e_1,e_3] = \frac{\varepsilon\delta\sqrt{A^2-B^2}}{2}e_1, \quad [e_1,e_4] = (\frac{A}{2} - \delta\frac{B}{2})e_1,$
$[e_2,e_4] = Ae_2 - \delta Ae_3, \quad [e_3,e_4] = Be_2 - \delta Be_3, \quad (A \neq \pm B),$

2) $[e_1,e_2] = \frac{\sqrt{A^2-B^2}B}{A}e_1, \quad [e_1,e_3] = \sqrt{A^2-B^2}e_1, \quad [e_2,e_4] = Ae_2 - Be_3,$
$[e_3,e_4] = Be_2 - \frac{B^2}{A}e_3, \quad (A \neq \pm B),$

a2) $\mathbb{R} \ltimes \mathbb{R}^3$ and $\mathfrak{g}$ is described by one of the following sets of conditions:

3) $[e_1,e_4] = Ae_1, \quad [e_2,e_4] = Ae_2 + Be_3, \quad [e_3,e_4] = Be_2 + Ae_3,$

4) $[e_1,e_4] = -\frac{A+B}{3}e_1, \quad [e_2,e_4] = -\frac{5A-B}{6}e_2 + Ae_3, \quad [e_3,e_4] = Be_2 - \frac{5B-A}{6}e_3,$

5) $[e_1,e_4] = \frac{2\sqrt{2}}{3}Ae_1 - Ae_3, \quad [e_2,e_4] = -\frac{\sqrt{2}}{3}Ae_2, \quad [e_3,e_4] = Ae_2 + \frac{\sqrt{2}}{6}Ae_3,$

6) $[e_1,e_4] = Ae_1 - \sqrt{C^2-AB-A^2-B^2}e_2,$
$[e_2,e_4] = -\sqrt{C^2-AB-A^2-B^2}e_1 - (A+B)e_2 - Ce_3, \quad [e_3,e_4] = Ce_2 + Be_3,$

7) $[e_1,e_4] = Ae_1 - Be_2, \quad [e_2,e_4] = Be_1 + Ae_2, \quad [e_3,e_4] = Ae_3,$

8) $[e_1,e_4] = -\frac{2A^2+5B^2}{6\sqrt{B^2+A^2}}e_1 - \frac{AB}{2\sqrt{B^2+A^2}}e_2 + Be_3,$
$[e_2,e_4] = -\frac{AB}{2\sqrt{B^2+A^2}}e_1 - \frac{5A^2+2B^2}{6\sqrt{B^2+A^2}}e_2 + Ae_3, \quad [e_3,e_4] = \frac{\sqrt{B^2+A^2}}{6}e_3,$

9) $[e_1,e_4] = \frac{A+B}{2}e_1 - \frac{\sqrt{6(B^2-A^2)}}{2}e_2 - \frac{\sqrt{6(B^2-A^2)}}{2}e_3,$
$[e_2,e_4] = Ae_2 + (B+2A)e_3, \quad [e_3,e_4] = (A+2B)e_2 + Be_3,$

10) $[e_1,e_4]=\frac{5\sqrt{A^2-B^2}}{6}e_1+Be_2+Ae_3, \quad [e_2,e_4]=\frac{B^2+2A^2}{6\sqrt{A^2-B^2}}e_2+\frac{AB}{2\sqrt{A^2-B^2}}e_3,$
$[e_3,e_4]=-\frac{AB}{2\sqrt{A^2-B^2}}e_2-\frac{2B^2+A^2}{6\sqrt{A^2-B^2}}e_3$

N2) $\{e_1,e_2,e_3,e_4\}$ *is a basis, with the inner product* g *on* $\mathfrak{g}$ *completely determined by* $g(e_1,e_1)=-g(e_2,e_2)=g(e_3,e_4)=g(e_4,e_3)=1$ *and* $g(e_i,e_j)=0$ *otherwise. In this case,* G *is isometric to one of the following semi-direct products* $\mathbb{R}\ltimes G_3$*:*

b1) $\mathbb{R}\ltimes H$ *and* $\mathfrak{g}$ *is described by one of the following sets of conditions:*

11) $[e_1,e_2]=A(e_1+e_2)+(2B-C-D)e_3,$
$[e_1,e_4]=D(e_1+e_2)+\frac{3B(C+D)-2B^2-AE-(C+D)^2}{A}e_3,$
$[e_2,e_4]=C(e_1+e_2)+Ee_3, \quad [e_3,e_4]=Be_3,$

12) $[e_1,e_2]=A(e_1+e_2)-(C+D)e_3,$
$[e_1,e_4]=C(e_1+e_2)-\frac{(C+D)^2-B(C+D)+AE}{A}e_3,$
$[e_2,e_4]=D(e_1+e_2)+Ee_3, \quad [e_3,e_4]=Be_3,$

13) $[e_1,e_2]=A(e_1-e_2)+Be_3, \quad [e_1,e_4]=2B(e_1-e_2)+\frac{B^2+AD}{A}e_3,$
$[e_2,e_4]=B(e_1-e_2)+De_3,$

14) $[e_1,e_2]=A(e_1+\varepsilon e_2)-\frac{CA+\varepsilon AD}{A}e_3,$
$[e_1,e_4]=D(e_1+\varepsilon e_2)-\frac{DC+\varepsilon D^2-EC}{A}e_3,$
$[e_2,e_4]=C(e_1+\varepsilon e_2)-\frac{C^2+\varepsilon CD-ED}{A}e_3, \quad [e_3,e_4]=Ee_3,$

15) $[e_1,e_2]=Ae_3, \quad [e_1,e_4]=\frac{A^2-(C-D)^2}{4B}e_1+De_2+Ee_3,$
$[e_2,e_4]=Ce_1+Be_2+Fe_3, \quad [e_3,e_4]=\frac{A^2-(C-D)^2+4B^2}{4B}e_3, \quad (A\neq 0),$

16) $[e_1,e_2]=\varepsilon(B-A)e_3, \quad [e_1,e_4]=Ce_1+Ae_2+De_3,$
$[e_2,e_4]=Be_1+Ee_3, \quad [e_3,e_4]=Ce_3, \quad (A\neq B),$

b2) Either $\mathbb{R}\ltimes E(2)$ *or* $\mathbb{R}\ltimes E(1,1)$*, with* $\mathfrak{g}$ *described by one of the following sets of conditions:*

17) $[e_1,e_2]=-\frac{A}{2}e_1+\frac{B}{2}e_2-\frac{CB-AD}{A}e_3, \quad [e_1,e_3]=Be_3,$
$[e_1,e_4]=Ce_1-\frac{CB}{A}e_2-\frac{2D(-CA+EA+DB)}{A^2}e_3, \quad [e_2,e_3]=Ae_3,$
$[e_2,e_4]=De_1-\frac{DB}{A}e_2-\frac{2C(-CA+EA+DB)}{A^2}e_3, \quad [e_3,e_4]=Ee_3,$

18) $[e_1,e_2]=-\frac{A}{2}(e_1-\varepsilon e_2)+(B-\varepsilon C)e_3, \quad [e_1,e_3]=\varepsilon Ae_3,$
$[e_1,e_4]=C(e_1-\varepsilon e_2)+De_3, \quad [e_2,e_3]=Ae_3,$

$[e_2,e_4]=B(e_1-\varepsilon e_2)+\frac{2B^2+\varepsilon(-4CB+AD+2EB)-2EC+2C^2}{A}e_3, \quad [e_3,e_4]=Ee_3,$

19) $[e_1,e_2]=\varepsilon Ae_2, \quad [e_1,e_3]=\varepsilon Ae_3, \quad [e_1,e_4]=\varepsilon Be_2-\frac{\varepsilon(2CA+B^2-2DB)}{2A}e_3,$
$[e_2,e_4]=Be_2+Ce_3, \quad [e_3,e_4]=Ae_2+De_3$

b3) $\mathbb{R}\ltimes\mathbb{R}^3$ *and* $\mathfrak{g}$ *is described by one of the following sets of conditions:*

20) $[e_1,e_4]=Ae_1+Be_2+Ce_3, \quad [e_2,e_4]=De_1+Ee_2+Fe_3,$
$[e_3,e_4]=\frac{-(B-D)^2+2A^2+2E^2}{2(A+E)}e_3,$

21) $[e_1,e_4]=\frac{-(A-C)^2}{4B}e_1+Ce_2+De_3, \quad [e_2,e_4]=Ae_1+Be_2+Ee_3,$
$[e_3,e_4]=\frac{-(A-C)^2+4B^2}{4B}e_3,$

22) $[e_1,e_4]=-Ae_1+(B+2\varepsilon A)e_2+Ce_3, \quad [e_2,e_4]=Be_1+Ae_2+De_3,$
$[e_3,e_4]=Ee_3,$

In the cases listed above, $\varepsilon=\pm1$ *and* $\delta=\pm1$.

REMARK 1. Comparing the results listed in Theorems 1 and 3, we may observe that the Einstein condition allows more cases for the neutral signature metrics than for the Lorentzian ones, in spite of the fact that Proposition 3 listed three possible cases, against just two in Proposition 4.

In fact, four-dimensional Einstein Lorentzian Lie groups G are isomorphic to either $\mathbb{R}\ltimes\mathbb{R}^3$ or $\mathbb{R}\ltimes H$, while left-invariant neutral Einstein metrics also occur on $\mathbb{R}\ltimes E(2)$ and $\mathbb{R}\ltimes E(1,1)$.

In particular, the derived Lie algebra $[\mathfrak{g},\mathfrak{g}]$ of a four-dimensional Einstein Lorentzian Lie algebra has a derived Lie algebra of dimension at most one, while in the case of four-dimensional Einstein neutral Lie algebras, it may also be two-dimensional.

We can then study curvature properties of left-invariant neutral Einstein metrics on four-dimensional Lie groups, as we already did for the Lorentzian case. Checking conformal flatness condition (3.3) and local symmetry condition (3.4) for cases **1)-22)** listed in Theorem 3, we obtained the following result.

THEOREM 4. *Consider four-dimensional neutral Einstein Lie groups, as classified in Theorem 3 up to isometries. Then, the locally symmetric, conformally flat, flat and Ricci-flat examples are listed in the following* Table II, *where the checkmark "✓" (respectively, "✗") means that the corresponding condition holds for all Lie algebras of that form (respectively, never holds for Lie algebras of that form).*

(G,g)	*Locally symmetric*	*Constant curvature*	*Flat*	*Ricci-flat*
1)	✓	✓	✗	✗
2)	✓	✗	✗	✗
3)	✓	✓	$A=0$	$A=0$
4)	$B=\pm A$	$B=\pm A$	$B=-A$	$B=-A$
5)	✗	✗	✗	✗
6)	$B=C=0$ *or* $A=B\pm C=0$	$B=C=0$ *or* $A=B\pm C=0$	$B=C=0$ *or* $A=B\pm C=0$	✓
7)	✓	✓	$A=0$	$A=0$
8)	✗	✗	✗	✗
9)	$B=\pm A$	$B=\pm A$	$B=-A$	$B=-A$
10)	✗	✗	✗	$B=\pm A$
11)	$E=-\frac{(C-B)D+(B-C)^2}{A}$	$E=-\frac{(C-B)D+(B-C)^2}{A}$ *or* $B-C-D=E=0$ *or* $B-C=E=0$	$E=-\frac{(C-B)D+(B-C)^2}{A}$ *or* $B-C-D=E=0$ *or* $B-C=E=0$	✓
12)	$E=\frac{BC-CD-D^2}{A}$	$E=\frac{BC-CD-D^2}{A}$	$E=\frac{BC-CD-D^2}{A}$	✓
13)	$D=\frac{B^2}{A}$	$D=\frac{B^2}{A}$	$D=\frac{B^2}{A}$	✓
14)	✓	✓	✓	✓
15)	$A=C+D$ *or* $A=\pm 2B$ *or* $(A,B)=(-C-D,\pm D)$ *or* $(A,B)=\frac{1}{3}(C+D,\pm(C-2D))$	$A=C+D$ *or* $A=\pm 2B$	$A=C+D$ *or* $A=\pm 2B$	✓
16)	$\varepsilon=1, A=0$ *or* $\varepsilon=-1, B=0$ *or* $\varepsilon=-1, (A,B)=(0,\pm C)$ *or* $\varepsilon=1, (B,C)=(2A,\pm A)$	$\varepsilon=1, A=0,$ *or* $\varepsilon=-1, B=0$	$\varepsilon=1, A=0$ *or* $\varepsilon=-1, B=0$	✓
17)	✓	✓	$B=\pm A$	$B=\pm A$
18)	$D=\frac{2B(C-\varepsilon B-E)}{A}$	$D=\frac{2B(C-\varepsilon B-E)}{A}$	$D=\frac{2B(C-\varepsilon B-E)}{A}$	✓
19)	✓	✗	✗	✗
20)	$(D,E)=(B,A)$ *or* $(D,E)=(-B,-\frac{B^2}{A})$ *or* $A=B=D=0$ *or* $A=\frac{\varepsilon}{2}(3B+D)$ *and* $E=\frac{\varepsilon}{2}(3D+B)$	$(D,E)=(B,A)$ *or* $(D,E)=(-B,-\frac{B^2}{A})$ *or* $A=B=D=0$ *or* $A=\frac{\varepsilon}{2}(3B+D)$ *and* $E=\frac{\varepsilon}{2}(3D+B)$	$(D,E)=(B,A)$ *or* $(D,E)=(-B,-\frac{B^2}{A})$ *or* $A=B=D=0$ *or* $A=\frac{\varepsilon}{2}(3B+D)$ *and* $E=\frac{\varepsilon}{2}(3D+B)$	✓
21)	$C=-A$	$C=-A$	$C=-A$	✓
22)	$A=0$ *or* $E=A+\varepsilon B$ *or* $E=2(A+\varepsilon B)$	$A=0$ *or* $E=2(A+\varepsilon B)$	$A=0$ *or* $E=2(A+\varepsilon B)$	✓

Table II: Geometry of 4D Einstein neutral Lie groups

REMARK 2. It is a well known fact that four-dimensional simply connected homogeneous *Riemannian* Einstein manifolds are symmetric [10]. The results listed in Tables I and II show that this result does not extend to metrics of different signature. In fact, do there exist left-invariant metrics (both Lorentzian and neutral) on four-dimensional Lie groups, which are Einstein, even Ricci-flat in some cases, but not (locally) symmetric.

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