Rend. Sem. Mat. Univ. Politec. Torino
Bruxelles-Torino Talks in PDE's –Turin, May 2–5, 2016
Vol. 74, 2 (2016), 31 – 41

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ON DIRICHLET DATA FOR THE SPECTRAL LAPLACIAN

Abstract. We present a construction for nontrivial harmonic functions associated to the *spectral fractional Laplacian* operator, that is a fractional power of the Dirichlet Laplacian giving rise to a nonlocal operator of fractional order. These harmonic functions present a divergent profile at the boundary of the prescribed domain, and they can be classified in terms of a singular boundary trace.

We introduce a notion of L^1-weak solution, in the spirit of Stampacchia, and we produce solutions of linear and nonlinear problems (possibly with measure data) where one prescribes such a singular boundary trace, therefore providing with a *nonhomogeneous* boundary value problem for this operator. We also present some results entailing the existence of *large* solutions in this context.

This is a summary of a joint work with Louis Dupaigne, see [3].

1. Introduction

Given a bounded domain Ω of the $\mathbb{R}^N$, the *spectral (or Navier) fractional Laplacian* operator $(-\triangle|_\Omega)^s$, $s \in (0,1)$, is defined as a fractional power of the Laplacian with homogeneous Dirichlet boundary conditions, see (2) below. This provides a nonlocal operator of elliptic type with *homogeneous* boundary conditions. Recent bibliography on this operator can be found *e.g.* in [8, 4].

One aspect of the theory is however left unanswered: the formulation of natural *nonhomogeneous* boundary conditions. We provide a well-posed weak formulation for linear problems of the form

$$(1) \qquad (-\triangle|_\Omega)^s u = \mu \quad \text{in } \Omega, \qquad \frac{u}{h_1} = \zeta \quad \text{on } \partial\Omega$$

where h_1 is a reference function, see (7) below, with prescribed singular behaviour at the boundary. Namely, h_1 is bounded above and below by constant multiples of $\delta^{-(2-2s)}$, where $\delta(x) := \operatorname{dist}(x, \partial\Omega)$ is the distance to the boundary. Unlike the classical Dirichlet problem for the Laplace operator, nonhomogeneous boundary conditions must be singular. In fact, for the special case of positive s-harmonic functions, the singular boundary condition was already identified in previous works emphasizing the probabilistic and potential theoretic aspects of the problem: see *e.g.* [11, 6, 10].

Turning to nonlinear problems, even more singular boundary conditions arise: in the above system, if $\mu = -u^p$ for suitable values of p, one may choose $\zeta = +\infty$, in the sense that the solution u will blow up at a higher rate than $\delta^{-(2-2s)}$.

DEFINITION 1. *Let $\Omega \subset \mathbb{R}^N$ a bounded domain and let $\{(\lambda_j, \varphi_j)\}_{j\in\mathbb{N}}$, $\varphi_j \in$*

$H^1_0(\Omega)\cap C^\infty(\Omega)$ and $-\triangle\varphi_j=\lambda_j\varphi_j$ in Ω. Given $s\in(0,1)$, consider the Hilbert space

$$H(2s):=\left\{v=\sum_{j=1}^{\infty}\widehat{v}_j\varphi_j\in L^2(\Omega):\|v\|^2_{H(2s)}=\sum_{j=0}^{\infty}\lambda_j^{2s}|\widehat{v}_j|^2<\infty\right\}.$$

The spectral fractional Laplacian of $u\in H(2s)$ is the function

$$(-\triangle|_\Omega)^s u \;=\; \sum_{j=1}^{\infty}\lambda_j^s\widehat{u}_j\,\varphi_j. \tag{2}$$

Alternatively, for almost every $x\in\Omega$,

$$(-\triangle|_\Omega)^s u(x) \;=\; p.v.\int_\Omega[u(x)-u(y)]J(x,y)\,dy+\kappa(x)u(x), \tag{3}$$

where, letting $p_\Omega(t,x,y)$ denote the heat kernel of $-\triangle|_\Omega$,

$$J(x,y)=\frac{s}{\Gamma(1-s)}\int_0^\infty\frac{p_\Omega(t,x,y)}{t^{1+s}}\,dt,\ \kappa(x)=\frac{s}{\Gamma(1-s)}\int_\Omega\left(1-\int_\Omega p_\Omega(t,x,y)\,dy\right)\frac{dt}{t^{1+s}} \tag{4}$$

are respectively the *jumping kernel* and the *killing measure*[*].

We assume from now on that Ω is of class $C^{1,1}$. In this case, sharp bounds are known for p_Ω, see (17), and provide in turn sharp estimates for $J(x,y)$, see (19), so that the right-hand side of (3) remains well-defined for every $x\in\Omega$ under the assumption that $u\in C^{2s+\varepsilon}_{loc}(\Omega)\cap L^1(\Omega,\delta(x)dx)$ for some $\varepsilon>0$. This allows us to *define* the spectral fractional Laplacian of functions which *do not* vanish on the boundary of Ω.

DEFINITION 2. *The Green function and the Poisson kernel of the spectral fractional Laplacian are defined respectively by*

$$G^s_\Omega(x,y) \;=\; \frac{1}{\Gamma(s)}\int_0^\infty p_\Omega(t,x,y)t^{s-1}\,dt,\qquad x,y\in\Omega,x\neq y,\,s\in(0,1], \tag{5}$$

and by

$$P^s_\Omega(x,y)\;:=\;-\frac{\partial}{\partial\nu_y}G^s_\Omega(x,y),\qquad x\in\Omega,y\in\partial\Omega. \tag{6}$$

where ν is the outward unit normal to $\partial\Omega$.

DEFINITION 3. *Consider the function space $\mathcal{T}(\Omega)\;:=\;(-\triangle|_\Omega)^{-s}C^\infty_c(\Omega)$ and*

$$h_1(x)=\int_{\partial\Omega}P^s_\Omega(x,y)\,d\sigma(y),\qquad x\in\Omega. \tag{7}$$

[*]In the language of potential theory of killed stochastic processes. Note that the integral in (3) must be understood in the sense of principal values. To see this, look at (19).

Given two Radon measures $\mu \in \mathcal{M}(\Omega)$ and $\zeta \in \mathcal{M}(\partial\Omega)$ with

$$\int_\Omega \delta(x)\, d|\mu|(x) \;<\; \infty, \qquad |\zeta|(\partial\Omega) \;<\; \infty, \tag{8}$$

a function $u \in L^1_{loc}(\Omega)$ is a weak solution to

$$(-\triangle|_\Omega)^s u = \mu \quad \text{in } \Omega, \qquad \frac{u}{h_1} = \zeta \quad \text{on } \partial\Omega \tag{9}$$

if, for any $\psi \in \mathcal{T}(\Omega)$,

$$\int_\Omega u\,(-\triangle|_\Omega)^s \psi \;=\; \int_\Omega \psi\, d\mu - \int_{\partial\Omega} \frac{\partial\psi}{\partial\nu}\, d\zeta. \tag{10}$$

We present some facts about harmonic functions in Section 2.1 with an eye kept on their singular boundary trace. We prove the well-posedness of (9) in Section 2.1. In Section 3 we solve nonlinear Dirichlet problems.

THEOREM 1. *Given two Radon measures $\mu \in \mathcal{M}(\Omega)$ and $\zeta \in \mathcal{M}(\partial\Omega)$ such that (8) holds, there exists a weak solution $u \in L^1_{loc}(\Omega)$ to (9). Moreover, for a.e. $x \in \Omega$,*

$$u(x) \;=\; \int_\Omega G^s_\Omega(x,y)\, d\mu(y) + \int_{\partial\Omega} P^s_\Omega(x,y)\, d\zeta(y). \tag{11}$$

THEOREM 2. *Let $g(x,t) : \Omega \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a Carathéodory function such that $g(x,0) = 0$, and $h : \mathbb{R}^+ \to \mathbb{R}^+$ a nondecreasing function with*

$$0 \leqslant g(x,t) \leqslant h(t) \quad \text{for a.e. } x \in \Omega \text{ and all } t > 0, \qquad h(\delta^{-(2-2s)})\delta \in L^1(\Omega).$$

Then, problem

$$(-\triangle|_\Omega)^s u = -g(x,u) \quad \text{in } \Omega, \qquad \frac{u}{h_1} = \zeta \quad \text{on } \partial\Omega \tag{12}$$

has a solution $u \in L^1(\Omega, \delta(x)dx)$ for any $\zeta \in C(\partial\Omega), \zeta \geqslant 0$. In addition, if $t \mapsto g(x,t)$ is nondecreasing then the solution is unique.

THEOREM 3. *Let $p \in \left(1+s, \frac{1}{1-s}\right)$. There exists a function $u \in L^1(\Omega, \delta(x)dx) \cap C^\infty(\Omega)$ solving*

$$(-\triangle|_\Omega)^s u = -u^p \quad \text{in } \Omega, \qquad \frac{u}{h_1} = +\infty \quad \text{on } \partial\Omega \tag{13}$$

in the following sense: the first equality holds pointwise and in the sense of distributions, the boundary condition is understood as a pointwise limit. In addition, there exists a constant $C = C(\Omega, N, s, p)$ such that $0 \leqslant u \leqslant C\delta^{-\frac{2s}{p-1}}$.

2. Green function and Poisson kernel

In the following three lemmas, we review some useful identities for the Green function defined by (5). Compare them also with [7, formulas (17) and (8) respectively].

LEMMA 1. *Let $f \in L^2(\Omega)$. For almost every $x \in \Omega$, $G^s_\Omega(x,\cdot)f \in L^1(\Omega)$ and*

$$(-\triangle|_\Omega)^{-s} f(x) = \int_\Omega G^s_\Omega(x,y) f(y)\, dy \qquad \text{for a.e. } x \in \Omega.$$

LEMMA 2. *For a.e. $x, y \in \Omega$, $\int_\Omega G^{1-s}_\Omega(x,\xi) G^s_\Omega(\xi,y)\, d\xi = G^1_\Omega(x,y)$.*

Proof. Clearly $(-\triangle|_\Omega)^{-s}(-\triangle|_\Omega)^{s-1}\varphi_j = \lambda_j^{-s}\lambda_j^{s-1}\varphi_j = (-\triangle|_\Omega)^{-1}\varphi_j$ for any eigenfunction φ_j, so $(-\triangle|_\Omega)^{-s} \circ (-\triangle|_\Omega)^{s-1} = (-\triangle|_\Omega)^{-1}$ in $L^2(\Omega)$. By the previous lemma and Fubini's theorem, we deduce that for $\varphi \in L^2(\Omega)$ and a.e. $x \in \Omega$,

$$\int_\Omega \int_\Omega G^{1-s}_\Omega(x,\xi) G^s_\Omega(\xi,y)\varphi(y)\, d\xi\, dy = \int_\Omega G^1_\Omega(x,y)\varphi(y)\, dy$$

and so (2) holds almost everywhere. □

LEMMA 3. *For any $\psi \in C^\infty_c(\Omega)$,*

$$(-\triangle|_\Omega)^s \psi = (-\triangle) \circ (-\triangle|_\Omega)^{s-1}\psi = (-\triangle|_\Omega)^{s-1} \circ (-\triangle)\psi \tag{14}$$

Proof. The identity clearly holds if ψ is an eigenfunction. If $\psi \in C^\infty_c(\Omega)$, its spectral coefficients have fast (more than algebraic) decay and the result follows by writing the spectral decomposition of ψ. □

LEMMA 4. *The function $P^s_\Omega(x,y) := -\frac{\partial}{\partial \nu_y} G^s_\Omega(x,y)$ is well-defined for $x \in \Omega, y \in \partial\Omega$ and $P^s_\Omega(x,\cdot) \in C(\partial\Omega)$ for any $x \in \Omega$. Furthermore, there exists a constant $C > 0$ depending on N, s, Ω only such that*

$$\frac{1}{C}\frac{\delta(x)}{|x-y|^{N+2-2s}} \leqslant P^s_\Omega(x,y) \leqslant C\frac{\delta(x)}{|x-y|^{N+2-2s}}. \tag{15}$$

and

$$\int_\Omega G^{1-s}_\Omega(x,\xi) P^s_\Omega(\xi,y)\, d\xi = P^1_\Omega(x,y). \tag{16}$$

Proof. The proof is technical. The interested reader can find it in [3]. A main ingredient is the sharp double-sided estimate for the heat kernel (cf. [5]):

$$\left[\frac{\delta(x)\delta(y)}{t} \wedge 1\right] \frac{1}{c_1 t^{N/2}} e^{-|x-y|^2/(c_2 t)} \leqslant p_\Omega(t,x,y) \leqslant \left[\frac{\delta(x)\delta(y)}{t} \wedge 1\right] \frac{c_1}{t^{N/2}} e^{-c_2|x-y|^2/t}. \tag{17}$$

Recall that the Poisson kernel of the Dirichlet Laplacian can be defined via the very same formula, just by considering $s = 1$. □

REMARK 1. Thanks to bound (17), the following estimates also hold:

$$\frac{1}{C|x-y|^{N-2s}}\left(1\wedge\frac{\delta(x)\,\delta(y)}{|x-y|^2}\right)\leqslant G_\Omega^s(x,y)\leqslant\frac{C}{|x-y|^{N-2s}}\left(1\wedge\frac{\delta(x)\,\delta(y)}{|x-y|^2}\right)\tag{18}$$

for some constant $C=C(\Omega,N,s)$, and

$$\frac{1}{C|x-y|^{N+2s}}\left(1\wedge\frac{\delta(x)\delta(y)}{|x-y|^2}\right)\leqslant J(x,y)\leqslant\frac{C}{|x-y|^{N+2s}}\left(1\wedge\frac{\delta(x)\delta(y)}{|x-y|^2}\right).\tag{19}$$

2.1. Harmonic functions

DEFINITION 4. *A function $h\in L^1(\Omega,\delta(x)dx)$ is s-harmonic in Ω if*

$$\int_\Omega h\,(-\triangle|_\Omega)^s\psi\;=\;0\qquad\text{for any }\psi\in C_c^\infty(\Omega).$$

LEMMA 5. *For any $\psi\in C_c^\infty(\Omega)$, $(-\triangle|_\Omega)^s\psi\in C_0^1(\overline{\Omega})$ and there exists a constant $C=C(s,N,\Omega,\psi)>0$ such that $|(-\triangle|_\Omega)^s\psi|\;\leqslant\;C\delta$ in Ω.*

Proof. One has

$$\left|\frac{(-\triangle|_\Omega)^s\psi}{\delta}\right|\leqslant\sum_{j=1}^{\infty}\lambda_j^s|\hat{\psi}_j|\left\|\frac{\varphi_j}{\delta}\right\|_{L^\infty(\Omega)}<\infty.$$

□

LEMMA 6. *The function $P_\Omega^s(\cdot,z)\in L^1(\Omega,\delta(x)dx)$ is s-harmonic in Ω.*

Proof. Thanks to (15), $P_\Omega^s(\cdot,z)\in L^1(\Omega,\delta(x)dx)$. Pick $\psi\in C_c^\infty(\Omega)$ and exploit (14). □

LEMMA 7. *For any finite Radon measure $\zeta\in\mathcal{M}(\partial\Omega)$, let*

$$h(x)\;=\;\int_{\partial\Omega}P_\Omega^s(x,z)\,d\zeta(z),\qquad x\in\Omega.\tag{20}$$

Then, h is s-harmonic in Ω.

Proof. Since $P_\Omega^s(x,\cdot)$ is continuous, h is well-defined. Pick $\psi\in C_c^\infty(\Omega)$:

$$\int_\Omega h(x)\,(-\triangle|_\Omega)^s\psi(x)\,dx=\int_{\partial\Omega}\left(\int_\Omega P_\Omega^s(x,z)\,(-\triangle|_\Omega)^s\psi(x)\,dx\right)d\zeta(z)=0$$

in view of Lemma 6. □

As a matter of fact, a computation shows how the reference function h_1 possess a precise boundary behaviour, which is

$$\frac{1}{C}\,\delta^{-(2-2s)}\leqslant h_1\leqslant C\delta^{-(2-2s)}.\tag{21}$$

for some constant $C = C(N,\Omega,s) > 0$. In the following we will use the notation $\mathbb{P}^s_\Omega g := \int_{\partial\Omega} P^s_\Omega(\cdot,\theta)\, g(\theta)\, d\sigma(\theta)$ where σ denotes the Hausdorff measure on $\partial\Omega$, whenever $g \in L^1(\Omega)$.

PROPOSITION 1. *Let $\zeta \in C(\partial\Omega)$. Then, for any $z \in \partial\Omega$,*

$$\frac{\mathbb{P}^s_\Omega \zeta(x)}{h_1(x)} \xrightarrow[x\to z]{x\in\Omega} \zeta(z) \qquad \textit{uniformly on } \partial\Omega. \tag{22}$$

Proof. Let us write

$$\left|\frac{\mathbb{P}^s_\Omega\zeta(x)}{h_1(x)} - \zeta(z)\right| = \left|\frac{1}{h_1(x)}\int_{\partial\Omega} P^s_\Omega(x,\theta)\,\zeta(\theta)\, d\sigma(\theta) - \frac{h_1(x)\,\zeta(z)}{h_1(x)}\right| \leqslant$$
$$\leqslant \frac{1}{h_1(x)}\int_{\partial\Omega} P^s_\Omega(x,\theta)|\zeta(\theta)-\zeta(z)|\, d\sigma(\theta) \leqslant C\delta(x)^{3-2s}\int_{\partial\Omega}\frac{|\zeta(\theta)-\zeta(z)|}{|x-\theta|^{N+2-2s}}\, d\sigma(\theta) \leqslant$$
$$\leqslant C\delta(x)\int_{\partial\Omega}\frac{|\zeta(\theta)-\zeta(z)|}{|x-\theta|^{N}}\, d\sigma(\theta).$$

It suffices now to repeat the computations in [1, Lemma 3.1.5] to show that the obtained quantity converges to 0 as $x \to z$. □

LEMMA 8. *$\mathcal{T}(\Omega) \subseteq C^1_0(\overline{\Omega}) \cap C^\infty(\Omega)$. Moreover, for any $\psi \in \mathcal{T}(\Omega)$ and $z \in \partial\Omega$,*

$$-\frac{\partial\psi}{\partial\nu}(z) = \int_\Omega P^s_\Omega(y,z)\,(-\triangle|_\Omega)^s\psi(y)\, dy. \tag{23}$$

Proof. Take $\psi \in \mathcal{T}(\Omega)$ and let $f = (-\triangle|_\Omega)^s\psi$. Since $f \in C^\infty_c(\Omega)$, the spectral coefficients of f have fast (more than algebraic) decay and so the same holds true for ψ. It follows that $\psi \in C^1_0(\overline{\Omega})$ and $\mathcal{T}(\Omega) \subseteq C^1_0(\overline{\Omega})$. By Lemma 1, for all $x \in \overline{\Omega}$, $\psi(x) = \int_\Omega G^s_\Omega(x,y)\, f(y)\, dy$. Using Lemma 4 and the dominated convergence theorem, (23) follows. Since $(-\triangle|_\Omega)^s$ is self-adjoint in $H(2s)$, we know that the equality $(-\triangle|_\Omega)^s\psi = f$ holds in $\mathcal{D}'(\Omega)$. □

LEMMA 9 (Maximum principle for classical solutions). *Let $u \in C^{2s+\varepsilon}_{loc}(\Omega) \cap L^1(\Omega,\delta(x)\,dx)$ such that*

$$(-\triangle|_\Omega)^s u \geqslant 0 \ \textit{ in } \Omega, \qquad \liminf_{x\to\partial\Omega} u(x) \geqslant 0.$$

Then $u \geqslant 0$ in Ω. In particular this holds when $u \in \mathcal{T}(\Omega)$.

Proof. Suppose $x^* \in \Omega$ such that $u(x^*) = \min_\Omega u < 0$. Then

$$(-\triangle|_\Omega)^s u(x^*) = \int_\Omega [u(x^*)-u(y)]\, J(x,y)\, dy + \kappa(x^*)u(x^*) < 0,$$

a contradiction. □

LEMMA 10. *Let $\mu \in \mathcal{M}(\Omega)$, $\zeta \in \mathcal{M}(\partial\Omega)$ be two Radon measures satisfying (8) with $\mu \geqslant 0$ and $\zeta \geqslant 0$. Consider $u \in L^1_{loc}(\Omega)$ a weak solution to the Dirichlet problem (9). Then $u \geqslant 0$ a.e. in Ω.*

Proof. Take $f \in C_c^\infty(\Omega)$, $f \geqslant 0$ and $\psi = (-\triangle|_\Omega)^{-s} f \in \mathcal{T}(\Omega)$. By Lemma 9, $\psi \geqslant 0$ in Ω and by Lemma 8 $-\frac{\partial\psi}{\partial\nu} \geqslant 0$ on $\partial\Omega$. Thus, by (10), $\int_\Omega uf \geqslant 0$. Since this is true for every $f \in C_c^\infty(\Omega)$, the result follows. □

Proof of Theorem 1. Uniqueness is a direct consequence of the comparison principle, Lemma 10. Let us prove that formula (11) defines the desired weak solution. Observe that if u is given by (11), then $u \in L^1(\Omega, \delta(x)dx)$. Indeed,

$$\begin{aligned}(24)\qquad \int_\Omega \left|\varphi_1(x)\int_\Omega G^s_\Omega(x,y)d\mu(y)\right| dx &\leqslant \int_\Omega\int_\Omega G^s_\Omega(x,y)\varphi_1(x)dx\, d|\mu|(y)\\ &= \frac{1}{\lambda_1^s}\int_\Omega \varphi_1(y)\, d|\mu|(y) \leqslant C\|\delta\mu\|_{\mathcal{M}(\Omega)}\end{aligned}$$

This, along with Lemma 7, proves that $u \in L^1(\Omega, \delta(x)dx)$. Now, pick $\psi \in \mathcal{T}(\Omega)$ and compute, via the Fubini's Theorem, Lemma 1 and Lemma 8,

$$\begin{aligned}&\int_\Omega u(x)(-\triangle|_\Omega)^s\psi(x)\,dx =\\ &= \int_\Omega\int_\Omega G^s_\Omega(x,y)d\mu(y)(-\triangle|_\Omega)^s\psi(x)\,dx + \int_\Omega\int_{\partial\Omega} P^s_\Omega(x,z)\,d\zeta(z)(-\triangle|_\Omega)^s\psi(x)\,dx\\ &= \int_\Omega\int_\Omega G^s_\Omega(x,y)(-\triangle|_\Omega)^s\psi(x)\,dx\,d\mu(y) + \int_{\partial\Omega}\int_\Omega P^s_\Omega(x,z)(-\triangle|_\Omega)^s\psi(x)\,dx\,d\zeta(z)\\ &= \int_\Omega \psi(y)\,d\mu(y) - \int_{\partial\Omega}\frac{\partial\psi}{\partial\nu}(z)\,d\zeta(z).\end{aligned}$$

□

3. The nonlinear problem

THEOREM 4. *Let $f(x,t) : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be a Carathéodory function. Assume that there exists a subsolution and a supersolution $\underline{u}, \overline{u} \in L^1(\Omega, \delta(x)dx) \cap L^\infty_{loc}(\Omega)$ to*

$$(25)\qquad (-\triangle|_\Omega)^s u = f(x,u) \quad \text{in } \Omega, \qquad \frac{u}{h_1} = 0 \quad \text{on } \partial\Omega$$

Assume in addition that $f(\cdot, v) \in L^1(\Omega, \delta(x)dx)$ for every $v \in L^1(\Omega, \delta(x)dx)$ such that $\underline{u} \leqslant v \leqslant \overline{u}$ a.e. Then, there exist weak solutions $u_1, u_2 \in L^1(\Omega, \delta(x)dx)$ in $[\underline{u}, \overline{u}]$ such that any solution in the interval $[\underline{u}, \overline{u}]$ satisfies

$$\underline{u} \leqslant u_1 \leqslant u \leqslant u_2 \leqslant \overline{u} \qquad a.e.$$

Moreover, if the nonlinearity f is decreasing in the second variable, then the solution is unique.

Proof. The proof can be performed by adapting the one in [9] to the fractional case. More details can be found in [3]. □

Proof of Theorem 2. Problem (12) is equivalent to

$$(26) \qquad (-\triangle|_\Omega)^s v = g(x, \mathbb{P}^s_\Omega \zeta - v) \quad \text{in } \Omega, \qquad \frac{v}{h_1} = 0 \quad \text{on } \partial\Omega$$

that possesses $\overline{u} = \mathbb{P}^s_\Omega \zeta$ as a supersolution and $\underline{u} = 0$ as a subsolution. Indeed, by equation (21) we have $0 \leqslant \mathbb{P}^s_\Omega \zeta \leqslant \|\zeta\|_{L^\infty(\Omega)} h_1 \leqslant C\|\zeta\|_{L^\infty(\Omega)} \delta^{-(2-2s)}$. Thus any $v \in L^1(\Omega, \delta(x)dx)$ such that $0 \leqslant v \leqslant \mathbb{P}^s_\Omega \zeta$ satisfies

$$g(x,v) \leqslant h(v) \leqslant h(c\delta^{-(2-2s)}) \in L^1(\Omega, \delta(x)dx).$$

So, all hypotheses of Theorem 4 are satisfied and the result follows. □

4. Large solutions

Consider the sequence $\{u_j\}_{j\in\mathbb{N}}$ built by solving

$$(27) \qquad (-\triangle|_\Omega)^s u_j = -u_j^p \quad \text{in } \Omega, \qquad \frac{u_j}{h_1} = j \quad \text{on } \partial\Omega.$$

Theorem 2 guarantees the existence of such a sequence if $\delta^{-(2-2s)p} \in L^1(\Omega, \delta(x)dx)$, i.e. $p < 1/(1-s)$. Moreover, $\{u_j\}_{j\in\mathbb{N}}$ is increasing with j, thus it admits a pointwise limit, possibly infinite.

LEMMA 11. *There exist $\delta_0, C > 0$ such that $(-\triangle|_\Omega)^s \delta^{-\alpha} \geqslant -C\delta^{-\alpha p}$, for $\delta < \delta_0$ and $\alpha = \frac{2s}{p-1}$.*

Proof. We use the expression in equation (3). Obviously,

$$(-\triangle|_\Omega)^s \delta^{-\alpha}(x) = \int_\Omega [\delta(x)^{-\alpha} - \delta(y)^{-\alpha}] J(x,y)\, dy + \delta(x)^{-\alpha}\kappa(x) \geqslant$$
$$\geqslant \int_\Omega [\delta(x)^{-\alpha} - \delta(y)^{-\alpha}] J(x,y)\, dy.$$

□

LEMMA 12. *If a function $v \in L^1(\Omega, \delta(x)dx)$ satisfies*

$$(28) \qquad (-\triangle|_\Omega)^s v \in L^\infty_{loc}(\Omega), \qquad (-\triangle|_\Omega)^s v(x) \geqslant -C\, v(x)^p, \quad \text{when } \delta(x) < \delta_0,$$

for some $C, \delta_0 > 0$, then there exists $\overline{u} \in L^1(\Omega, \delta(x)dx)$ such that

$$(29) \qquad (-\triangle|_\Omega)^s \overline{u}(x) \geqslant -\overline{u}(x)^p, \qquad \text{throughout } \Omega.$$

Proof. Let $\lambda := C^{1/(p-1)} \vee 1$ and $\Omega_0 = \{x \in \Omega : \delta(x) < \delta_0\}$, then

$$(-\triangle|_\Omega)^s(\lambda v) \geqslant -(\lambda v)^p, \qquad \text{in } \Omega_0.$$

Let also $\mu := \lambda \|(-\triangle|_\Omega)^s v\|_{L^\infty(\Omega\setminus\Omega_0)}$ and define $\overline{u} = \mu \mathbb{G}_\Omega^s 1 + \lambda v$. On $\overline{u}$ we have $(-\triangle|_\Omega)^s \overline{u} = \mu + \lambda(-\triangle|_\Omega)^s v \geqslant \lambda|(-\triangle|_\Omega)^s v| + \lambda(-\triangle|_\Omega)^s v \geqslant -\overline{u}^p$ throughout Ω. □

COROLLARY 1. *There exists a function $\overline{u} \in L^1(\Omega, \delta(x)dx)$ such that*

$$(-\triangle|_\Omega)^s \overline{u} \geqslant -\overline{u}^p, \qquad \text{in } \Omega,$$

holds in a pointwise sense. Moreover, $\overline{u} \asymp \delta^{-2s/(p-1)}$.

Proof. Apply Lemma 12 with $v = \delta^{-2s/(p-1)}$. □

LEMMA 13. *For any $j \in \mathbb{N}$, the solution u_j to problem (27) satisfies the upper bound $u_j \leqslant \overline{u}$, in Ω, where $\overline{u}$ is provided by Corollary 1.*

Proof. Write $u_j = jh_1 - v_j$ where

$$(-\triangle|_\Omega)^s v_j = (jh_1 - v_j)^p \quad \text{in } \Omega, \qquad \frac{v_j}{h_1} = 0 \quad \text{on } \partial\Omega.$$

and $0 \leqslant v_j \leqslant jh_1$. Since $(jh_1 - v_j)^p \in L^\infty_{loc}(\Omega)$, we deduce that $v_j \in C^\alpha_{loc}(\Omega)$ for any $\alpha \in (0, 2s)$. Now, we have that, by the boundary behaviour of $\overline{u}$ stated in Corollary 1, $u_j \leqslant \overline{u}$ close enough to $\partial\Omega$ (depending on the value of j) and

$$(-\triangle|_\Omega)^s(\overline{u} - u_j) \geqslant u_j^p - \overline{u}^p, \qquad \text{in } \Omega.$$

Since $u_j^p - \overline{u}^p \in C(\Omega)$ and $\lim_{x\to\partial\Omega} u_j^p - \overline{u}^p = -\infty$, then there exists $x_0 \in \Omega$ such that $u_j(x_0)^p - \overline{u}(x_0)^p = m =: \max_{x\in\Omega}(u_j(x)^p - \overline{u}(x)^p)$. If $m > 0$ then also $(-\triangle|_\Omega)^s(\overline{u} - u_j)(x_0) \geqslant m > 0$: this is a contradiction, as Definition 3 implies. Thus $m \leqslant 0$ and $u_j \leqslant \overline{u}$ throughout Ω. □

THEOREM 5. *For any $p \in \left(1+s, \frac{1}{1-s}\right)$ there exists $u \in L^1(\Omega, \delta(x)dx)$ solving*

$$(-\triangle|_\Omega)^s u = -u^p \quad \text{in } \Omega, \qquad \delta^{2-2s} u = +\infty \quad \text{on } \partial\Omega.$$

Proof. Consider the sequence $\{u_j\}_{j\in\mathbb{N}}$ provided by problem 27: it is increasing and locally bounded by Lemma 13, so it has a pointwise limit $u \leqslant \overline{u}$, where $\overline{u}$ is the function provided by Corollary 1. Since $p > 1+s$ and $\overline{u} \leqslant C\delta^{-2s/(p-1)}$, then $u \in L^1(\Omega, \delta(x)dx)$. Pick now $\psi \in C^\infty_c(\Omega)$, and recall that $\delta^{-1}(-\triangle|_\Omega)^s\psi \in L^\infty(\Omega)$: we have, by dominated convergence,

$$\int_\Omega u_j(-\triangle|_\Omega)^s\psi \xrightarrow[j\uparrow\infty]{} \int_\Omega u(-\triangle|_\Omega)^s\psi, \qquad \int_\Omega u_j^p\psi \xrightarrow[j\uparrow\infty]{} \int_\Omega u^p\psi$$

so we deduce

$$\int_\Omega u(-\triangle|_\Omega)^s\psi = -\int_\Omega u^p\psi.$$

Note now that for any compact $K \subset\subset \Omega$, applying some known elliptic regularity estimates we get for any $\alpha \in (0,2s)$

$$\|u_j\|_{C^\alpha(K)} \leqslant C\left(\|u_j\|^p_{L^\infty(K)} + \|u_j\|_{L^1(\Omega,\delta(x)dx)}\right) \leqslant C\left(\|\overline{u}\|^p_{L^\infty(K)} + \|\overline{u}\|_{L^1(\Omega,\delta(x)dx)}\right)$$

which means that $\{u_j\}_{j\in\mathbb{N}}$ is equibounded and equicontinuous in $C(K)$. By the Ascoli-Arzelà Theorem, its pointwise limit u will be in $C(K)$ too. Now, since

$$(-\triangle|_\Omega)^s u = -u^p \qquad \text{in } \mathcal{D}'(\Omega),$$

by bootstrapping the interior regularity we deduce $u \in C^\infty(\Omega)$. □

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AMS Subject Classification: Primary: 35B40; Secondary: 35B30, 45P05, 35C15

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Lavoro pervenuto in redazione il 11.11.2016.