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DERIVATIONS OF RINGS AND BANACH ALGEBRAS INVOLVING COMMUTATOR

Abstract. In the present paper we obtain the commutativity of a semiprime ring $\mathcal{R}$ admitting a derivation d such that $(d([u^m,v^n]))^\ell = \pm[u^p,v^q]_k$ for all $u,v \in \mathcal{R}$ and for fixed positive integers m,n,p,q,k,ℓ. Finally, we apply the above purely ring theoretic result to Banach algebras and obtain a noncommutative version of the Singer-Wermer theorem. In particular, we prove that if $\mathfrak{B}$ is a noncommutative Banach algebra which admits a continuous linear derivation $d : \mathfrak{B} \to \mathfrak{B}$ such that $(d([u^m,v^n]))^\ell \pm [u^p,v^q]_k \in rad(\mathfrak{B})$ for all $u,v \in \mathfrak{B}$, where m,n,p,q,k,ℓ are fixed positive integers, then $d(\mathfrak{B}) \subseteq rad(\mathfrak{B})$.

1. Introduction

Throughout this paper $\mathcal{R}$ will denote a prime ring with center $Z(\mathcal{R})$. The Martindale quotient ring and the Utumi quotient ring of $\mathcal{R}$ are denoted by $Q = Q(\mathcal{R})$ and $\mathfrak{A} = \mathfrak{A}(\mathcal{R})$ respectively. The center of $\mathfrak{A}$ is called the extended centroid of $\mathcal{R}$ and is denoted by $C = Z(\mathfrak{A})$ (we refer the reader to [6] for these objects). By a Banach algebra $\mathfrak{B}$ we mean an algebra equipped with a norm $\| \cdot \|$ that makes it into a Banach space and additionally satisfies the inequality $\| uv \| \leqslant \| u \| \| v \|$ for all $u,v \in \mathfrak{B}$ (see [7]). The Jacobson radical of $\mathfrak{B}$, denoted by $rad(\mathfrak{B})$, is the intersection of all the primitive ideals of $\mathcal{B}$. An algebra $\mathcal{B}$ is called semi-simple Banach algebra if $rad(\mathcal{B}) = 0$.

For any $u,v \in \mathcal{R}$, the symbol $[u,v]$ represents the commutator $uv - vu$. Given $u,v \in \mathcal{R}$, set $[u,v]_0 = u$, $[u,v]_1 = uv - vu$ and by induction $[u,v]_k = [[u,v]_{k-1},v]$ for $k > 1$. The ring $\mathcal{R}$ is said to satisfy the Engel condition if $[u,v]_k = 0$ for some fixed $k > 1$. A ring $\mathcal{R}$ is said to be prime if for any $u,v \in \mathcal{R}$, $u\mathcal{R}v = \{0\}$ implies either $u = 0$ or $v = 0$, and is said to be semiprime if for any $u \in \mathcal{R}$, $u\mathcal{R}u = \{0\}$ implies $u = 0$. An additive mapping $d : \mathcal{R} \to \mathcal{R}$ is called a derivation on $\mathcal{R}$ if $d(uv) = d(u)v + ud(v)$ holds for all $u,v \in \mathcal{R}$. The mapping $I_a : \mathcal{R} \to \mathcal{R}$ given by $I_a(u) = [a,u]$ for all $u \in \mathcal{R}$ and for some fixed $a \in \mathcal{R}$ is known as the inner derivation in $\mathcal{R}$ determined by a.

During the past few decades, many authors have given fascinating results concerning the relationship between the commutativity of a ring and certain specific types of derivations under some suitable restrictions (see [5], where further references can be found). The beginning of the Engel type identity with derivation was first seen in the well-known paper by Posner [26] who proved that a prime ring admitting a nonzero derivation d such that $[d(u),u] \in Z(\mathcal{R})$ for all $u \in \mathcal{R}$, must be commutative. Since then there has been an ongoing interest in the study of Engel type identities with derivations acting on two-sided ideals and Lie ideals of prime and semiprime rings (see [11]). In the year 1992, Daif and Bell [10, Theorem 3] showed that if I is a nonzero ideal in a semiprime ring $\mathcal{R}$ and $d : \mathcal{R} \to \mathcal{R}$ is a derivation such that $d[u,v] = [u,v]$ holds for

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all $u,v \in I$, then I is contained into the centre of ring $\mathcal{R}$. If $\mathcal{R}$ is a prime ring, this implies that $\mathcal{R}$ is commutative. Following the same line, Huang [16, Theorem 1], obtained that if I is a nonzero ideal in a prime ring $\mathcal{R}$ and $d : \mathcal{R} \to \mathcal{R}$ is a derivation such that $(d[u,v])^m = [u,v]_n$ holds for all $u,v \in I$ and m,n fixed positive integers, then $\mathcal{R}$ is commutative. Further several authors have investigated commutativity of prime and semiprime rings satisfying certain differential identities in rings (for references see [1],[2], [4],[14],[15], [22],[24]).

In view of the above results it is quite natural to ask about the commutativity of a ring $\mathcal{R}$ if $\mathcal{R}$ satisfies $(d([u^m,v^n]))^\ell = \pm[u^p,v^q]_k$ for all $u,v \in \mathcal{R}$. In this paper we have investigated this identity and obtained the commutativity of a prime ring $\mathcal{R}$. In fact, we also prove the commutativity of a semiprime ring $\mathcal{R}$.

2. The Result on prime and semiprime rings

We know that any derivation d of a semiprime ring $\mathcal{R}$ can be uniquely extended to a derivation of its Utumi quotient ring $\mathfrak{A}$ (maximal right ring of quotient) and hence can be defined on the whole $\mathfrak{A}$ (Beidar et al., [6, Proposition 2.5.1]). Further, the Utumi quotient ring of a semiprime ring $\mathcal{R}$ is also semiprime. Moreover, we shall use the fact that the extended centroid $\mathcal{C}$ of a semiprime ring $\mathcal{R}$ coincides with the center of its Martindale quotient ring (Chuang, [8, pp. 724]). Also $\mathcal{R}$ is assumed to be a $\mathcal{B}$-algebra which is orthogonally complete, where $\mathcal{B}$ is the set of all the idempotents in $\mathcal{C}$. For any maximal ideal $\mathcal{P}$ of $\mathcal{B}$, $\mathcal{P}\mathcal{R}$ forms a minimal prime ideal of $\mathcal{R}$, which is invariant under any nonzero derivation of $\mathcal{R}$ (Chuang, [9, pp. 42]). We shall use the following facts without any specific mention.

FACT 1. [6, proposition 2.5.1]Suppose $\mathcal{R}$ is a semiprime ring and d be any derivation on $\mathcal{R}$, then d can be uniquely extended to its left Utumi quotient ring $\mathfrak{A}$, and so any derivation d of $\mathcal{R}$ can be defined on the whole $\mathfrak{A}$.

FACT 2. [8, Theorem 2] Let $\mathcal{R}$ be a prime ring and and $\mathfrak{I}$ be an ideal of $\mathcal{R}$, then $\mathfrak{I}$, $\mathcal{R}$ and $\mathcal{Q}$ satisfy the same generalized polynomial identities with coefficients in $\mathcal{Q}$.

FACT 3. [19, Theorem 2] Suppose $\mathcal{R}$ is a prime ring admitting a nonzero derivation d and $\mathfrak{I}$ be a nonzero two-sided ideal of $\mathcal{R}$. If $f(u_1,\dots,u_n,d(u_1),\dots,d(u_n))$ is a differential identity in $\mathfrak{I}$, that is

$$f(s_1,\dots,s_n,d(s_1),\dots,d(s_n)) = 0 \text{ for all } s_1,\dots,s_n \in \mathfrak{I}.$$

Then one of the following holds:

1. For the Martindale quotient ring Q of $\mathcal{R}$, d is an inner derivation in Q, and therefore there exists $b \in Q$ such that $d(u) = [b,u]$ for all $u \in \mathcal{R}$. Moreover, $\mathfrak{I}$ satisfies the generalized polynomial identity

   $$f(s_1,\dots,s_n,[b,s_1],\dots,[b,s_n]) \text{ for all } s_1,\dots,s_n \in \mathfrak{I}.$$

   or

2. $\mathfrak{I}$ satisfies the generalized polynomial identity

$$f(u_1,\ldots,u_n,v_1,\ldots,v_n).$$

We shall start with the following result which plays an important role in establishing the proof of our main Theorem. The proof of Lemma 1 can be seen in [3].

LEMMA 1. *Let $\mathcal{R}$ be a ring satisfying an identity $q(X)$, where $q(X)$ is a polynomial in the finite number of non-commuting indeterminates, its coefficients being integers with highest common factor 1. If there exists no prime number p for which the ring of 2×2 matrices over $GF(p)$ satisfies $q(X)=0$, then $\mathcal{R}$ has nil commutator ideal and the nilpotent elements of $\mathcal{R}$ forms an ideal.*

LEMMA 2. *Let $\mathcal{R}$ be a ring and $r\in\mathcal{R}$ such that $r^2=0$. Then $[(ru)^m,(ur)^n]_k=(ru)^m(ur)^{kn}$ for all $u\in\mathcal{R}$ and fixed positive integers m,n,k.*

Proof. We proceed by induction on k. For $k=1$, we have $[(ru)^m,(ur)^n]=(ru)^m(ur)^n-(ur)^n(ru)^m$. Using the given hypothesis we get $[(ru)^m,(ur)^n]=(ru)^m(ur)^{1n}$ for all $u\in\mathcal{R}$. Thus the result is true for $k=1$. Now for $k>1$ assume that the result is true for $k-1$ i.e.;

$$[(ru)^m,(ur)^n]_{k-1}=(ru)^m(ur)^{(k-1)n}$$

for all $u\in\mathcal{R}$. Now

$$[(ru)^m,(ur)^n]_k=[[(ru)^m,(ur)^n]_{k-1},(ur)^n]$$

for all $u\in\mathcal{R}$. Then by the induction hypothesis, we find that

$$[(ru)^m,(ur)^n]_k=[(ru)^m(ur)^{(k-1)n},(ur)^n]=(ru)^m(ur)^{kn}$$

$u\in\mathcal{R}$. Thus the result is true for k as well. Hence the result is true for every positive integer k. □

LEMMA 3. *Let $\mathcal{R}$ be a semiprime ring such that $[u^m,v^n]_k=0$ for all $u,v\in\mathcal{R}$ and fixed integers $m\geqslant 1$, $n\geqslant 1$, $k\geqslant 1$. Then $\mathcal{R}$ is commutative.*

Proof. Given that

$$[u^m,v^n]_k=0 \text{ for all } u,v\in\mathcal{R}. \tag{1}$$

If $\mathcal{R}$ is a semiprime ring satisfying the polynomial identity (1), then $\mathcal{R}$ is isomorphic to a subdirect sum of prime rings $\mathcal{R}\alpha$ each of which as a homomorphic image of $\mathcal{R}$ satisfies the hypothesis placed on $\mathcal{R}$. Therefore, it can be assumed that $\mathcal{R}$ is a prime ring satisfying the identity (1). First we shall show that $\mathcal{R}$ has no nonzero nilpotent elements. Let $r\in\mathcal{R}$ such that $r^2=0$. Using our hypothesis, we find that

$$[(ru)^m,(ur)^n]_k=0 \text{ for all } u\in\mathcal{R}.$$

By Lemma 2, we have $(ru)^m(ur)^{kn}=0$, for all $u\in\mathcal{R}$. Now compute

$$\begin{aligned}(ru)^{m+kn+1} &= \{(ru)^{m+kn}r+(ru)^m(ur)^{kn}\}u\\ &= \{r(ur+u)\}^m\{(ur+u)r\}^{kn}u\\ &= 0.\end{aligned}$$

Therefore $(ru)^{m+kn+1}=0$ for all $u\in\mathcal{R}$. If $r\mathcal{R}\neq 0$, then it is to be noted that $r\mathcal{R}$ is a nil right ideal satisfying the identity $w^{m+kn+1}=0$ for all $w\in r\mathcal{R}$. Thus by the application of Lemma 1.1 of Herstein [13], we see that $r\mathcal{R}=0$ and hence the primeness of $\mathcal{R}$ forces that $r=0$.

Now if we consider $u=\begin{pmatrix}1&0\\0&0\end{pmatrix}$, $v=\begin{pmatrix}1&1\\0&0\end{pmatrix}$, we find that no 2×2 matrix ring over $GF(p)$, p a prime, satisfies the relation(1). Hence by Lemma 1, $\mathcal{R}$ contains a nil commutator ideal. But since $\mathcal{R}$ has no nonzero nilpotent elements, $\mathcal{R}$ does not contain any nonzero nil ideal and hence $\mathcal{R}$ is commutative. □

THEOREM 1. *Let $\mathcal{R}$ be a prime ring and m,n,p,q,k,ℓ be fixed positive integers. If $\mathcal{R}$ admits a derivation d such that $(d([u^m,v^n]))^\ell=\pm[u^p,v^q]_k$ for all $u,v\in\mathcal{R}$, then $\mathcal{R}$ is commutative.*

Proof. If $d=0$, then $[u^p,v^q]_k=0$ for all $u,v\in\mathcal{R}$. By Lemma 3, $\mathcal{R}$ is commutative.

Now we assume d is a nonzero derivation satisfying $(d([u^m,v^n]))^\ell=\pm[u^p,v^q]_k$ for all $u,v\in\mathcal{R}$ which can be rewritten as $([d(u^m),v^n]+[u^m,d(v^n)])^\ell=\pm[u^p,v^q]_k$. By Fact 3, we divide the proof into two cases:

Case 1. If d is Q-outer, then $\mathcal{R}$ satisfies the polynomial identity

$$\left(\left[\sum_{s=0}^{m-1}u^szu^{m-1-s},v^n\right]+\left[u^m,\sum_{s=0}^{n-1}v^swv^{n-1-s}\right]\right)^\ell=\pm[u^p,v^q]_k$$

for all $u,v,z,w\in\mathcal{R}$. In particular, for $z=w=0$, we obtain the identity $[u^p,v^q]_k=0$ for all $u,v\in\mathcal{R}$. By using Lemma 3, $\mathcal{R}$ is commutative.

Case 2. Let d be Q-inner derivation, i.e., there is an element $b\in Q$, such that $d(u)=[b,u]$ for all $u\in\mathcal{R}$. It follows that $([[b,u^m],v^n]+[u^m,[b,v^n]])^\ell=\pm[u^p,v^q]_k$ for all $u,v\in\mathcal{R}$. By Fact 2, $\mathcal{R}$ and Q satisfy the same generalized polynomial identities (GPIs), and hence we have $([[b,u^m],v^n]+[u^m,[b,v^n]])^\ell=\pm[u^p,v^q]_k$ for all $u,v\in Q$. If the center $\mathcal{C}$ of Q is infinite, we see that $([[b,u^m],v^n]+[u^m,[b,v^n]])^\ell=\pm[u^p,v^q]_k$ for all $u,v\in Q\bigotimes_{\mathcal{C}}\overline{\mathcal{C}}$, where $\overline{\mathcal{C}}$ is the algebraic closure of $\mathcal{C}$.

Since $\mathcal{R}$ is prime, in view of [9, pp.38] Q is also prime and in the light of [6, Proposition 2.2.2] $\mathcal{R}\subseteq Q\subseteq\mathfrak{A}$. By [6, Remark 2.3.1] $\mathcal{C}$ satisfies $Z(Q)=\mathcal{C}=Z(\mathfrak{A})$. But by [6, Theorem 2.1.11] we know that the Utumi quotient ring of the Utumi quotient ring of $\mathcal{R}$ coincides with $\mathfrak{A}$ i.e., $\mathfrak{A}(\mathfrak{A})=\mathfrak{A}$, and hence $\mathcal{C}(\mathfrak{A})=Z(\mathfrak{A}(\mathfrak{A}))=Z(\mathfrak{A})=\mathcal{C}$. Now in view of [6, Proposition 2.1.10], the above chain of inclusion yields that $\mathfrak{A}=\mathfrak{A}(\mathcal{R})\subseteq\mathfrak{A}(Q)\subseteq\mathfrak{A}(\mathfrak{A})=\mathfrak{A}$ and hence $\mathcal{C}=\mathcal{C}(\mathcal{R})\subseteq\mathcal{C}(Q)\subseteq\mathcal{C}(\mathfrak{A})=\mathcal{C}$. Since a ring is said to be centrally closed if it coincides with its central closure (see [6, pp.68]), we conclude that Q is centrally closed because $\mathcal{C}(Q)Q=\mathcal{C}Q=Q$. By [6, pp.70] $\mathcal{C}$ is a field and the obvious inclusion $\mathcal{C} \subseteq Q$ makes Q a $\mathcal{C}$-algebra. In the light of the above argument, Q is a closed prime algebra over $\mathcal{C}$ and $\overline{\mathcal{C}}$ is a field extension of $\mathcal{C}$. Now apply [12, Theorem 3.5] to conclude that $Q \bigotimes_{\mathcal{C}} \overline{\mathcal{C}}$ is prime and centrally closed over $\overline{\mathcal{C}}$. Hence we may replace $\mathcal{R}$ by Q or $Q \bigotimes_{\mathcal{C}} \overline{\mathcal{C}}$ according as $\mathcal{C}$ is finite or infinite.

Hence it can be seen that $\mathcal{R}$ is centrally closed over $\mathcal{C}$ (i.e., $\mathcal{R}\mathcal{C} = \mathcal{R}$) which is either algebraically closed or finite and

$$([[b,u^m],v^n]+[u^m,[b,v^n]])^{\ell} = \pm[u^p,v^q]_k \text{ for all } u,v \in \mathcal{R}.$$

Now using Martindale's [23, Theorem 3], we see that $\mathcal{R}\mathcal{C}$ (and so $\mathcal{R}$) is a primitive ring having nonzero socle H, where $\mathcal{C}$ as the associated division ring. By using Jacobson's theorem [17, pp.75], we see that $\mathcal{R}$ is isomorphic to a dense ring of linear transformations on some vector space V over $\mathcal{C}$. Moreover, H consists of the finite rank linear transformations in $\mathcal{R}$. Let's assume that $dim_{\mathcal{C}} V \geqslant 2$, otherwise we are done.

Now our job is to prove that for any $t \in V$, t and bt are linearly $\mathcal{C}$-dependent. If we assume that $bt = 0$, then t, bt are $\mathcal{C}$-dependent. So we shall start by assuming that $bt \neq 0$. If t and bt are linearly $\mathcal{C}$-independent for some $t \in V$, by the density of $\mathcal{R}$ there exist $u,v \in \mathcal{R}$ such that

$$ut = t,\ ubt = t;$$

$$vt = 0,\ vbt = bt.$$

We can easily see that $0 = (([[b,u^m],v^n]+[u^m,[b,v^n]])^{\ell} \mp [u^p,v^q]_k)t = (-1)^{\ell} t \neq 0$, a contradiction. So we conclude that t and bt are linearly $\mathcal{C}$-dependent for all $t \in V$. Hence for each $t \in V$, $bt = t\alpha_t$ for some $\alpha_t \in \mathcal{C}$. Now we prove that α_t does not depend on the choice of $t \in V$. Since $dim_{\mathcal{C}} V \geqslant 2$ there exists $w \in V$ such that t and w are linearly independent over $\mathcal{C}$. Now there exist $\alpha_t, \alpha_w, \alpha_{t+w} \in \mathcal{C}$ such that

$$bt = t\alpha_t, bw = w\alpha_w, b(t+w) = (t+w)\alpha_{t+w},$$

and hence,

$$t\alpha_t + w\alpha_w = b(t+w) = (t+w)\alpha_{t+w}.$$

This implies

$$t(\alpha_t - \alpha_{t+w}) + w(\alpha_w - \alpha_{t+w}) = 0.$$

Since t and w are linearly independent over $\mathcal{C}$, it follows that $\alpha_t = \alpha_{t+w} = \alpha_w$. Therefore there exists $\alpha \in \mathcal{C}$ such that $bt = t\alpha$ for all $t \in V$. Now let $r \in \mathcal{R}$, $t \in V$. Since $bt = t\alpha$,

$$[b,r]t = (br)t - (rb)t = b(rt) - r(bt) = (rt)\alpha - r(t\alpha) = 0,$$

that is, $[b,\mathcal{R}]V = 0$. Since V is a faithful irreducible $\mathcal{R}$-module, $[b,\mathcal{R}] = 0$, i.e.; $b \in Z(\mathcal{R})$, and hence $d = 0$, a contradiction. □

THEOREM 2. *Let $\mathcal{R}$ be a semiprime ring and m,n,p,q,k,ℓ be fixed positive integers. If $\mathcal{R}$ admits a derivation d such that $(d([u^m,v^n]))^{\ell} = \pm[u^p,v^q]_k$ for all $u,v \in \mathcal{R}$, then $\mathcal{R}$ is commutative.*

Proof. By Beidar [6], every derivation on a semiprime ring $\mathcal{R}$ can be defined on the whole Utumi quotient ring $\mathfrak{A}$ of $\mathcal{R}$. It is to be noted that $\mathcal{R}$ and $\mathfrak{A}$ satisfy the same differential identities and hence $(d([u^m,v^n]))^\ell = \pm[u^p,v^q]_k$ for all $u,v \in \mathfrak{A}$.

Now suppose that $\mathcal{B}$ is the complete Boolean algebra of idempotents in $\mathcal{C}$, then by [9, Fact 1(2)], $\mathfrak{A}$ is an orthogonal complete $\mathcal{B}$-algebra. If M be a maximal ideal of $\mathcal{B}$, then by Chuang [9, Fact 1(5)], $M\mathfrak{A}$ is a prime ideal of $\mathfrak{A}$ which is invariant under d. Denote $\overline{\mathfrak{A}} = \mathfrak{A}/M\mathfrak{A}$ and suppose $\overline{d}$ is a derivation induced by d on $\overline{\mathfrak{A}}$, i.e., $\overline{d}(\overline{u}) = \overline{d(u)}$ for all $u \in \mathfrak{A}$. Then $\overline{d}$ has same properties in $\overline{\mathfrak{A}}$ as d has on $\mathfrak{A}$. In particular, as $\overline{\mathfrak{A}}$ is prime and by making use of Theorem 1, we see that $\overline{\mathfrak{A}}$ is commutative. Thus, for any maximal ideal M of $\mathcal{B}$, $[\mathfrak{A},\mathfrak{A}] \subseteq M\mathfrak{A}$ and hence $[\mathfrak{A},\mathfrak{A}] \subseteq \cap_M M\mathfrak{A} = 0$, where $M\mathfrak{A}$ runs over all prime ideals of $\mathfrak{A}$. In particular, $[\mathcal{R},\mathcal{R}] = 0$ and therefore $\mathcal{R}$ is commutative. □

We close this section by the following corollaries.

COROLLARY 1. *[10, Theorem 2] Suppose that $\mathcal{R}$ is a semiprime ring and $d : \mathcal{R} \to \mathcal{R}$ be a derivation satisfying $d([u,v]) = \pm[u,v]$ for all $u,v \in \mathcal{R}$, then $\mathcal{R}$ is commutative.*

COROLLARY 2. *[16, Theorem 3.1] Suppose $\mathcal{R}$ is a semiprime ring and $d : \mathcal{R} \to \mathcal{R}$ be a derivation satisfying $(d([u,v]))^\ell = \pm[u,v]_k$ for all $u,v \in \mathcal{R}$ and for all fixed positive integers k,ℓ, then $\mathcal{R}$ is commutative.*

3. The Result on prime Banach algebra

Let us introduce the background of our investigation. Singer and Wermer [28] obtained an essential result which started the examination on the ranges of derivations on Banach algebras. In [28] , Singer and Wermer showed that the range of a continuous derivation on a commutative Banach algebra is contained in Jacobson radical of that algebra. In particular a commutative semisimple commutative Banach algebra does not admit a non-zero derivation. In the same paper they also conjectured that the continuity of the derivation is not necessary. In [29] Thomas acknowledged this conjecture. The same result of Singer and Wermer does not hold in noncommutative Banach algebras because of inner derivations. Hence, in connection with Singer-Wermer theorem, it is natural to ask what will happen if Banach algebra is noncommutative. The first solution to this problem was established by Sinclair in [27], who showed that the primitive ideals of a Banach algebra remain invariant under every continuous derivation of the Banach algebra. In [20], Kim obtained that if a continuous linear Jordan derivation d on a noncommutative Banach algebra $\mathfrak{B}$ satisfies $d(u)[d(u),u]d(u) \in rad(\mathfrak{B})$ for all $u \in \mathfrak{B}$, then $d(\mathfrak{B}) \subseteq rad(\mathfrak{B})$. Very recently, Park [25] also established that $d(\mathfrak{B}) \subseteq rad(\mathfrak{B})$ on a noncommutative Banach algebra, if d is a linear continuous derivation on $\mathfrak{B}$ and satisfies $[[d(u),u],d(u)] \in rad(\mathfrak{B})$ for all $u \in \mathfrak{B}$. In the meanwhile many authors also established fascinating results about derivations satisfying certain suitable conditions in Banach algebras (see [30],[31] where further references can be found). Motivated by these results, in this Section, we use the above mentioned ring theoretic results and show that if a non commutative Banach algebra admits a continuous linear derivation

d satisfying $(d([u^m,u^n]))^\ell \pm [u^p,u^q]_k \in rad(\mathfrak{B})$ for all $u,v \in \mathfrak{B}$ and m,n,p,q,k,ℓ, fixed positive integers, then $d(\mathfrak{B}) \subseteq rad(\mathfrak{B})$.

THEOREM 3. *Let* $\mathfrak{B}$ *be a noncommutative Banach algebra and* m,n,p,q,k,ℓ *be fixed positive integers. Suppose* $d : \mathfrak{B} \to \mathfrak{B}$ *is a continuous linear derivation such that* $(d([u^m,v^n]))^\ell \pm [u^p,v^q]_k \in rad(\mathfrak{B})$ *for all* $u,v \in \mathfrak{B}$*, then* $d(\mathfrak{B}) \subseteq rad(\mathfrak{B})$.

Proof. We know that every primitive ideal of a Banach algebra is invariant under the continuous linear derivation on $\mathfrak{B}$. So, for any primitive ideal $P \subseteq \mathfrak{B}$, we define a linear derivation $d_P : \mathfrak{B}/P \to \mathfrak{B}/P$, by $d_P(\hat{u}) = d(u) + P$, $\hat{u} = u + P$ for all $u \in \mathfrak{B}$, where $\mathfrak{B}/P$ is a factor Banach algebra. As P is a primitive ideal, $\mathfrak{B}/P$ is prime and hence semiprime. The hypothesis $(d_p([u^m,v^n]))^\ell \pm [u^p,v^q]_k \in rad(\mathfrak{B})$ reduces to $(d_p([\overline{u}^m,\overline{v}^n]))^\ell \pm [\overline{u^p},\overline{v^q}]_k = 0$ for all $\overline{u},\overline{v} \in \mathfrak{B}/P$. Moreover, by [18], $\mathfrak{B}/P$ is semisimple. So, d_P is continuous and $d_P(\mathfrak{B}/P) \subseteq rad(\mathfrak{B}/P)$. Again by the semisimplicity of $\mathfrak{B}/P$ we see that $d_P = 0$ on $\mathfrak{B}/P$. Thus, we obtain that $d(\mathfrak{B}) \subseteq P$ for every primitive ideal P and hence $d(\mathfrak{B}) \subseteq rad(\mathfrak{B})$, where $rad(\mathfrak{B})$ is the intersection of all primitive ideals. This completes the proof of the theorem. $\square$

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References

[1] ALI S. AND KHAN A. N., *On rings and algebras with derivations*, Journal of Algebra and Its Applications, **15** 6 (2016), 1650107.

[2] ASHRAF M., REHMAN N. AND ARIF M., *A note on commutativity of semiprime Banach algebras*, Beitr. Algebra Geom., **57** 3 (2016), 553-560.

[3] BELL H. E., *On some commutativity theorems of Herstein*, Arch. Math. (Basel), **24** 1 (1973), 34-38.

[4] BELL H. E., *On the commutativity of prime rings with derivation*, Quaest. Math., **22** 3 (1999), 329-335.

[5] BELL H. E. AND DAIF M. N., *On derivations and commutativity in prime rings*, Acta Math. Hung., **66** (1995), 337-343.

[6] BEIDAR M. N., MARTINDALE III W. S. AND MIKHALEV A. V., *Rings with Generalized Identities*, Pure and Applied Mathematics, Marcel Dekker 196, New York, (1996).

[7] BONSALL F. F. AND DUNCAN J., *Complete Normed Algebras*, (Springer-Verlag, New York,(1973)).

[8] CHUANG C. L., *GPIs having coefficients in Utumi quotient rings*, Proc. Amer. Math. Soc., **103** (1988), 723-728.

[9] CHUANG C. L., *Hypercentral derivations*, J.Algebra **166** (1994), 34-71.

[10] DAIF M. N. AND BELL H. E., *Remarks on derivations on semiprime rings*, Int. J. Math. Math. Sci., **15** 1 (1992), 205-206.

[11] DHARA B. AND SHARMA R. K., *Vanishing power values of commutators with derivations*, Sib. Math. J., **50** (2009), 60-65.

[12] ERICKSON T. S., MARTINDALE III W. S. AND OSBORN J. M., *Prime nonassociative algebras*, Pac. J. Math., **60** (1975), 49-63.

[13] HERSTEIN I. N., *Topics in Ring Theory*, Univ. of Chicago Press, USA (1969).

[14] HERSTEIN I. N., *Center-like elements in prime rings*, J. Algebra **60** (1979), 567-574.

[15] HONGAN M., *A note on semiprime rings with derivation* , Internat. J. Math. Sci., **20** 2 (1997), 413-415.

[16] HUANG S., *Derivations with Engel conditions in prime and semiprime rings*, Czechoslovak Mathematical Journal, **61** (136) (2011), 1135-1140.

[17] JACOBSON N. , *Structure of Rings*, Colloquium Publications 37, Amer. Math. Soc. VII, Provindence, RI, (1956).

[18] JOHNSON B. E. AND SINCLAIR A. M., *Continuity of derivations and a problem of Kaplansky*, Amer. J. Math., **90** (1968), 1067-1073.

[19] KHARCHENKO V. K., *Differential identities of prime rings*, Algebra Logic, **17** (1979), 155-168.

[20] KIM B. D., *On the deriavtions of semiprime rings and noncommutative Banach algebras.*, Atct Math. Sinica, **16** (2000), 21-28.

[21] LEE T. K., *Semiprime rings with differential identities* , Bull. Inst. Math. Acad. Sinica., **20** 1 (1992), 27-38.

[22] LEE T. K., *Semiprime rings with hypercentral derivations* , Can. Math. Bull., **38** 4 (1995), 445-449 .

[23] MARTINDALE III W. S., *Prime rings satisfying a generalized polynomial identity* , J. Algebra., **12** (1969), 576-584.

[24] MAYNE J. H., *Centralizing mappings of prime rings* , Can. Math. Bull., **27** (1984), 122-126.

[25] PARK K. H., *On derivations in noncommutative semiprime rings and Banach algebras*, Bull. Korean Math. Soc., **42** (2005), 671-678.

[26] POSNER E. C., *Derivation in prime rings*, Proc. Amer. Math. Soc., **8** (1957), 1093-1100.

[27] SINCLAIR A. M., *Continuous derivations on Banach algebras*, Proc. Amer. Math. Soc., **20** (1969), 166-170.

[28] SINGER I. M. AND WERMER J., *Derivations on commutative normed algebras*, Math. Ann., **129** (1955), 260-264.

[29] Thomas M. P., *The image of a derivation is contained in the radical*, Ann. of Math., **128** 3 (1988), 435-460.

[30] VUKMAN J., *A result concerning derivations in noncommutative Banach algebras*, Glas. Mat. Ser.III, **26** 46, (1-2), (1991), 83-88.

[31] VUKMAN J., *On derivations in prime rings and Banach algebras*, Proc. Amer. Math. Soc., **116** (1992), 877-884.

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