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LATTICES, COHOMOLOGY AND MODELS OF 6-DIMENSIONAL ALMOST ABELIAN SOLVMANIFOLDS

Abstract. We construct lattices on six-dimensional non-completely solvable almost abelian Lie groups, for which the Mostow condition does not hold. For the corresponding compact quotients, we compute the deRham cohomology (which does not agree in general with the Lie algebra one) and a minimal model. We show that some of these solvmanifolds admit non-invariant symplectic structures and we study formality and Lefschetz properties.

1. Introduction

A solvmanifold M is a compact homogeneous space $M = G/\Gamma$, where G is a connected and simply connected solvable Lie group and Γ is a lattice in G (that is, a discrete subgroup with compact quotient space). In the special case of nilmanifolds (i.e., when the solvable Lie group is nilpotent), if the structure constants are rational, a lattice can always be found [17], while for solvmanifolds the existence of Γ is harder to establish. Lattices determine the topology of solvmanifolds and actually coincide with their fundamental groups (indeed solvmanifolds are Eilenberg-MacLane spaces of type $K(\pi,1)$, i.e. all homotopy groups vanish besides the first). Actually, lattices of solvmanifolds yield their diffeomorphism class (cf. Theorem 4). Much of the rich structure of solvmanifolds is encoded by the Mostow fibration (see Section 2) $N/\Gamma_N = (N\Gamma)/\Gamma \hookrightarrow G/\Gamma \longrightarrow G/(N\Gamma) = \mathbb{T}^k$, where $\mathbb{T}^k$ is a (k-dimensional) torus and N the nilradical of G (the largest nilpotent normal subgroup of G). In general, the Mostow bundle is not principal.

An important special case is when the Lie algebra $\mathfrak{g}$ of G has an abelian ideal of codimension one. In this case the Mostow bundle is a torus bundle over S^1 (actually a mapping torus, cf. [2]), G is called *almost abelian* and G can be written as a semidirect product $\mathbb{R} \ltimes_\varphi \mathbb{R}^n$. The action φ of $\mathbb{R}$ on $\mathbb{R}^n$ is represented by a family of matrices $\varphi(t)$, which encode the monodromy or "twist" in the Mostow bundle (cf. [1]). In particular the Lie algebra $\mathfrak{g}$ has the form $\mathbb{R} \ltimes_{\mathrm{ad}_{X_{n+1}}} \mathbb{R}^n$, where we consider $\mathbb{R}^n$ generated by $\{X_1,...,X_n\}$ and $\mathbb{R}$ by X_{n+1}, and $\varphi(t) = e^{t\,\mathrm{ad}_{X_{n+1}}}$. Moreover, a lattice can always be represented as $\Gamma = \mathbb{Z} \ltimes \mathbb{Z}^n$ (cf. [12]).

In this paper, we find lattices in six-dimensional almost abelian solvable Lie groups, using a criterium of [3] (Proposition 3). The cases we deal with correspond to situations when the deRham cohomology does not agree in general with the Chevalley-Eilenberg cohomology $H^*(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$. Namely the *Mostow condition* does not hold (see [20], [24, Corollary 7.29] and Section 3). Intuitively, in these cases there is some extra twist that modifies the topology and it turns out, in particular, that the cohomology depends on the lattice and not on the solvable Lie algebra only (unlike when the Mostow condition holds). We use two methods to compute cohomology and minimal models

- the *modification of the solvable Lie group* [13, 5] (Section 3). This consists in altering the Lie group G to obtain a new $\tilde{G}$ in such a way that $\tilde{G}/\tilde{\Gamma}$ is diffeomorphic to $G/\tilde{\Gamma}$ (where $\tilde{\Gamma}$ is a finite-index subgroup in Γ, whose algebraic closure is connected) and $H^*(G/\tilde{\Gamma}) \cong H^*(\tilde{\mathfrak{g}})$, where $\tilde{\mathfrak{g}}$ is the Lie algebra of $\tilde{G}$;
- the *Oprea-Tralle method* [22, 23], that consists in applying a result of Felix and Thomas [7] giving a Koszul-Sullivan model for non-nilpotent fibrations.

We summarize the results in Table 1, listing six-dimensional non-completely solvable unimodular, almost abelian Lie groups [3] (see Subsection 3.1) which admit a lattice $\Gamma_{\bar{t}}$ for some choice of $\bar{t} \in \mathbb{R}$ and of the parameters. We use the same notation as in [3]. For each group in Table 1 we study *formality* (**F**), *existence of invariant symplectic structures* (**IS**), *existence of non-invariant symplectic structures induced by ones on the modified Lie algebra* (**S**) and the *Hard Lefschetz property* (**HL**). Minimal models are computed in Section 5 where we prove the following

THEOREM 1. *$G_{6.8}^{p=0}/\Gamma$, $G_{6.11}^{p=0}/\Gamma$, $G_{5.17}^{p,-p,r} \times \mathbb{R}/\Gamma$ and $G_{3.5}^{0} \times \mathbb{R}^3/\Gamma$ are formal, while $G_{6.10}^{a=0}/\Gamma$, $G_{5.14}^{0} \times \mathbb{R}/\Gamma$, $G_{5.18}^{0} \times \mathbb{R}/\Gamma$ are not formal, for every lattice $\Gamma = \Gamma_{\bar{t}}$ considered in Table 1.*

Some of our results answer open questions on formality, the hard Lefschetz property and the cohomology of six-dimensional solvmanifolds (see [3, Proposition 6.18], and in the decomposable case [3, Table 6.3]). Note that there are examples where the cohomology depends strongly on the lattice: for example $H^*(G/\Gamma_\pi) \ncong H^*(G/\Gamma_{2\pi}) \ncong H^*(\mathfrak{g})$, for $G = G_{5.18}^{0} \times \mathbb{R}$.

In theoretical physics, in the context of string theory, both nilmanifolds and solvmanifolds are extensively used as compactification manifolds. Six-dimensional almost abelian solvmanifolds were considered by Andriot, Goi, Minasian and Petrini [1] in string backgrounds whose internal type II compactifications are solvmanifolds. They discuss solutions of the supersymmetry (SUSY) equations, and the twist construction of solvmanifolds which serve as internal spaces. In our paper we try to fill in the constraints on the solutions they observe, which are due to the absence of an isomorphism between the cohomology groups $H^*(\mathfrak{g})$ and $H^*(G/\Gamma)$ for non-completely solvable manifolds (and, more specifically, solvmanifolds not satisfying the Mostow condition). By [11], solutions of the supersymmetry (SUSY) equations of type IIA possess a symplectic half-flat structure, whereas solutions of the IIB system admit a half-flat structure (see e.g. [4] for the definition of half-flat structure, cf. also Section 4).

In Section 4, we prove the following

Proposition 2. *We have the following behaviour concerning half-flatness of (invariant) symplectic structures for the above solvmanifolds:*

- *$G_{6.10}^{a=0}/\Gamma_{2\pi}$ and $G_{5.14}^{0} \times \mathbb{R}/\Gamma_{2\pi}$ admit (non-)invariant symplectic forms which are not half-flat.*

Table 7.1: Six-dimensional almost abelian solvmanifolds admitting lattices (for some value of the parameter) but not satisfying the Mostow condition

G	$\Gamma_{\bar{t}}$	$H^*(\mathfrak{g})$	$H^*(G/\Gamma_{\bar{t}})$	**F**	**IS**	**S**	**HL**
$G^{p=0}_{6.8}$	$\bar{t}=2\pi$	$b_1=1,b_2=1,b_3=2$	$b_1=3,b_2=3,b_3=2$	Yes	No	No	$\diagdown$
	$\bar{t}=\pi,\frac{\pi}{2},\frac{\pi}{3}$		$b_1=1,b_2=1,b_3=2$	Yes	No	$\diagdown$	$\diagdown$
$G^{a=0}_{6.10}$	$\bar{t}=2\pi$	$b_1=2,b_2=3,b_3=4$	$b_1=4,b_2=7,b_3=8$	No	Yes	Yes	No$^\times$
	$\bar{t}=\pi,\frac{\pi}{2},\frac{\pi}{3}$		$b_1=2,b_2=3,b_3=4$	No	Yes	$\diagdown$	No*
$G^{p=0}_{6.11}$	$\bar{t}=2\pi$	$b_1=1,b_2=1,b_3=2$	$b_1=3,b_2=3,b_3=2$	Yes	No	No	$\diagdown$
	$\bar{t}=\pi,\frac{\pi}{2},\frac{\pi}{3}$		$b_1=1,b_2=1,b_3=2$	Yes	No	$\diagdown$	$\diagdown$
$G^0_{5.14}\times\mathbb{R}$	$\bar{t}=2\pi$	$b_1=3,b_2=5,b_3=6$	$b_1=5,b_2=11,b_3=14$	No	Yes	Yes	No$^\times$
	$\bar{t}=\pi,\frac{\pi}{2},\frac{\pi}{3}$		$b_1=3,b_2=5,b_3=6$	No	Yes	$\diagdown$	No*
$G^{p,-p,r}_{5.17}\times\mathbb{R}$ $r=\frac{r_1}{r_2}\in\mathbb{Q}$	$\bar{t}=2\pi r_2$	if $p\neq 0,r\neq\pm1$ $b_1=2,b_2=1,b_3=0$	$p\neq 0: b_1=2,b_2=5,b_3=8$ $p=0: b_1=6,b_2=15,b_3=20$	Yes	Yes	Yes	Yes$^\times$
	$\bar{t}=\pi$, r even	if $p=0,r\neq\pm1$ or $p\neq 0,r=\pm1$ $b_1=2,b_2=3,b_3=4$	$p\neq 0: b_1=2,b_2=1,b_3=0$ $p=0: b_1=4,b_2=7,b_3=8$	Yes	Yes	$\diagdown$	Yes*
	$\bar{t}=\pi$, r odd		$p\neq 0: b_1=2,b_2=5,b_3=8$ $p=0: b_1=2,b_2=7,b_3=12$	Yes	Yes	$\diagdown$	Yes*
	$\bar{t}=\frac{\pi}{2},r\equiv_4 0$		$p=0: b_1=4,b_2=7,b_3=8$	Yes	Yes	$\diagdown$	Yes*
	$\bar{t}=\frac{\pi}{2}$, $r\equiv_4 1,3$	if $p=0,r=\pm1$ $b_1=2,b_2=5,b_3=8$	$p\neq 0: b_1=2,b_2=3,b_3=4$ $p=0: b_1=2,b_2=5,b_3=8$	Yes	Yes	$\diagdown$	Yes*
	$\bar{t}=\frac{\pi}{2},r\equiv_4 2$		$p=0: b_1=2,b_2=3,b_3=4$	Yes	Yes	$\diagdown$	Yes*
$G^0_{5.18}\times\mathbb{R}$	$\bar{t}=2\pi$	$b_1=2,b_2=3,b_3=4$	$b_1=4,b_2=9,b_3=12$	No	Yes	Yes	No$^\times$
	$\bar{t}=\pi$,		$b_1=2,b_2=5,b_3=8$	No	Yes	$\diagdown$	No*
	$\bar{t}=\frac{\pi}{2},\frac{\pi}{3}$		$b_1=2,b_2=3,b_3=4$	No	Yes	$\diagdown$	No*
$G^0_{3.5}\times\mathbb{R}^3$	$\bar{t}=2\pi$	$b_1=4,b_2=7,b_3=8$	$b_1=6,b_2=15,b_3=20$	Yes	Yes	Yes	Yes$^\times$
	$\bar{t}=\pi,\frac{\pi}{2},\frac{\pi}{3}$		$b_1=4,b_2=7,b_3=8$	Yes	Yes	$\diagdown$	Yes*

$^\times$ for both the invariant and non-invariant symplectic structures considered.
* for the invariant symplectic structures.

$\diagdown$

- $G^{p,-p,r}_{5.17}\times\mathbb{R}/\Gamma_{2\pi r_2}$ *($r=\frac{r_1}{r_2}\in\mathbb{Q}$) admits an invariant symplectic form which is* *half-flat only for* $p \geqslant 0$ *and* $r = 1$, *and a non-invariant symplectic half-flat form.*

- $G^0_{5.18} \times \mathbb{R}/\Gamma_{2\pi}$ *and* $G^0_{3.5} \times \mathbb{R}^3/\Gamma_{2\pi}$ *admit non-invariant symplectic half-flat forms.*

2. The Mostow bundle and almost abelian solvmanifolds

Let $M = G/\Gamma$ be a solvmanifold and N the nilradical of G (of course, N is G if and only if M is a nilmanifold). Then $\Gamma_N := \Gamma \cap N$ is a lattice in N, $\Gamma N = N\Gamma$ is closed in G and $G/(N\Gamma) =: \mathbb{T}^k$ is a torus. Thus we have the so-called *Mostow fibration* [19]:

$$\begin{tikzcd} N/\Gamma_N = (N\Gamma)/\Gamma \arrow[r, hook] & G/\Gamma \arrow[d] \\ & \mathbb{T}^k = G/(N\Gamma) \end{tikzcd}$$

In six dimensions, the nilradical $\mathfrak{n}$ can have any dimension from 3 to 6. Dimension 6 corresponds clearly to nilmanifolds. In the codimension-one case the Mostow fibration is simpler. A connected and simply-connected solvable Lie group G with nilradical N is called *almost nilpotent* if its nilradical has codimension one. The group G is then given by the semi-direct product $G = \mathbb{R} \ltimes_\varphi N$, where φ is some action on N depending on the direction $\mathbb{R}$

$$(t_1, x_1) \cdot (t_2, x_2) = (t_1 \cdot t_2, x_1 \cdot \varphi(t_1)(x_2)) \qquad (t_i, x_i) \in \mathbb{R} \times N \, .$$

In general, we label by t the coordinate on $\mathbb{R}$ and by $X_{n+1} = \partial_t$, $n = \dim N$, the corresponding vector of the algebra. From a geometrical point of view, $\varphi(t)$ encodes the monodromy of the Mostow bundle. An *almost abelian solvable group* is an almost nilpotent group whose nilradical is abelian $N = \mathbb{R}^n$. In this case, the action of $\mathbb{R}$ on N is given by

$$\varphi(t) = e^{t \ \mathrm{ad} X_{n+1}} \, .$$

In general, finding lattices in solvable Lie groups is a hard task. Only a necessary criterium is known, namely that G be unimodular [18, Lemma 6.2]. A nice feature of almost abelian solvable groups is that there is a criterion for the existence of a lattice [3]

Proposition 3. *Let* $G = \mathbb{R} \ltimes_\varphi \mathbb{R}^n$ *be an almost abelian solvable Lie group. Then* G *admits a lattice if and only if there exists a* $t_0 \neq 0$ *for which* $\varphi(t_0)$ *can be conjugated to an integer matrix.*

We shall call *almost abelian solvmanifold* the quotient of an almost abelian solvable Lie group by a lattice. Lattices of solvmanifolds determine the diffeomorphism class. Indeed,

THEOREM 4. *[24, Theorem 3.6] Let* G_i/Γ_i *be solvmanifolds for* $i \in \{1, 2\}$ *and* $\psi : \Gamma_1 \to \Gamma_2$ *an isomorphism. Then there exists a diffeomorphism* $\Psi : G_1 \to G_2$ *such that* *(i)* $\Psi|_{\Gamma_1} = \psi$,

(ii) $\Psi(p\gamma) = \Psi(p)\psi(\gamma)$, *for any* $\gamma \in \Gamma_1$ *and any* $p \in G_1$.

As a consequence two compact solvmanifolds with isomorphic fundamental groups are diffeomorphic.

3. Modification of the cohomology

If the algebraic closures $\mathcal{A}(\mathrm{Ad}_G(G))$ and $\mathcal{A}(\mathrm{Ad}_G(\Gamma))$ are equal, one says that G and Γ satisfy the *Mostow condition* (see [24] for more details and definitions). In this case, the de Rham cohomology $H^*(M)$ of the compact solvmanifold $M = G/\Gamma$ can be computed by the Chevalley-Eilenberg cohomology $H^*(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of G (see [20] and [24, Corollary 7.29]); actually, one has an isomorphism $H^*(M) \cong H^*(\mathfrak{g})$. A special case is provided by nilmanifolds (Nomizu's Theorem, [21]) and more generally if G is *completely solvable* [14], i.e. the linear operators $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$, $X \in \mathfrak{g}$ have only real eigenvalues. For almost abelian solvmanifolds, Gorbatsevich found a criterion to decide whether the Mostow condition holds [12]:

Proposition 5. *The Mostow condition is satisfied if and only if* πi *cannot be written as a rational linear combination of the eigenvalues of* $t_0 \mathrm{ad}_{X_{n+1}}$, *where* Γ *is generated by* t_0.

Let $M = G/\Gamma$ be a solvmanifold. By [24, Theorem 6.11, p.93] it is not restrictive to suppose that $\mathcal{A}(\mathrm{Ad}_G(\Gamma))$ is connected. Otherwise we could restrict to a finite-index subgroup $\tilde{\Gamma}$ of Γ. This is equivalent to passing from $M = G/\Gamma$ to $G/\tilde{\Gamma}$, which is a finite covering of M. Borel's density theorem (see e.g. [24, Theorem 5.5]) says there exists a compact torus $\mathbb{T}_{cpt}$ such that $\mathbb{T}_{cpt}\mathcal{A}(\mathrm{Ad}_G(\Gamma)) = \mathcal{A}(\mathrm{Ad}_G(G))$. The main step of the "modification method" is the following

THEOREM 6. *[5] Let G be a solvable simply connected Lie group,* Γ *a lattice in G such that* G/Γ *is a solvmanifold and* $\mathcal{A}(\mathrm{Ad}_G(\Gamma))$ *is connected. Suppose* $\mathcal{A}(\mathrm{Ad}_G(G)) = \mathbb{T}_c\mathcal{A}(\mathrm{Ad}_G(\Gamma))$, *with* $\mathbb{T}_c$ *the maximal compact torus of* $\mathcal{A}(\mathrm{Ad}_G(G))$. *Then there exists a normal simply connected subgroup* $\tilde{G}$ *of* $\mathbb{T}_c \ltimes G$ *such that* $\mathcal{A}(\mathrm{Ad}_{\tilde{G}}(\tilde{G})) = \mathcal{A}(\mathrm{Ad}_{\tilde{G}}(\Gamma))$.

The Mostow condition holds for $\tilde{G}$, so $H^*(\tilde{G}/\Gamma) = H^*(\tilde{\mathfrak{g}})$. The modified solvable group $\tilde{G}$ is obtained from G by killing the action of subtorus $\mathbb{S}_c$ that we get by comparing the compact and $\mathbb{C}$-diagonalizable parts of $\mathcal{A}(\mathrm{Ad}_G(G))$ and $\mathcal{A}(\mathrm{Ad}_G(\Gamma))$. More precisely, let $\overline{\mathbb{S}}_c$ be a maximal compact torus of $\mathcal{A}(\mathrm{Ad}_G(\tilde{\Gamma}))$ contained in $\mathbb{T}_c$. Let $\mathbb{S}_c$ be a sub-torus of $\mathbb{T}_c$ complementary to $\overline{\mathbb{S}}_c$ so that $\mathbb{T}_c = \mathbb{S}_c \times \overline{\mathbb{S}}_c$. Let σ be the composite of the homomorphisms

$$\sigma : G \xrightarrow{\mathrm{Ad}} \mathcal{A}(\mathrm{Ad}_G(G)) \xrightarrow{\mathrm{proj}} \mathbb{T}_c \xrightarrow{\mathrm{proj}} \mathbb{S}_c \xrightarrow{x \to x^{-1}} \mathbb{S}_c.$$

One uses σ to get rid of $\mathbb{S}_c$ (see [5]). It turns out that $\tilde{G}$ is diffeomorphic to G, both are simply connected and, by Theorem 4, $\tilde{G}/\Gamma$ is diffeomorphic to G/Γ. Therefore $H^*(G/\Gamma) = H^*(\tilde{G}/\Gamma)$ and we get

Corollary 7. *Let G be a solvable simply connected Lie group and Γ a lattice in G such that G/Γ is a solvmanifold and $\mathcal{A}(\mathrm{Ad}_G(\Gamma))$ is connected. Then*

$$H^*(G/\Gamma) = H^*(\tilde{G}/\Gamma) = H^*(\tilde{\mathfrak{g}}),$$

where $\tilde{\mathfrak{g}}$ is the Lie algebra of $\tilde{G}$.

Observe that the lattice Γ has not been modified. Indeed, as already remarked, G/Γ is an Eilenberg-MacLane space, so its topology is determined by the fundamental group Γ only.

REMARK 1. The Lie algebra $\tilde{\mathfrak{g}}$ of $\tilde{G}$ can be identified with

$$\tilde{\mathfrak{g}} = \{(X_{\mathfrak{s}}, X) \mid X \in \mathfrak{g}\}$$

with Lie bracket:

$$[(X_{\mathfrak{s}}, X), (Y_{\mathfrak{s}}, Y)] = (0, [X,Y] - \mathrm{ad}(X_{\mathfrak{s}})(Y) + \mathrm{ad}(Y_{\mathfrak{s}})(X)).$$

where $X_{\mathfrak{s}}$ is the image $\sigma_*(X)$ of $X \in \mathfrak{g}$. (see [5, Proposition 6.1])

In the general case of a lattice Γ, the method runs as follows. Given $M = G/\Gamma$, there is a finite covering space $\tilde{M} = G/\tilde{\Gamma}$, i.e. $\Gamma/\tilde{\Gamma}$ is a finite group, with $\mathcal{A}(\mathrm{Ad}_G(\tilde{\Gamma})$ connected. Hence $H^*(G/\Gamma) \cong H^*(G/\tilde{\Gamma})^{\Gamma/\tilde{\Gamma}}$ (the invariant elements under the action of the finite group $\Gamma/\tilde{\Gamma}$).

REMARK 2. There is a natural injection $H^*(\mathfrak{g}) \hookrightarrow H^*(G/\Gamma)$, [22, Theorem 3.2.10]. Hence cohomology classes in $H^*(\mathfrak{g})$ correspond to cohomology classes of invariant forms in $H^*(G/\Gamma)$.

3.1. Six-dimensional almost abelian Lie groups

We are interested in six-dimensional, unimodular almost abelian Lie groups which are not completely solvable. There are eleven such Lie groups that can admit a lattice and their Lie algebras are [3]:

$\mathfrak{g}_{6.8}^{a,b,c,p}$ $[X_1,X_6] = aX_1,\ [X_2,X_6] = bX_2,\ [X_3,X_6] = cX_3,\ [X_4,X_6] = pX_4 - X_5,$
$[X_5,X_6] = X_4 + pX_5, \qquad a+b+c+2p = 0, \quad 0 < |c| \leqslant |b| \leqslant |a|.$

$\mathfrak{g}_{6.9}^{a,b,p}$ $[X_1,X_6] = aX_1,\ [X_2,X_6] = bX_2,\ [X_3,X_6] = X_2 + bX_3,$
$[X_4,X_6] = pX_4 - X_5,\ [X_5,X_6] = X_4 + pX_5, \qquad a+2b+2p = 0, \quad a \neq 0.$

$\mathfrak{g}_{6.10}^{a,-\frac{3}{2}a}$	$[X_1,X_6]=aX_1,\ [X_2,X_6]=X_1+aX_2,\ [X_3,X_6]=X_2+aX_3,$ $[X_4,X_6]=-\frac{3}{2}aX_4-X_5,\ [X_5,X_6]=X_4-\frac{3}{2}aX_5.$
$\mathfrak{g}_{6.11}^{a,p,q,s}$	$[X_1,X_6]=aX_1,\ [X_2,X_6]=pX_2-x_3,\ [X_3,X_6]=X_2+pX_3,$ $[X_4,X_6]=qX_4-sX_5,\ [X_5,X_6]=sX_4+qX_5,\quad a+2p+2q=0,\quad as\neq 0.$
$\mathfrak{g}_{6.12}^{-4p,p}$	$[X_1,X_6]=-4pX_1,\ [X_2,X_6]=pX_2-X_3,\ [X_3,X_6]=X_2+pX_3,$ $[X_4,X_6]=X_2+pX_4-X_5,\ [X_5,X_6]=X_3+X_4+pX_5,\qquad p\neq 0.$
$\mathfrak{g}_{5.13}^{-1-2q,q,r}\oplus\mathbb{R}$	$[X_1,X_5]=X_1,\ [X_2,X_5]=(-1-2q)X_2,\ [X_3,X_5]=qX_3-rX_4,$ $[X_4,X_5]=rX_3+qX_4,\quad q\neq -\frac{1}{2},\quad r\neq 0,\quad -1\leqslant q\leqslant 0.$
$\mathfrak{g}_{5.14}^{0}\oplus\mathbb{R}$	$[X_2,X_5]=X_1,\ [X_3,X_5]=-X_4,\ [X_4,X_5]=X_3.$
$\mathfrak{g}_{5.17}^{p,-p,r}\oplus\mathbb{R}$	$[X_1,X_5]=pX_1-X_2,\ [X_2,X_5]=X_1+pX_2,\ [X_3,X_5]=-pX_3-rX_4,$ $[X_4,X_5]=rX_3-pX_4,\quad r\neq 0.$
$\mathfrak{g}_{5.18}^{0}\oplus\mathbb{R}$	$[X_1,X_5]=-X_2,\ [X_2,X_5]=X_1,\ [X_3,X_5]=X_1-X_4,\ [X_4,X_5]=X_2+X_3.$
$\mathfrak{g}_{4.6}^{-2p,p}\oplus\mathbb{R}^2$	$[X_1,X_4]=-2pX_1,\ [X_2,X_4]=pX_2-X_3,\ [X_3,X_4]=X_2+pX_3,\quad p>0.$
$\mathfrak{g}_{3.5}^{0}\oplus\mathbb{R}^3$	$[X_1,X_3]=-X_2,\ [X_2,X_3]=X_1.$

Next, we apply Proposition 3 to determine for which values of $t=\bar{t}$ tha map $\varphi(\bar{t})=\exp(\bar{t}\,\mathrm{ad}_{X_6})$ determines a lattice $\Gamma_{\bar{t}}$ in G. Note in particular that as consequence of Proposition 3, both the characteristic polynomial and the minimal polynomial of $\exp(\bar{t}\,\mathrm{ad}_{X_{n+1}})$ must have integer coefficients. For computations we used the software Maple. To illustrate the method, we develop in detail the case $G_{6.8}^{a,b,c,p}$, just writing down the final results for the other cases.

- $G_{6.8}^{a,b,c,p}$: From the structure equations of $\mathfrak{g}_{6.8}^{a,b,c,p}$, we get

$$\exp(t\,\mathrm{ad}_{X_6})=\begin{pmatrix} e^{t(-b-c-2p)} & 0 & 0 & 0 & 0\\ 0 & e^{tb} & 0 & 0 & 0\\ 0 & 0 & e^{tc} & 0 & 0\\ 0 & 0 & 0 & e^{tp}\cos t & e^{tp}\sin t\\ 0 & 0 & 0 & -e^{tp}\sin t & e^{tp}\cos t \end{pmatrix}$$

The eigenvalues of $t\,\mathrm{ad}_{X_6}$ are

$$t(p\pm i),\ t\,b,\ t\,c,-(b+c+2p)t\ ,$$

so the Mostow condition does not hold when $t=\bar{t}$ is a rational multiple of π. To apply Corollary 7 we need to have $\mathcal{A}(\mathrm{Ad}_G(\Gamma_{\bar{t}}))$ connected. Using the same arguments as in the proof of Proposition 5 (see [12]), one can see that this is the case for $\bar{t}=2\pi$. Indeed, using the Jordan decomposition into semisimple and nilpotent parts, the only blocks whose algebraic closures are not in general connected are the subgroups given by exponentiating the roots of the complex eigenvalues. They are the cyclic subgroups $\begin{pmatrix} \cos(n\bar{t}) & \sin(n\bar{t}) \\ -\sin(n\bar{t}) & \cos(n\bar{t}) \end{pmatrix}$, $n \in \mathbb{Z}$, for $t = \bar{t} \in \mathbb{Q}\pi$. The above subgroups are connected only if trivial, i.e., for $\bar{t} = 2\pi$.

Let us consider then $\Gamma_{\bar{t}}$ for $\bar{t} = 2\pi$. Setting $e^{2\pi b} = w$, $e^{2\pi c} = v$, $e^{-2\pi p} = k$, we have

$$\exp(2\pi\,\mathrm{ad}_{X_6}) = \begin{pmatrix} \frac{k^2}{wv} & 0 & 0 & 0 & 0 \\ 0 & w & 0 & 0 & 0 \\ 0 & 0 & v & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{k} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{k} \end{pmatrix}$$

Its minimal polynomial is

$$\begin{aligned} \mathrm{MinPol}\,(x) \quad = \quad & k - \frac{k^3wv + wk^2 + k^2v + v^2w^2}{wvk}x + \frac{vk^3 + wk^3 + w^2v^2k + k^2 + wv^2 + w^2v}{wvk}x^2 + \\ & - \frac{k^3 + kwv^2 + kw^2v + w}{wvk}x^3 + x^4 \end{aligned}$$

So, it can have integer coefficients only if $k \in \mathbb{Z}$. We set $w + v = r$, $wv = s$ and the coefficients p_i of x^i in $\mathrm{MinPol}\,(x)$ become:

$$p_1 = \frac{k^3s + k^2r + s^2}{ks} = k^2 + \frac{k^2r + s^2}{ks},$$
$$p_2 = \frac{k^3r + kr^2 + k^2 + rs}{ks}, \qquad p_3 = \frac{k^3 + krs + s}{ks}.$$

Hence $p_1 \in \mathbb{Z}$ if and only if $q_1 = \frac{k^2r+s^2}{ks} \in \mathbb{Z}$ and $p_2 - kq_1 = \frac{k^2+rs}{ks}$. If $p_1, p_2 \in \mathbb{Z}$ then with $h := p_2 - kq_1 \in \mathbb{Z}$ and $s = \frac{k^2}{hk-r}$ we have $p_3 = \frac{hk^2+1}{k} = hk + \frac{1}{k}$. So $p_3 \in \mathbb{Z}$ if and only if $\frac{1}{k} \in \mathbb{Z}$, but $k \in \mathbb{Z}$, so $k = 1$ and $p = 0$. *Therefore* $\Gamma_{2\pi}$ *is not a lattice for* $p \neq 0$.

Next we check the existence of a lattice for $p = 0$. The characteristic polynomial of $\exp(2\pi\,\mathrm{ad}_{X_6})$ has coefficients

$$a_0 = -1 \quad a_1 = 2 + \frac{r + s^2}{s} \quad a_2 = -1 - \frac{2s^2 + 2r + rs + 1}{s} = -1 - 2\frac{s^2 + r}{s} - \frac{rs + 1}{s}$$
$$a_3 = 1 + \frac{2rs + 2 + s^2 + r}{s} = 1 + 2\frac{rs + 1}{s} + \frac{s^2 + r}{s} \qquad a_4 = -2 - \frac{rs + 1}{s}$$

So $a_1, a_2, a_3, a_4 \in \mathbb{Z}$ if and only if $\dfrac{s^2 + r}{s}, \dfrac{rs + 1}{s} \in \mathbb{Z}$ and we must check that the solutions make w, v positive. For this we consider the system

$$(1) \qquad \begin{cases} \dfrac{s^2 + r}{s} = h_1 \\ \dfrac{rs + 1}{s} = h_2 \\ r > 0 \\ 0 < s \leqslant \frac{r^2}{4} \end{cases}$$

that admits solutions for some values of the integers h_1 and h_2 (for example $h_1 = 5, h_2 = 6$). In particular we can not accept $\{s = r-1\}$, because these correspond to $b = 0$ or $c = 0$, nor $\{s = 1\}$, because it corresponds to $a = 0$. Thus, for $p = 0$, we can find values of b and c (and $a = -b-c$) such that the characteristic polynomial of $\exp(2\pi \mathrm{ad}_{X_6})$ has integer coefficients and we can check by direct computation that $\exp(2\pi\,\mathrm{ad}_{X_6})$ is conjugate to $A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -h_1 & 0 & 0 \\ 0 & 1 & h_2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$. Therefore, *for some choice of the parameters b and c, $\Gamma_{2\pi}$ is a lattice*. We denote the group $G_{6.8}^{a,b,c,0}$, with the above choices of a, b, c, by $G_{6.8}^{p=0}$ for short.

Next we verify the Mostow condition: the eigenvalues of $2\pi\mathrm{ad}_{X_6}$ are

$$\pm 2\pi i,\ 2\pi b,\ 2\pi c,\ -(b+c)2\pi,$$

so we can easily find a linear combination in $\mathbb{Q}$ that gives πi. Hence, by Proposition 5 the Mostow condition does not hold.

To compute the cohomology we have then to apply the modification method. The Lie group $G_{6.8}^{p=0}$ is defined by the map

$$\exp(t\,\mathrm{ad}_{X_6}) = \begin{pmatrix} e^{t(-b-c)} & 0 & 0 & 0 & 0 \\ 0 & e^{tb} & 0 & 0 & 0 \\ 0 & 0 & e^{tc} & 0 & 0 \\ 0 & 0 & 0 & \cos t & \sin t \\ 0 & 0 & 0 & -\sin t & \cos t \end{pmatrix}$$

By definition the sub-torus $\mathbb{S}_c$ is the compact part of the $\mathbb{C}$-diagonalizable part, that is the product of $\mathbb{S}_c$ and the $\mathbb{R}$-diagonalizable torus, so it is just the circle given by the block $\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$. Therefore $\tilde{G}_{6.8}^{p=0}$ is defined by $\begin{pmatrix} e^{t(-b-c)} & 0 & 0 & 0 & 0 \\ 0 & e^{tb} & 0 & 0 & 0 \\ 0 & 0 & e^{tc} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$.

Hence, the structure constants of $\tilde{\mathfrak{g}}_{6.8}^{p=0}$ are

$$[X_1, X_6] = -(b+c)X_1,\ [X_2, X_6] = bX_2,\ [X_3, X_6] = cX_3.$$

Consequently $\tilde{\mathfrak{g}}_{6.8}^{p=0}$ *is isomorphic to the decomposable solvable Lie algebra* $\mathfrak{g}_{4.5}^{k,-k-1} \oplus \mathbb{R}^2$ *(for some k)*, cf. [3, Appendix A]. By Corollary 7 the cohomology of $G_{6.8}^{p=0}/\Gamma_{2\pi}$ is given by the cohomology groups of $\tilde{\mathfrak{g}}_{6.8}^{p=0}$. If we set $(\tilde{\mathfrak{g}}_{6.8}^{p=0})^* = \langle \alpha^1, \ldots, \alpha^6 \rangle$, where α^i are the dual forms of X_i $(i = 1, \ldots, 6)$, they are

$$H^1(G_{6.8}^{p=0}/\Gamma_{2\pi}) = H^1(\tilde{\mathfrak{g}}_{6.8}^{p=0}) = \langle \alpha^4, \alpha^5, \alpha^6 \rangle$$
$$H^2(G_{6.8}^{p=0}/\Gamma_{2\pi}) = H^2(\tilde{\mathfrak{g}}_{6.8}^{p=0}) = \langle \alpha^{45}, \alpha^{46}, \alpha^{56} \rangle$$
$$H^3(G_{6.8}^{p=0}/\Gamma_{2\pi}) = H^3(\tilde{\mathfrak{g}}_{6.8}^{p=0}) = \langle \alpha^{123}, \alpha^{456} \rangle$$

Here and in the sequel, for the sake of simplicity, we do not use any special symbol for the cohomology class and just write one representative.

Next, *let us investigate if there are integer values k for which* $\Gamma_{2\pi/k}$ *is a lattice in* $G_{6.8}^{p=0}$.

$$\exp(2\pi/k\,\mathrm{ad}_{X_6}) = \begin{pmatrix} e^{2\pi(-b-c)/k} & 0 & 0 & 0 & 0 \\ 0 & e^{2\pi b/k} & 0 & 0 & 0 \\ 0 & 0 & e^{2\pi c/k} & 0 & 0 \\ 0 & 0 & 0 & \cos 2\pi/k & \sin 2\pi/k \\ 0 & 0 & 0 & -\sin 2\pi/k & \cos 2\pi/k \end{pmatrix}$$

We set $e^{2\pi b/k} = w$, $e^{2\pi c/k} = v$, $\cos 2\pi/k = u/2$, and $w+v=r$, $wv=s$. Then the coefficients of the characteristic polynomial of $\exp(2\pi/k\,\mathrm{ad}_{X_6})$ become:

$$a_1 = \frac{us+r+s^2}{s} = u + \frac{r+s^2}{s} \qquad a_2 = -1 - \frac{ur+us^2+1+rs}{s}$$
$$a_3 = 1 + \frac{u+urs+s^2+r}{s} \qquad a_4 = -\frac{1+rs+us}{s} = -\frac{1+rs}{s} - u$$

Thus $a_2 = -ua_1 + a_4 + u + u^2 - 1$ and $a_3 = -ua_4 + a_1 - u - u^2 + 1$, so if $a_1, a_2, a_3, a_4 \in \mathbb{Z}$, then $a_1 + a_4$ and $a_2 + a_3$ are integers and $u \in \mathbb{Q}$. Therefore, if $\cos 2\pi/k$ is not rational, $\Gamma_{2\pi/k}$ is not a lattice. If $u \in \mathbb{Q}$, the characteristic polynomial has integer coefficients if and only if the same system (1) as the one for $\bar{t} = 2\pi$ admits a solution. Again by direct computation we can check that the matrix A is conjugate to $\exp(\bar{t}\,\mathrm{ad}_{X_6})$, for every $\bar{t}$ such that $\cos\bar{t} = \pm 1, 0, \pm\frac{1}{2}$.

Hence *we have a lattice in* $G_{6.8}^{p=0}$ *for* $\bar{t} = \frac{2\pi}{k}$ *such that* $\cos\bar{t} = \pm 1, 0, \pm\frac{1}{2}$.

We compute the cohomology groups by finding the invariants of the action of $\Gamma_{\bar{t}}/\Gamma_{2\pi}$ for $\bar{t} = \frac{\pi}{2}, \frac{\pi}{3}, \pi$. For the other cases we get the same result for the cohomology.

For $\bar{t} = \frac{\pi}{2}$, let

$$\psi_2 := \exp(\pi/2\,\mathrm{ad}_{X_6})^t = \begin{pmatrix} e^{\pi(-b-c)/2} & 0 & 0 & 0 & 0 \\ 0 & e^{\pi b/2} & 0 & 0 & 0 \\ 0 & 0 & e^{\pi c/2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Hence

$$\psi_2\alpha^4 = \alpha^5,\ \psi_2\alpha^5 = -\alpha^4,\ \psi_2\alpha^6 = \alpha^6 \quad \Longrightarrow \quad H^1(G_{6.8}^{p=0}/\Gamma_{\pi/2}) = \langle\alpha^6\rangle,$$
$$\psi_2\alpha^{45} = \alpha^{45},\ \psi_2\alpha^{46} = \alpha^{56},\ \psi_2\alpha^{56} = -\alpha^{46} \quad \Longrightarrow \quad H^2(G_{6.8}^{p=0}/\Gamma_{\pi/2}) = \langle\alpha^{45}\rangle,$$
$$\psi_2\alpha^{123} = \alpha^{123},\ \psi_2\alpha^{456} = \alpha^{456} \quad \Longrightarrow \quad H^3(G_{6.8}^{p=0}/\Gamma_{\pi/2}) = \langle\alpha^{123},\ \alpha^{456}\rangle.$$

Similarly, one gets

$$H^1(G_{6.8}^{p=0}/\Gamma_{\pi/3}) = \langle\alpha^6\rangle, \qquad H^1(G_{6.8}^{p=0}/\Gamma_{\pi}) = \langle\alpha^6\rangle$$
$$H^2(G_{6.8}^{p=0}/\Gamma_{\pi/3}) = \langle\alpha^{45}\rangle, \qquad H^2(G_{6.8}^{p=0}/\Gamma_{\pi}) = \langle\alpha^{45}\rangle$$
$$H^3(G_{6.8}^{p=0}/\Gamma_{\pi/3}) = \langle\alpha^{123},\ \alpha^{456}\rangle. \qquad H^3(G_{6.8}^{p=0}/\Gamma_{\pi}) = \langle\alpha^{123},\ \alpha^{456}\rangle$$

• $G^{a,b,p}_{6.9}$: again $\exp(t\,\mathrm{ad}_{X_6})$ has a pair of complex-conjugate roots. One would get a lattice $\Gamma_{\bar t}$ violating the Mostow condition and $\mathcal{A}(\mathrm{Ad}_G(\Gamma_{\bar t}))$ connected for $\bar t = 2\pi$, but one can show that *there is no lattice for* $\bar t = 2\pi$.

• $G^{a}_{6.10}$: $\exp(t\,\mathrm{ad}_{X_6})$ has a pair complex-conjugate roots. *If* $a \neq 0$ *there is no lattice* $t = 2\pi$*, but* $\Gamma_{2\pi}$ *is a lattice for* $G^0_{6.10}$. The eigenvalues of $2\pi\mathrm{ad}_{X_6}$ are $\pm 2\pi i$, so the Mostow condition does not hold. The Lie algebra $\tilde{\mathfrak{g}}^0_{6.10}$ has structure constants $[X_2, X_6] = X_1$, $[X_3, X_6] = X_2$ and *is isomorphic to* $\mathfrak{g}_{4.1} \oplus \mathbb{R}^2$, cf. [3, Appendix A]. The cohomology groups of $G^0_{6.10}/\Gamma_{2\pi}$ are

$$\begin{aligned}
H^1(G^0_{6.10}/\Gamma_{2\pi}) &= H^1(\tilde{\mathfrak{g}}^0_{6.10}) = \langle \alpha^3, \alpha^4, \alpha^5, \alpha^6 \rangle\\
H^2(G^0_{6.10}/\Gamma_{2\pi}) &= H^2(\tilde{\mathfrak{g}}^0_{6.10}) = \langle \alpha^{16}, \alpha^{23}, \alpha^{34}, \alpha^{35}, \alpha^{45}, \alpha^{46}, \alpha^{56} \rangle\\
H^3(G^0_{6.10}/\Gamma_{2\pi}) &= H^3(\tilde{\mathfrak{g}}^0_{6.10}) = \langle \alpha^{123}, \alpha^{126}, \alpha^{146}, \alpha^{156}, \alpha^{234}, \alpha^{235}, \alpha^{345}, \alpha^{456} \rangle
\end{aligned}$$

The subgroups $\Gamma_{2\pi/k}$ ($k \in \mathbb{Z}$) are also lattices if and only if $2\cos\left(\frac{2\pi}{k}\right) \in \mathbb{Z}$. In all these cases we have

$$\begin{aligned}
H^1(G^0_{6.10}/\Gamma_{\bar t}) &= \langle \alpha^3, \alpha^6 \rangle\\
H^2(G^0_{6.10}/\Gamma_{\bar t}) &= \langle \alpha^{16}, \alpha^{23}, \alpha^{45} \rangle\\
H^3(G^0_{6.10}/\Gamma_{\bar t}) &= \langle \alpha^{123}, \alpha^{126}, \alpha^{345}, \alpha^{456} \rangle.
\end{aligned}$$

REMARK 3. The lattice Γ_π was found in [3, Proposition 6.18]. Part (ii) states that if there is a lattice in $G^0_{6.10}$ for which the corresponding solvmanifold has $b_1 = 2$ and $b_2 = 3$, then the latter is symplectic and not formal. Here we show that Γ_π is such a lattice. We will deal about symplectic structures and formality in Section 4.

• $G^{a,p,q,s}_{6.11}$: from the structure equations of $\mathfrak{g}^{a,p,q,s}_{6.11}$ we get

$$\mathrm{ad}_{X_6} = \begin{pmatrix} -2(p+q) & 0 & 0 & 0 & 0\\ 0 & p & 1 & 0 & 0\\ 0 & -1 & p & 0 & 0\\ 0 & 0 & 0 & q & s\\ 0 & 0 & 0 & -s & q \end{pmatrix}.$$

We have two non-diagonal blocks with a couple of complex-conjugate roots. Hence there could be several situations where the Mostow condition fails.

(i): *If* $s \in \mathbb{Q}$ *(say* $s = \frac{s_1}{s_2}$*), then* $\Gamma_{2\pi s_2}$ *would be the right choice of parameter to have* $\mathcal{A}(\mathrm{Ad}_G(\Gamma_{2\pi s_2}))$ *connected. But* $\Gamma_{2\pi s_2}$ *is not a lattice in* $G^{a,p,q,s}_{6.11}$.

Proof. Exponentiating we get

$$\exp(2\pi s_2\,\mathrm{ad}_{X_6}) = \begin{pmatrix} e^{-4(p+q)\pi s_2} & 0 & 0 & 0 & 0\\ 0 & e^{2p\pi s_2} & 0 & 0 & 0\\ 0 & 0 & e^{2p\pi s_2} & 0 & 0\\ 0 & 0 & 0 & e^{2q\pi s_2} & 0\\ 0 & 0 & 0 & 0 & e^{2q\pi s_2} \end{pmatrix}.$$

We set $e^{-2(p+q)\pi s_2} = \alpha$ and $e^{-2q\pi s_2} + e^{-2p\pi s_2} = \beta$, so its minimal polynomial

$$\operatorname{MinPol}(x) = -\alpha + \frac{(\alpha^2\beta+1)\,x}{\alpha} - \frac{(\alpha^3+\beta)\,x^2}{\alpha} + x^3$$

can have integer coefficients only if $\alpha \in \mathbb{Z}$. Then $\frac{\alpha^2\beta+1}{\alpha} = \alpha\beta + \frac{1}{\alpha} \in \mathbb{Z}$ implies $\beta \in \mathbb{Q}$. But then $\frac{\alpha^3+\beta}{\alpha} = \alpha^2 + \frac{\beta}{\alpha} \in \mathbb{Z}$ implies $\frac{\beta}{\alpha} \in \mathbb{Z}$ and so $\beta \in \mathbb{Z}$. Therefore if α and β are not both integers we have no lattice $\Gamma_{2\pi s_2}$. Suppose $\alpha, \beta \in \mathbb{Z}$, then $\beta + \frac{1}{\alpha} \in \mathbb{Z}$ only if $\alpha = 1$ that is $a = p + q = 0$, but this value is not acceptable, so $\Gamma_{2\pi s_2}$ is not a lattice. $\square$

(ii): *If s is irrational one can look for lattices* $\Gamma_{\bar{t}}$ *with* $\mathcal{A}(\mathrm{Ad}_G(\Gamma_{\bar{t}}))$ *connected for* $\bar{t} = 2\pi$. *For* $p \neq 0$ *there is no lattice for* $\bar{t} = 2\pi$, *but* $\Gamma_{2\pi}$ *is a lattice for* $G_{6.11}^{a,0,q,s}$ *for some value of q and s (recall,* $a + 2p + 2q = 0$*). We denote the group* $G_{6.11}^{a,0,q,s}$ *for these choices of the parameters by* $G_{6.11}^{p=0}$.

The Mostow condition does not hold, and after modification $\tilde{\mathfrak{g}}_{6.11}^{p=0}$ has structure equations $[X_1, X_6] = -2qX_1$, $[X_4, X_6] = qX_4 - sX_5$, $[X_5, X_6] = sX_4 + qX_5$, so it is isomorphic to $\mathfrak{g}_{4.6}^{-2k,k} \oplus \mathbb{R}^2$ for some k, cf. [3, Appendix A]. The Lie algebra $\tilde{\mathfrak{g}}_{6.11}^{p=0}$ is not completely solvable, but satisfies the Mostow condition for our choice of lattice. The cohomology groups are

$$\begin{aligned} H^1(G_{6.11}^{p=0}/\Gamma_{2\pi}) &= H^1(\tilde{\mathfrak{g}}_{6.11}^{p=0}) = \langle \alpha^2, \alpha^3, \alpha^6 \rangle \\ H^2(G_{6.11}^{p=0}/\Gamma_{2\pi}) &= H^2(\tilde{\mathfrak{g}}_{6.11}^{p=0}) = \langle \alpha^{23}, \alpha^{26}, \alpha^{36} \rangle \\ H^3(G_{6.11}^{p=0}/\Gamma_{2\pi}) &= H^3(\tilde{\mathfrak{g}}_{6.11}^{p=0}) = \langle \alpha^{145}, \alpha^{236} \rangle \end{aligned}$$

The subgroups $\Gamma_{2\pi/k}$ *(*$k \in \mathbb{Z}$*) are lattices if and only if* $2\cos\left(\frac{2\pi}{k}\right) \in \mathbb{Z}$. In all these cases

$$\begin{aligned} H^1(G_{6.11}^{p=0}/\Gamma_{\bar{t}}) &= \langle \alpha^6 \rangle \\ H^2(G_{6.11}^{p=0}/\Gamma_{\bar{t}}) &= \langle \alpha^{23} \rangle \\ H^3(G_{6.11}^{p=0}/\Gamma_{\bar{t}}) &= \langle \alpha^{145}, \alpha^{236} \rangle. \end{aligned}$$

- $G_{6.12}^{-4p,p}$: *one can show that there is no lattice for* $t = 2\pi$.

- $G_{5.13}^{-1-2q,q,r} \times \mathbb{R}$: we must consider two cases. If $r \in \mathbb{R} \setminus \mathbb{Q}$ then $\mathcal{A}(\mathrm{Ad}(\Gamma_{2\pi}))$ is connected, whilst if $r = \frac{r_1}{r_2} \in \mathbb{Q}$, $\mathcal{A}(\mathrm{Ad}(\Gamma_{2\pi r_2}))$ is connected, but *one can show that there is no lattice for either values of* t.

- $G_{5.14}^{0} \times \mathbb{R}$: *for* $\bar{t} = 2\pi$ $\mathcal{A}(\mathrm{Ad}(\Gamma_{\bar{t}}))$ *is connected and* $\Gamma_{\bar{t}}$ *is a lattice, then the only non-zero bracket of the Lie algebra* $\tilde{\mathfrak{g}} \cong \mathfrak{g}_{3.1} \oplus \mathbb{R}^3$ *is* $[X_2, X_5] = X_1$ and the cohomology groups are

$$\begin{aligned} H^1(G_{5.14}^{0} \times \mathbb{R}/\Gamma_{2\pi}) = H^1(\tilde{\mathfrak{g}}) = {}& \langle \alpha^2, \alpha^3, \alpha^4, \alpha^5, \alpha^6 \rangle, \\ H^2(G_{5.14}^{0} \times \mathbb{R}/\Gamma_{2\pi}) = H^2(\tilde{\mathfrak{g}}) = {}& \langle \alpha^{12}, \alpha^{15}, \alpha^{23}, \alpha^{24}, \alpha^{26}, \alpha^{34}, \alpha^{35}, \\ & \alpha^{36}, \alpha^{45}, \alpha^{46}, \alpha^{56} \rangle, \\ H^3(G_{5.14}^{0} \times \mathbb{R}/\Gamma_{2\pi}) = H^3(\tilde{\mathfrak{g}}) = {}& \langle \alpha^{123}, \alpha^{124}, \alpha^{125}, \alpha^{126}, \alpha^{135}, \alpha^{145}, \\ & \alpha^{156}, \alpha^{234}, \alpha^{236}, \alpha^{246}, \alpha^{345}, \alpha^{346}, \alpha^{356}, \alpha^{456} \rangle. \end{aligned}$$

The subgroups $\Gamma_{2\pi/k}$ ($k\in\mathbb{Z}$) are again lattices if and only if $2\cos\left(\frac{2\pi}{k}\right)\in\mathbb{Z}$, in particular for all these values we have the same invariants and the cohomology groups are:

$$\begin{aligned}
H^1(G^0_{5.14}\times\mathbb{R}/\Gamma_{\frac{2\pi}{k}}) &= \langle\alpha^2,\,\alpha^5,\,\alpha^6\rangle,\\
H^2(G^0_{5.14}\times\mathbb{R}/\Gamma_{\frac{2\pi}{k}}) &= \langle\alpha^{12},\,\alpha^{15},\,\alpha^{26},\,\alpha^{34},\,\alpha^{56}\rangle,\\
H^3(G^0_{5.14}\times\mathbb{R}/\Gamma_{\frac{2\pi}{k}}) &= \langle\alpha^{125},\,\alpha^{126},\,\alpha^{156},\,\alpha^{234},\,\alpha^{345},\,\alpha^{346}\rangle.
\end{aligned}$$

We note that these groups are isomorphic to the cohomology groups of the Lie algebra $\mathfrak{g}^0_{5.14}\oplus\mathbb{R}$.

• $G^{p,-p,r}_{5.17}\times\mathbb{R}$: again we must consider two different cases: *if $r\in\mathbb{R}\setminus\mathbb{Q}$ then $\mathcal{A}(\mathrm{Ad}\,(\Gamma_{2\pi}))$ is connected, but we have no lattice. If $r=\frac{r_1}{r_2}\in\mathbb{Q}$ then $\mathcal{A}(\mathrm{Ad}\,(\Gamma_{2\pi r_2}))$ is connected and $\Gamma_{2\pi r_2}$ is a lattice if and only if $e^{2\pi p r_2}+e^{-2\pi p r_2}=h\in\mathbb{Z}$.*

So for these values of p and r_2 the Lie algebra $\tilde{\mathfrak{g}}$ is $\mathbb{R}^6$ for $p=0$, while for $p\neq 0$ the non zero brackets in $\tilde{\mathfrak{g}}$ are given by

$$[X_1,X_5]=pX_1,\ [X_2,X_5]=pX_2,\ [X_3,X_5]=-pX_3,\ [X_4,X_5]=-pX_4.$$

Thus if $p\neq 0$, $\tilde{\mathfrak{g}}$ is isomorphic to $\mathfrak{g}^{1,-1,-1}_{5.7}\oplus\mathbb{R}$. The cohomology groups of the solvmanifold $G^{p,-p,r}_{5.17}\times\mathbb{R}/\Gamma_{2\pi}$ are

$$\begin{aligned}
H^1(G^{p,-p,r}_{5.17}\times\mathbb{R}/\Gamma_{2\pi r_2}) &= H^1(\tilde{\mathfrak{g}}) = \langle\alpha^5,\,\alpha^6\rangle,\\
H^2(G^{p,-p,r}_{5.17}\times\mathbb{R}/\Gamma_{2\pi r_2}) &= H^2(\tilde{\mathfrak{g}}) = \langle\alpha^{13},\,\alpha^{14},\,\alpha^{23},\,\alpha^{24},\,\alpha^{56}\rangle,\\
H^3(G^{p,-p,r}_{5.17}\times\mathbb{R}/\Gamma_{2\pi r_2}) &= H^3(\tilde{\mathfrak{g}}) = \langle\alpha^{135},\,\alpha^{136},\,\alpha^{145},\,\alpha^{146},\,\alpha^{235},\,\alpha^{236},\\
&\qquad\qquad\qquad\qquad \alpha^{245},\,\alpha^{246}\rangle.
\end{aligned}$$

To study other lattices we consider $r\in\mathbb{Z}$ and then $t=\frac{2\pi}{k}$: the characteristic polynomial of $\exp(t\,\mathrm{ad}_{X_5})$ has coefficients depending strongly on the relationship between k and r, so it is difficult to determine for which values of k they are integers, in general. For this reason we consider only particular values of k:

(a) $k=2$: if r is even we have a lattice if and only if $h-2=n^2$ for some $n\in\mathbb{Z}$, and the cohomology groups of the solvmanifold are:

if $p\neq 0$

$$\begin{aligned}
H^1(G^{p,-p,r}_{5.17}\times\mathbb{R}/\Gamma_\pi) &= \langle\alpha^5,\alpha^6\rangle,\\
H^2(G^{p,-p,r}_{5.17}\times\mathbb{R}/\Gamma_\pi) &= \langle\alpha^{56}\rangle,\\
H^3(G^{p,-p,r}_{5.17}\times\mathbb{R}/\Gamma_\pi) &= \{0\};
\end{aligned}$$

if $p=0$

$$\begin{aligned}
H^1(G^{p,-p,r}_{5.17}\times\mathbb{R}/\Gamma_\pi) &= \langle\alpha^3,\alpha^4,\alpha^5,\alpha^6\rangle,\\
H^2(G^{p,-p,r}_{5.17}\times\mathbb{R}/\Gamma_\pi) &= \langle\alpha^{12},\alpha^{34},\alpha^{35},\alpha^{36},\alpha^{45},\alpha^{46},\alpha^{56}\rangle,\\
H^3(G^{p,-p,r}_{5.17}\times\mathbb{R}/\Gamma_\pi) &= \langle\alpha^{123},\alpha^{124},\alpha^{125},\alpha^{126},\alpha^{345},\alpha^{346},\alpha^{356},\alpha^{456}\rangle.
\end{aligned}$$

Note that for $p\neq 0$ these groups are isomorphic to the cohomology groups of the Lie algebra.

If r is odd we have a lattice if there is an integer n such that $h+2=n^2$, and

if $p \neq 0$
$$\begin{aligned}
&H^1(G_{5.17}^{p,-p,r} \times \mathbb{R}/\Gamma_\pi) = \langle \alpha^5, \alpha^6 \rangle,\\
&H^2(G_{5.17}^{p,-p,r} \times \mathbb{R}/\Gamma_\pi) = \langle \alpha^{13}, \alpha^{14}, \alpha^{23}, \alpha^{24}, \alpha^{56} \rangle,\\
&H^3(G_{5.17}^{p,-p,r} \times \mathbb{R}/\Gamma_\pi) = \langle \alpha^{135}, \alpha^{136}, \alpha^{145}, \alpha^{146}, \alpha^{235}, \alpha^{236}, \alpha^{245}, \alpha^{246} \rangle;
\end{aligned}$$

if $p = 0$
$$\begin{aligned}
&H^1(G_{5.17}^{p,-p,r} \times \mathbb{R}/\Gamma_\pi) = \langle \alpha^5, \alpha^6 \rangle,\\
&H^2(G_{5.17}^{p,-p,r} \times \mathbb{R}/\Gamma_\pi) = \langle \alpha^{12}, \alpha^{13}, \alpha^{14}, \alpha^{23}, \alpha^{24}, \alpha^{34}, \alpha^{56} \rangle,\\
&H^3(G_{5.17}^{p,-p,r} \times \mathbb{R}/\Gamma_\pi) = \langle \alpha^{125}, \alpha^{126}, \alpha^{135}, \alpha^{136}, \alpha^{145}, \alpha^{146}, \alpha^{235},\\
&\qquad \alpha^{236}, \alpha^{245}, \alpha^{246}, \alpha^{345}, \alpha^{346} \rangle.
\end{aligned}$$

(b) $k = 4$: if $r \equiv 0 \mod 4$ the characteristic polynomial has integer coefficients if and only if $p = 0$, and for this value our matrix has integer entries, so there is a lattice:
$$\begin{aligned}
&H^1(G_{5.17}^{p,-p,r} \times \mathbb{R}/\Gamma_{\frac{\pi}{2}}) = \langle \alpha^3, \alpha^4, \alpha^5, \alpha^6 \rangle,\\
&H^2(G_{5.17}^{p,-p,r} \times \mathbb{R}/\Gamma_{\frac{\pi}{2}}) = \langle \alpha^{12}, \alpha^{34}, \alpha^{35}, \alpha^{36}, \alpha^{45}, \alpha^{46}, \alpha^{56} \rangle,\\
&H^3(G_{5.17}^{p,-p,r} \times \mathbb{R}/\Gamma_{\frac{\pi}{2}}) = \langle \alpha^{123}, \alpha^{124}, \alpha^{125}, \alpha^{126}, \alpha^{345}, \alpha^{346}, \alpha^{356}, \alpha^{456} \rangle.
\end{aligned}$$

If $r \equiv 1 \mod 4$ again we have a lattice only if $h + 2 = n^2$ for some $n \in \mathbb{Z}$, and

if $p \neq 0$
$$\begin{aligned}
&H^1(G_{5.17}^{p,-p,r} \times \mathbb{R}/\Gamma_{\frac{\pi}{2}}) = \langle \alpha^5, \alpha^6 \rangle,\\
&H^2(G_{5.17}^{p,-p,r} \times \mathbb{R}/\Gamma_{\frac{\pi}{2}}) = \langle \alpha^{13} + \alpha^{24},\ \alpha^{14} - \alpha^{23},\ \alpha^{56} \rangle,\\
&H^3(G_{5.17}^{p,-p,r} \times \mathbb{R}/\Gamma_{\frac{\pi}{2}}) = \langle \alpha^{135} + \alpha^{245}, \alpha^{145} - \alpha^{235}, \alpha^{146} - \alpha^{236}, \alpha^{136} + \alpha^{246} \rangle;
\end{aligned}$$

if $p = 0$
$$\begin{aligned}
&H^1(G_{5.17}^{p,-p,r} \times \mathbb{R}/\Gamma_{\frac{\pi}{2}}) = \langle \alpha^5, \alpha^6 \rangle,\\
&H^2(G_{5.17}^{p,-p,r} \times \mathbb{R}/\Gamma_{\frac{\pi}{2}}) = \langle \alpha^{12}, \alpha^{13} + \alpha^{24},\ \alpha^{14} - \alpha^{23},\ \alpha^{34}, \alpha^{56} \rangle,\\
&H^3(G_{5.17}^{p,-p,r} \times \mathbb{R}/\Gamma_{\frac{\pi}{2}}) = \langle \alpha^{125}, \alpha^{126}, \alpha^{135} + \alpha^{245},\ \alpha^{145} - \alpha^{235},\\
&\qquad \alpha^{146} - \alpha^{236},\ \alpha^{136} + \alpha^{246}, \alpha^{345}, \alpha^{346} \rangle.
\end{aligned}$$

For $r = 1$ these are isomorphic to the Lie algebra cohomology.

If $r \equiv 2 \mod 4$ then again there is a lattice only if $p = 0$ and we have an isomorphism with the invariant cohomology groups:
$$\begin{aligned}
&H^1(G_{5.17}^{p,-p,r} \times \mathbb{R}/\Gamma_{\frac{\pi}{2}}) = \langle \alpha^5, \alpha^6 \rangle,\\
&H^2(G_{5.17}^{p,-p,r} \times \mathbb{R}/\Gamma_{\frac{\pi}{2}}) = \langle \alpha^{12}, \alpha^{34}, \alpha^{56} \rangle,\\
&H^3(G_{5.17}^{p,-p,r} \times \mathbb{R}/\Gamma_{\frac{\pi}{2}}) = \langle \alpha^{125}, \alpha^{126}, \alpha^{345}, \alpha^{346} \rangle.
\end{aligned}$$

If $r \equiv 3 \mod 4$ we get the same coefficients as for $r \equiv 1 \mod 4$, and we have a lattice only if $h + 2 = n^2$ for some $n \in \mathbb{Z}$.

if $p \neq 0$
$$\begin{aligned}
&H^1(G_{5.17}^{p,-p,r} \times \mathbb{R}/\Gamma_{\frac{\pi}{2}}) = \langle \alpha^5, \alpha^6 \rangle,\\
&H^2(G_{5.17}^{p,-p,r} \times \mathbb{R}/\Gamma_{\frac{\pi}{2}}) = \langle \alpha^{14} + \alpha^{23},\ \alpha^{13} - \alpha^{24},\ \alpha^{56} \rangle,\\
&H^3(G_{5.17}^{p,-p,r} \times \mathbb{R}/\Gamma_{\frac{\pi}{2}}) = \langle \alpha^{145} + \alpha^{235}, \alpha^{135} - \alpha^{245}, \alpha^{136} - \alpha^{246}, \alpha^{146} + \alpha^{236} \rangle;
\end{aligned}$$

$$\begin{aligned}
\textbf{if } p=0 \quad & H^1(G_{5.17}^{p,-p,r}\times\mathbb{R}/\Gamma_{\frac{\pi}{2}}) = \langle\alpha^5,\alpha^6\rangle,\\
& H^2(G_{5.17}^{p,-p,r}\times\mathbb{R}/\Gamma_{\frac{\pi}{2}}) = \langle\alpha^{12},\alpha^{14}+\alpha^{23},\ \alpha^{13}-\alpha^{24},\ \alpha^{34},\alpha^{56}\rangle,\\
& H^3(G_{5.17}^{p,-p,r}\times\mathbb{R}/\Gamma_{\frac{\pi}{2}}) = \langle\alpha^{125},\alpha^{126},\alpha^{145}+\alpha^{235},\alpha^{135}-\alpha^{245},\\
& \qquad \alpha^{136}-\alpha^{246},\ \alpha^{146}+\alpha^{236},\alpha^{345},\alpha^{346}\rangle.
\end{aligned}$$

Again we have the isomorphism with the invariant cohomology, but only for $r=-1$.

- $G_{5.18}^0\times\mathbb{R}$: for $t=2\pi$ $\mathcal{A}(\mathrm{Ad}(\Gamma_t))$ is connected, *there is a lattice and* $\tilde{\mathfrak{g}}\cong\mathfrak{g}_{5.1}\oplus\mathbb{R}$, so

$$\begin{aligned}
H^1(G_{5.18}^0\times\mathbb{R}/\Gamma_{2\pi}) = H^1(\tilde{\mathfrak{g}}) = {} & \langle\alpha^3,\ \alpha^4,\ \alpha^5,\ \alpha^6\rangle\\
H^2(G_{5.18}^0\times\mathbb{R}/\Gamma_{2\pi}) = H^2(\tilde{\mathfrak{g}}) = {} & \langle\alpha^{13},\ \alpha^{15},\ \alpha^{14}+\alpha^{23},\ \alpha^{24},\ \alpha^{25},\ \alpha^{34},\ \alpha^{36},\ \alpha^{46},\ \alpha^{56}\rangle\\
H^3(G_{5.18}^0\times\mathbb{R}/\Gamma_{2\pi}) = H^3(\tilde{\mathfrak{g}}) = {} & \langle\alpha^{125},\ \alpha^{134},\ \alpha^{135},\ \alpha^{136},\ \alpha^{146}+\alpha^{236},\ \alpha^{156},\ \alpha^{234},\\
& \alpha^{235},\ \alpha^{245},\ \alpha^{246},\ \alpha^{256},\ \alpha^{346}\rangle
\end{aligned}$$

Again we can have other lattices $\Gamma_{2\pi/k}$ only for $k=2,3,4,6$ and

$$\begin{aligned}
k=2 \qquad & H^1(G_{5.18}^0\times\mathbb{R}/\Gamma_{\pi}) = \langle\alpha^5,\ \alpha^6\rangle\\
& H^2(G_{5.18}^0\times\mathbb{R}/\Gamma_{\pi}) = \langle\alpha^{13},\ \alpha^{14}+\alpha^{23},\ \alpha^{24},\ \alpha^{34},\ \alpha^{56}\rangle\\
& H^3(G_{5.18}^0\times\mathbb{R}/\Gamma_{\pi}) = \langle\alpha^{125},\alpha^{135},\alpha^{136},\alpha^{146}+\alpha^{236},\alpha^{235},\alpha^{245},\alpha^{246},\alpha^{346}\rangle\\
k=3,4,6 \qquad & H^1(G_{5.18}^0\times\mathbb{R}/\Gamma_{\frac{2\pi}{k}}) = \langle\alpha^5,\ \alpha^6\rangle\\
& H^2(G_{5.18}^0\times\mathbb{R}/\Gamma_{\frac{2\pi}{k}}) = \langle\alpha^{13}+\alpha^{24},\ \alpha^{34},\ \alpha^{56}\rangle\\
& H^3(G_{5.18}^0\times\mathbb{R}/\Gamma_{\frac{2\pi}{k}}) = \langle\alpha^{125},\ \alpha^{135}+\alpha^{245},\ \alpha^{136}+\alpha^{246},\ \alpha^{346}\rangle
\end{aligned}$$

The last case is isomorphic to the cohomology of the Lie algebra.

- $G_{4.6}^{-2p,p}\times\mathbb{R}^2$: *for* $t=2\pi$ $\mathcal{A}(\mathrm{Ad}(\Gamma_t))$ *is connected, but there is no lattice.*

- $G_{3.5}^0\times\mathbb{R}^3$: *for* $t=2\pi$, $\mathcal{A}(\mathrm{Ad}(\Gamma_t))$ *is obviously connected and we have a lattice, in particular* $\tilde{\mathfrak{g}}_{3.5}^0\times\mathbb{R}^3\cong\mathbb{R}^6$, *so* $G_{3.5}^0\times\mathbb{R}^3/\Gamma_{2\pi}$ *is diffeomorphic to a 6-torus.*

Again, the subgroups $\Gamma_{2\pi/k}$ ($k\in\mathbb{Z}$) are lattices if and only if $2\cos\left(\frac{2\pi}{k}\right)\in\mathbb{Z}$. In particular for all these values the cohomology groups are always isomorphic to the invariant ones:

$$\begin{aligned}
& H^1(G_{3.5}^0\times\mathbb{R}^3/\Gamma_{\frac{2\pi}{k}}) = \langle\alpha^3,\alpha^4,\alpha^5,\alpha^6\rangle,\\
& H^2(G_{3.5}^0\times\mathbb{R}^3/\Gamma_{\frac{2\pi}{k}}) = \langle\alpha^{12},\alpha^{34},\alpha^{35},\alpha^{36},\alpha^{45},\alpha^{46},\alpha^{56}\rangle,\\
& H^3(G_{3.5}^0\times\mathbb{R}^3/\Gamma_{\frac{2\pi}{k}}) = \langle\alpha^{123},\alpha^{124},\alpha^{125},\alpha^{126},\alpha^{345},\alpha^{346},\alpha^{356},\alpha^{456}\rangle.
\end{aligned}$$

We list the Lie algebras $\mathfrak{g}$ and modified Lie algebras $\tilde{\mathfrak{g}}$ in Table 3.1.

Table 7.2: Deformed Lie algebras

$\mathfrak{g}$	$\tilde{\mathfrak{g}}$
$\mathfrak{g}_{6.8}^{p=0}$	$\mathfrak{g}_{4.5} \oplus \mathbb{R}^2$
$\mathfrak{g}_{6.10}^{a=0}$	$\mathfrak{g}_{4.1} \oplus \mathbb{R}^2$
$\mathfrak{g}_{6.11}^{p=0}$	$\mathfrak{g}_{4.6}^{-2k,k} \oplus \mathbb{R}^2$
$\mathfrak{g}_{5.14}^{0} \oplus \mathbb{R}$	$\mathfrak{g}_{3.1} \oplus \mathbb{R}^3$
$\mathfrak{g}_{5.17}^{p,-p,r} \oplus \mathbb{R}$	$\mathbb{R}^6, \quad p = 0$ $\mathfrak{g}_{5.7}^{1,-1,-1} \oplus \mathbb{R}, \quad p \neq 0$
$\mathfrak{g}_{5.18}^{0} \oplus \mathbb{R}$	$\mathfrak{g}_{5.1} \oplus \mathbb{R}$
$\mathfrak{g}_{3.5}^{0} \oplus \mathbb{R}^3$	$\mathbb{R}^6$

4. Symplectic structures and Lefschetz properties

Let us study symplectic structures on the solvmanifolds of concern. In general, since $\tilde{G}/\Gamma$ is diffeomorphic to G/Γ (Theorem 4), symplectic structures on the modified Lie algebra $\tilde{\mathfrak{g}}$ yield non-G-invariant symplectic structures $\tilde{\omega}$ on G/Γ (where Γ is the lattice for which $\mathcal{A}(\mathrm{Ad}_G(\Gamma))$ is connected). Recall from the previous Sections that these G/Γ cover solvmanifolds $G/\bar{\Gamma}$. Observe that in general the symplectic forms $\tilde{\omega}$ are defined only on the covering G/Γ and not on $G/\bar{\Gamma}$.

Let us start with the indecomposable case. We know from the classification of symplectic structures on six-dimensional solvable Lie algebras (see [16]) that only $G_{6.10}^{a=0}/\Gamma_t$ have invariant symplectic structures (inherited by $\mathfrak{g}_{6.10}^{a=0}$). The generic invariant symplectic form is

$$\omega = \omega_{1,6}\alpha^{16} + \omega_{2,3}\alpha^{23} + \omega_{2,6}\alpha^{26} + \omega_{3,6}\alpha^{36} + \omega_{4,5}\alpha^{45} + \omega_{4,6}\alpha^{46} + \omega_{5,6}\alpha^{56} \tag{2}$$

with $\omega_{1,6}\omega_{2,3}\omega_{4,5} \neq 0$ ($\det(\omega_{ij}) \neq 0$).

Next, let us look for non-invariant symplectic structures. In the case of $G_{6.8}^{p=0}$ we have that also $\tilde{\mathfrak{g}}_{6.8}^{p=0}$ does not admit symplectic structures. As for $G_{6.10}^{a=0}$, symplectic forms on the modified $\tilde{\mathfrak{g}}_{6.10}^{a=0}$ (isomorphic to $\mathfrak{g}_{4.1} \oplus \mathbb{R}^2$) yield the (in general non-$G_{6.10}^{a=0}$-invariant) symplectic form on $G_{6.10}^{a=0}/\Gamma_{2\pi}$

$$\tilde{\omega} = \omega + \eta\,,$$

where ω is given by (2) and $\eta = \omega_{3,4}\alpha^{34} + \omega_{3,5}\alpha^{35}$ is the "new part" (recall that the invariant cohomology, i.e. the cohomology of the Lie algebra $\mathfrak{g}_{6.10}^{a=0}$, is contained in the deRham cohomology of $G^{a=0}_{6.10}/\Gamma_{2\pi}$, cf. Remark 2). Again $\tilde{\mathfrak{g}}^{p=0}_{6.11}$ does not admit symplectic structures.

Consider now the decomposable case. The Lie algebra $\mathfrak{g} = \mathfrak{g}^0_{5.14} \oplus \mathbb{R}$ admits the symplectic form

$$\omega = \omega_{1,2}\alpha^{12} + \omega_{1,5}\alpha^{15} + \omega_{2,5}\alpha^{25} + \omega_{2,6}\alpha^{26} + \omega_{3,4}\alpha^{34} + \omega_{3,5}\alpha^{35} + \omega_{4,5}\alpha^{45} + \omega_{5,6}\alpha^{56},$$

with $\det(\omega_{i,j}) \neq 0$,

$$\tilde{\omega} = \omega + \omega_{2,3}\alpha^{23} + \omega_{2,4}\alpha^{24} + \omega_{3,6}\alpha^{36} + \omega_{4,6}\alpha^{46} \text{ with } \det(\tilde{\omega}_{i,j}) \neq 0\,.$$

The Lie algebra $\mathfrak{g} = \mathfrak{g}^{p,-p,r}_{5.17} \oplus \mathbb{R}$ admits symplectic structures only for particular values of the parameters p and r, [16], but:
for $p = 0$: $\tilde{\mathfrak{g}}$ is isomorphic to $\mathbb{R}^6$ so it is symplectic;
for $p \neq 0$: $\tilde{\mathfrak{g}}$ has generic symplectic form

$$\begin{aligned}\tilde{\omega} = {} & \omega_{1,3}\alpha^{13} + \omega_{1,4}\alpha^{14} + \omega_{1,5}\alpha^{15} + \omega_{2,3}\alpha^{23} + \omega_{2,4}\alpha^{24} + \omega_{2,5}\alpha^{25} + \\ & \omega_{3,5}\alpha^{35} + \omega_{4,5}\alpha^{45} + \omega_{5,6}\alpha^{56}\end{aligned}$$

with $\det(\tilde{\omega}_{i,j}) \neq 0$. The Lie algebra $\mathfrak{g} = \mathfrak{g}^0_{5.18} \oplus \mathbb{R}$ admits the symplectic form

$$\omega = \omega_{1,3}(\alpha^{13} + \alpha^{24}) + \omega_{1,5}\alpha^{15} + \omega_{2,5}\alpha^{25} + \omega_{3,4}\alpha^{34} + \omega_{3,5}\alpha^{35} + \omega_{4,5}\alpha^{45} + \omega_{5,6}\alpha^{56},$$

with $\omega_{1,3}\omega_{5,6} \neq 0$, but the solvmanifold $G^0_{5.18} \times \mathbb{R}/\Gamma_{2\pi}$ also has a non-invariant symplectic structure inherited by $\tilde{\mathfrak{g}}$:

$$\tilde{\omega} = \omega + \omega_{1,3}\alpha^{13} + \omega_{1,4}(\alpha^{14} + \alpha^{23}) + \omega_{3,6}\alpha^{36} + \omega_{4,6}\alpha^{46},$$

with $\det(\tilde{\omega}_{i,j}) \neq 0$. The Lie algebra $\mathfrak{g} = \mathfrak{g}^0_{3.5} \oplus \mathbb{R}^3$ admits symplectic structures, but $\tilde{\mathfrak{g}}$ is isomorphic to $\mathbb{R}^6$, so $G^0_{3.5} \times \mathbb{R}^3/\Gamma_{2\pi}$ admits obviously a non-invariant symplectic structure too.

DEFINITION 1. *An $SU(3)$ structure on a six-dimensional manifold M (i.e., an $SU(3)$ reduction of the frame bundle of M) defines a non-degenerate 2-form ω, an almost complex structure J and a complex volume form Ψ. The SU(3) structure is called* half-flat *if $\omega \wedge \omega$ and the real part of Ψ are closed [4]. If in addition ω is closed, the half-flat structure is called* symplectic.

Proof of Proposition 2. We use the classification of [8] together with the above discussion on symplectic forms, possibly coming from forms on the modified Lie algebra (cf. Table 2). By [8, Proposition 4.2], there is no $4 \oplus 2$ decomposable Lie algebra admitting symplectic half-flat structures. Hence the symplectic forms on $G^{a=0}_{6.10}/\Gamma_{2\pi}$ we found are not half-flat (recall that $\tilde{\mathfrak{g}}^{a=0}_{6.10}$ is isomorphic to $\mathfrak{g}_{4.1} \oplus \mathbb{R}^2$). By [8, Proposition 4.3], the $5 \oplus 1$ decomposable Lie algebras having symplectic half-flat structures are $\mathfrak{g}_{5.1} \oplus \mathbb{R}$ (isomorphic to $\mathfrak{h}_3$ in the notation of [8]), $\mathfrak{g}^{p,-p,r}_{5.17} \oplus \mathbb{R}$ for $p \geqslant 0$ and $r = 1$, and $\mathfrak{g}^{p,q,r}_{5.7} \oplus \mathbb{R}$ for $p = q = -1$ and $r = 1$. □

Next we consider the hard Lefschetz property. Recall that a symplectic manifold (M^{2n},ω) fulfils the *hard Lefschetz property* if for every $0 \leqslant k \leqslant n$

$$\begin{array}{rccc} L^{n-k}: & H^k(M) & \to & H^{2n-k}(M) \\ & [\alpha] & \mapsto & [\omega^{n-k} \wedge \alpha] \end{array}$$

is an isomorphism. More in general, (M^{2n},ω) is called *s-Lefschetz* if L^{n-k} is an isomorphism for all $k \leqslant s$ [9]. The property of being 0-Lefschetz is equivalent to being *cohomologically symplectic*, i.e. there exists $\omega \in H^2(M)$ such that $\omega^n \neq 0$.

We need to consider $G^{a=0}_{6.10}/\Gamma_{\bar{t}}$, $G^0_{5.14} \times \mathbb{R}/\Gamma_{\bar{t}}$, $G^0_{5.18} \times \mathbb{R}/\Gamma_{\bar{t}}$, $G^{p,-p,r}_{5.17} \times \mathbb{R}/\Gamma_{\bar{t}}$, $G^0_{3.5} \times \mathbb{R}^3/\Gamma_{\bar{t}}$ (for the above choices of $\bar{t}$). As a start, we consider the generic (non-invariant) symplectic form $\tilde{\omega}$ on G/Γ with $\mathcal{A}(\mathrm{Ad}_G(\Gamma))$ connected.

Proposition 8. *The hard Lefschetz property does not hold for the symplectic form* $\tilde{\omega}$ *on* $G^{a=0}_{6.10}/\Gamma_{2\pi}$, $G^0_{5.14} \times \mathbb{R}/\Gamma_{2\pi}$, $G^0_{5.18} \times \mathbb{R}/\Gamma_{2\pi}$. *More in general, these solvmanifolds are 0-Lefschetz but neither* 1- *nor* 2-*Lefschetz.*
The hard Lefschetz property holds for the symplectic form $\tilde{\omega}$ *on* $G^{p,-p,r}_{5.17} \times \mathbb{R}/\Gamma_{2\pi r_2}$ $(r = \frac{r_1}{r_2} \in \mathbb{Q})$, $G^0_{3.5} \times \mathbb{R}^3/\Gamma_{2\pi}$.

Proof. We prove the statement for the first solvmanifold only, because the other cases are quite similar and the proposition comes from the direct computation of the morphisms L^{n-k}. First, $G^{a=0}_{6.10}/\Gamma_{2\pi}$ is 0-Lefschetz because cohomologically symplectic. By direct computation we find that for every $\alpha \in \Lambda^2\Omega(G/\Gamma)$, α^{1235} never appears in $\tilde{\omega} \wedge \alpha$; as it is a generator of $H^4(G/\Gamma)$, $L^1 : H^2(M) \to H^4(M)$ cannot be an isomorphism. This implies that $G^{a=0}_{6.10}/\Gamma_{2\pi}$ is not 2-Lefschetz. Moreover, $\tilde{\omega}^2 \wedge \alpha^3$ is cohomologous to zero, so $L^2 : H^1(M) \to H^5(M)$ cannot be an isomorphism (recall that α^3 is a generator of $H^1(G^0_{6.10}/\Gamma_{2\pi})$) and $G^{a=0}_{6.10}/\Gamma_{2\pi}$ is not 1-Lefschetz.
The rest of the proposition is quite obvious because $\tilde{\mathfrak{g}}$ is in both cases isomorphic to $\mathbb{R}^6$. □

If we consider the invariant symplectic form ω, then one can see that the hard Lefschetz property holds for $(G^{0,0,r}_{5.17} \times \mathbb{R}/\Gamma_{\bar{t}},\omega)$ and $(G^{p,-p,\pm 1}_{5.17} \times \mathbb{R}/\Gamma_{\bar{t}},\omega)$, but not for $(G^0_{5.14} \times \mathbb{R}/\Gamma_{\bar{t}},\omega)$, $(G^0_{5.18} \times \mathbb{R}/\Gamma_{\bar{t}},\omega)$ [3, Proposition 7.12] and $(G^{a=0}_{6.10}/\Gamma_{\bar{t}},\omega)$ [3, Proposition 7.9].

5. Minimal Models and formality

We now compute the minimal model of the above solvmanifolds. We shall use a method developed by Oprea and Tralle [22, 23] that exploits the Mostow fibration.

THEOREM 9. *[22, 23] Let* $F \to E \to B$ *be a fibration and let* U *be the largest* $\pi_1(B)$*-submodule of* $H^*(F,\mathbb{Q})$ *on which* $\pi_1(B)$ *acts nilpotently. Suppose that* $H^*(F)$ *is a vector space of finite type and* B *is a nilpotent space. Then in the Sullivan model of* *the fibration*

$$\begin{tikzcd}
\mathcal{A}(B) \arrow[r] & \mathcal{A}(E) \arrow[r] & \mathcal{A}(F) \\
(\Lambda X, d_X) \arrow[r] \arrow[u] & (\Lambda(X \oplus Y), D) \arrow[r] \arrow[u] & (\Lambda Y, d_Y) \arrow[u, "\alpha"']
\end{tikzcd}$$

the homomorphism $\alpha : (\Lambda Y, d_Y) \to \mathcal{A}(F)$ *of differential graded algebras induces an isomorphism* $\alpha^* : H^*(\Lambda Y, d_Y) \to U$.

In the case of the Mostow fibration

$$N/\Gamma_N = (N\Gamma)/\Gamma \hookrightarrow G/\Gamma \longrightarrow G/(N\Gamma) = \mathbb{T}^k,$$

we can construct the minimal model $(\Lambda(X \oplus Y), D)$ of the solvmanifold using the models of the base $\mathbb{T}^k$ (for almost abelian solvmanifolds $k = 1$, i.e., a circle S^1) and the fibre N/Γ_N (actually its submodule U). In general finding U is very difficult, but when the solvmanifold is almost nilpotent (in particular almost abelian), the monodromy action of $\mathbb{Z} \cong \pi_1(S^1)$ on $H^*(N/\Gamma_N)$ is exploited by the (transpose of) twist action that defines the semi-direct sum $\mathfrak{g} = \mathbb{R} \ltimes \mathfrak{n}$, that in our case is just $\exp(t\,\mathrm{ad}_{X_6})$ (see [22, Theorems 3.7 and 3.8]). Unfortunately, in some of our examples we cannot find the model uniquely using this method, because there are different choices for constructing $(\Lambda(X \oplus Y), D)$. However, we can identify the right one using the cohomology groups from the previous computations.

We write down the computations explicitly only for some of the solvmanifolds, trying to show all possible different cases, while for the others we only provide the minimal model.

- $G_{6.8}^{p=0}/\Gamma_{2\pi}$:

$$U = \begin{cases} \langle \alpha^4, \alpha^5 \rangle \subset H^1(\mathfrak{n}) \\ \langle \alpha^{45} \rangle \subset H^2(\mathfrak{n}) \\ \langle \alpha^{123} \rangle \subset H^3(\mathfrak{n}) \\ \langle \alpha^{1234}, \alpha^{1235} \rangle \subset H^4(\mathfrak{n}) \\ \langle \alpha^{12345} \rangle = H^5(\mathfrak{n}) \end{cases},$$

and a minimal model for U is $\mathcal{M}_U = (\Lambda(x_1, y_1, z_3), 0)$. The minimal model of the base S^1 is $(\Lambda(A_1), 0)$ and the minimal model of the solvmanifold is $\mathcal{M} = (\Lambda(A_1, x_1, y_1, z_3), 0)$.

- $G_{6.8}^{p=0}/\Gamma_{\pi, \frac{\pi}{2}, \frac{\pi}{3}}$:

$$\mathcal{M} = (\Lambda(A_1, x_2, \beta_3, y_3), D), \qquad DA = Dx = Dy = 0,\ D\beta = x^2.$$

- $G_{6.10}^{a=0}/\Gamma_{2\pi}$:

$$U = H^*(\mathfrak{n}) \quad \Rightarrow \quad \mathcal{M}_U = (\Lambda(x_1, y_1, z_1, p_1, q_1), 0)$$

The minimal model of the solvmanifold is $\mathcal{M} = (\Lambda(A_1, x_1, y_1, z_1, p_1, q_1), D)$, but we have 7 different choices for D:

1. $D \equiv 0$,
2. $DA = Dx = Dy = Dz = Dp = 0,\ Dq = Ax,$
3. $DA = Dx = Dy = Dz = 0\ Dp = Ax,\ Dq = Ay,$
4. $DA = Dx = Dy = Dz = 0\ Dp = Ax,\ Dq = Ap,$
5. $DA = Dx = Dy = 0\ Dz = Ax\ Dp = Ay,\ Dq = Az,$
6. $DA = Dx = Dy = 0\ Dz = Ax\ Dp = Az,\ Dq = Ap,$
7. $DA = Dx = 0\ Dy = Ax\ Dz = Ay\ Dp = Az,\ Dq = Ap.$

Computing the cohomology groups of these commutative differential graded algebras (CDGAs) and comparing with those of $G^{a=0}_{6.10}/\Gamma_{2\pi}$, we find that (4) is the right one.

- $G^{a=0}_{6.10}/\Gamma_{\pi,\frac{\pi}{2},\frac{\pi}{3}}$:

$$U = \begin{cases} \langle \alpha^1, \alpha^2, \alpha^3 \rangle \subset H^1(\mathfrak{n}) \\ \langle \alpha^{12}, \alpha^{13}, \alpha^{23}, \alpha^{45} \rangle \subset H^2(\mathfrak{n}) \\ \langle \alpha^{123}, \alpha^{145}, \alpha^{245}, \alpha^{345} \rangle \subset H^3(\mathfrak{n}) \\ \langle \alpha^{1245}, \alpha^{1345}, \alpha^{2345} \rangle \subset H^4(\mathfrak{n}) \\ \langle \alpha^{12345} \rangle = H^5(\mathfrak{n}) \end{cases} \Rightarrow \quad \begin{array}{c} \mathcal{M}_U = (\Lambda(x_1, y_1, z_1, t_2, \beta_3), d), \\ dx = dy = dz = dt = 0,\ d\beta = t^2 \end{array}$$

The minimal model of the solvmanifold is $\mathcal{M} = (\Lambda(A_1, x_1, y_1, z_1, t_2, \beta_3), D)$, but we have 13 different choices for D. Fortunately, only the following are not isomorphic with each other:

1. $DA = Dx = Dy = Dz = Dt = 0,\ D\beta = t^2,$
2. $DA = Dx = Dy = 0,\ Dz = Ay\ Dt = 0,\ D\beta = t^2,$
3. $DA = Dx = 0,\ Dy = Ax,\ Dz = Ay\ Dt = 0,\ D\beta = t^2.$

Computing the cohomology groups of these CDGAs and comparing with those of $G^{a=0}_{6.10}/\Gamma_{\pi,\frac{\pi}{2},\frac{\pi}{3}}$, we find that (3) is the right one.

REMARK 4. The model (7) in the case of $G^{a=0}_{6.10}/\Gamma_{2\pi}$ has cohomology groups isomorphic to the cohomology groups of $G^{a=0}_{6.10}/\Gamma_{\pi,\frac{\pi}{2},\frac{\pi}{3}}$, and conversely the first model in $G^{a=0}_{6.10}/\Gamma_{\pi,\frac{\pi}{2},\frac{\pi}{3}}$ has the same cohomology as $G^{a=0}_{6.10}/\Gamma_{2\pi}$.

- $G^{p=0}_{6.11}/\Gamma_{2\pi}$: $\mathcal{M} = (\Lambda(A_1, x_1, y_1, z_3), 0)$
- $G^{p=0}_{6.11}/\Gamma_{\pi,\frac{\pi}{2},\frac{\pi}{3}}$: $\mathcal{M} = (\Lambda(A_1, x_2, \beta_3, y_3), D)$, $DA = Dx = Dy = 0$, $D\beta = x^2$.

- $G^{0}_{5.14} \times \mathbb{R}/\Gamma_{2\pi}$:

$$\mathcal{M} = (\Lambda(u_1,A_1,x_1,y_1,z_1,t_1),D),\ Du = DA = Dx = Dy = Dz = 0,\ Dt = Ax\,.$$

- $G^{0}_{5.14} \times \mathbb{R}/\Gamma_{\frac{2\pi}{k}}$:

$$\mathcal{M} = (\Lambda(u_1,A_1,x_1,y_1,z_2,\beta_3),D),\ Du = DA = Dx = Dz = 0,\ Dy = Ax, D\beta = z^2$$

- $G^{p,-p,r}_{5.17} \times \mathbb{R}/\Gamma_{2\pi r_2}$ $(r = \frac{r_1}{r_2} \in \mathbb{Q})$:

If $p \neq 0$: $U = \begin{cases} \langle \alpha^{13}, \alpha^{14}, \alpha^{23}, \alpha^{24} \rangle \subset H^2(\mathfrak{n}) \\ \langle \alpha^{1234} \rangle = H^4(\mathfrak{n}) \end{cases}$.

Listing all generators in $\mathcal{M}_U$ is almost impossible in this case: in every degree we need to add several generators to get the isomorphism in cohomology, but in this way we increase the number of generators. Let us denote by $\mathcal{M}^n$ the subalgebra of $\mathcal{M}$ generated by generators of $\mathcal{M}$ of degree n. Then $\mathcal{M}^1_U = \{0\}$, $\mathcal{M}^2_U = (\Lambda(x_2,y_2,z_2,t_2),0)$, and for any $n > 2$ $\mathcal{M}^n_U$ can be computed inductively ([6, Theorem 2.24]). The minimal model of the solvmanifold is

$$\begin{aligned}\mathcal{M} = \ & (\Lambda(u_1,A_1,\mathcal{M}_U),D),\ Du = DA = Dx = Dy = Dz = Dt = 0,\\ & D|_{\mathcal{M}^n_U} \equiv d\ \forall n > 2\,.\end{aligned}$$

If $p = 0$:

$$U = H^*(\mathfrak{n}) \quad \Rightarrow \quad \mathcal{M}_U = (\Lambda(x_1,y_1,z_1,t_1),0).$$

The minimal model of the solvmanifold is $\mathcal{M} = (\Lambda(u_1,A_1,x_1,y_1,z_1,t_1),D)$, with different choices for D. But we know that it is isomorphic to $\mathbb{R}^6$, so $D \equiv 0$.

- $G^{p,-p,r}_{5.17} \times \mathbb{R}/\Gamma_{\pi}$:

r even:

if $p \neq 0$, then $\mathcal{M} = (\Lambda(u_1,A_1,x_4,\beta_7),D)$, $Du = DA = Dx = 0$, $D\beta = x^2$.

if $p = 0$, then

$$\mathcal{M} = (\Lambda(u_1,A_1,x_1,y_1,z_2,\beta_3),D),\ Du = DA = Dx = Dy = Dz = 0,\ D\beta = z^2\,.$$

r odd:

if $p \neq 0$ we have the same model of $t = 2\pi$.

If $p = 0$ we are not able to list all generators of $\mathcal{M}_U$, but as in the case of $t = 2\pi r_2$, $p \neq 0$ we have

$$\begin{aligned}\mathcal{M} = \ & (\Lambda(u_1,A_1,\mathcal{M}_U),D),\ Du = DA = Dx = Dy = Dz = Dt = Ds = Dq = 0\\ & D|_{\mathcal{M}^n_U} \equiv d\ \forall n > 2\,.\end{aligned}$$

- $G^{p,-p,r}_{5.17} \times \mathbb{R}/\Gamma_{\frac{\pi}{2}}$:

$r \equiv 0 \mod 4$: $p = 0$ and we have the same computation of the case $t = \pi$ with r even.
$r \equiv 1 \mod 4$: $\mathcal{M} = (\Lambda(u_1, A_1, \mathcal{M}_U), D)$, $Du = DA = 0$, $D|_{\mathcal{M}_U} \equiv d$.
$r \equiv 2 \mod 4$: $p = 0$ and

$$\mathcal{M} = (\Lambda(u_1, A_1, x_2, y_2 \beta_3, \gamma_3), D),\ Du = DA = Dx = Dy = 0, D\beta = x^2, D\gamma = y^2 .$$

$r \equiv 3 \mod 4$: the model is the same as for $r \equiv_4 1$.

- $G^0_{5.18} \times \mathbb{R}/\Gamma_{2\pi}$:

$$\mathcal{M} = (\Lambda(u_1, A_1, x_1, y_1, z_1, t_1), D),\ Du = DA = Dx = Dy = 0,\ Dz = Ax,\ Dt = Ay .$$

- $G^0_{5.18} \times \mathbb{R}/\Gamma_{\pi}$:

$$\begin{aligned} \mathcal{M} = \ & (\Lambda(u_1, A_1, \mathcal{M}_U), D), \quad Du = DA = Dx = Dy = Dz = Dt = 0,\ Ds = Ax, \\ & Dq = Ay, D|_{\mathcal{M}_U^n} \equiv d\ \forall n > 2 . \end{aligned}$$

- $G^0_{5.18} \times \mathbb{R}/\Gamma_{\frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}}$:

$$\begin{aligned} \mathcal{M} = \ & (\Lambda(u_1, A_1, \mathcal{M}_U), D), \quad Du = DA = Dx = Dy = 0,\ Dz = Ax,\ Dt = Ay, \\ & D|_{\mathcal{M}_U^n} \equiv d\ \forall n > 2 . \end{aligned}$$

- $G^0_{3.5} \times \mathbb{R}^3/\Gamma_{2\pi}$: $\mathcal{M} = (\Lambda(w_1, v_1, A_1, x_1, y_1), 0)$.
- $G^0_{3.5} \times \mathbb{R}^3/\Gamma_{\frac{2\pi}{k}}$:

$$\mathcal{M} = (\Lambda(w_1, v_1, u_1, A_1, x_2, \beta_3), D),\ Dw = Dv = Du = DA = Dx = 0,\ D\beta = x^2 .$$

Next, we use these models to decide which solvmanifolds are formal. Recall that a manifold is *formal* if its minimal model is formal, meaning there exists a homomorphism $\psi : \mathcal{M} \to H^*(\mathcal{M})$ of CGDAs that induces the identity in cohomology. In particular, every closed generator must be sent to its cohomology class, while the others must be sent to zero. Alas, this construction does not always give the identity, even in higher cohomology.

Proof of Theorem 1. If we have the explicit computation of the model of the solvmanifold we can define ψ and see directly if the map induced in cohomology is the identity or not.
For example, in the case $G^{a=0}_{6.10}/\Gamma_{2\pi}$ we have

$$\psi : A \mapsto [A], x \mapsto [x], y \mapsto [y], z \mapsto [z], p \mapsto 0, q \mapsto 0 ,$$

so $\psi([Ap]) = \psi([0]) = 0$; as $[Ap] \neq 0$, ψ^* is not the identity. A similar proof can be carried out for the other solvmanifolds, of which we have the explicit minimal model.

In absence of the model, we consider a weaker notion of formality introduced by M. Fernández and V. Muñoz in [9], namely s-formality. We shall define it as follows (see [10, Lemma 2.7]). Let M be a manifold with minimal model $\mathcal{M} = (\bigwedge V, d)$. Then M is s-formal if and only if there is a map of differential algebras $\phi : (\bigwedge V^{\leqslant s}, d) \to (H^*(M), d = 0)$, such that the induced $\phi^* : H^*(\bigwedge V^{\leqslant s}, d) \to H^*(M)$ equals $i^* : H^*(\bigwedge V^{\leqslant s}, d) \to (H^*(\bigwedge V, d) = H^*(M)$, induced by the inclusion $i : (\bigwedge V^{\leqslant s}, d) \to (\bigwedge V, d)$.
To study formality we use the following fact

THEOREM 10. *[9] Let M be a connected and orientable compact smooth manifold of dimension $2n$ or $2n-1$. Then M is formal if and only if is $(n-1)$-formal.*

We can apply this theorem to the CDGA $\mathcal{M}_U$ because the manifold M in the hypothesis can be replaced by a real CDGA $\mathcal{A}$ with the following properties:

- $H^0(\mathcal{A}) = \mathbb{R}$;
- $H^i(\mathcal{A}) = 0$ for any $i > \dim(\mathcal{A})$;
- $H^{\dim(M)-i}(\mathcal{A}) \cong H^i(\mathcal{A})$ (Poincaré duality).

$\mathcal{M}_U$ has dimension 4 and always satisfies these properties, so to prove that it is formal we must only prove that it is 1-formal. In the cases where there is no explicit model, $\mathcal{M}_U$ is always simply connected because $U^1 = \{0\}$, so it is 1-formal and hence formal. Now we use the formality of $(\mathcal{M}_U, d_{\mathcal{M}_U})$ to study the formality of the model of $(\mathcal{M}, D)$: if $\mathcal{M}$ has differential D such that $D|_{\mathcal{M}_U} \equiv d_{\mathcal{M}_U}$, then it is obviously formal, otherwise we can show that it is non-formal by defining ψ as in the case of $G_{6.10}^{a=0}/\Gamma_{2\pi}$.
□

In particular one can verify that all non-formal solvmanifolds considered are 0-formal but not 1-formal.

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