Rendiconti Sem. Mat. Univ. Pol. Torino
Vol. 76, 1 (2018), 81 – 92

L. Di Terlizzi - A. M. Pastore

INVARIANT SUBMANIFOLDS OF SOME CLASSES OF CONTACT AND OF PARACONTACT METRIC MANIFOLDS

Abstract. We consider contact metric manifolds such that its (1,1)-tensor field anticommutes with the Jacobi operator. These manifolds admit two paracontact metric structures compatible with the contact form. We discuss some geometric properties of invariant submanifolds of such paracontact metric manifolds. Finally, we consider submanifolds of non-Sasakian contact metric (κ,μ)-spaces.

Keywords and phrases. Contact manifolds, paracontact manifolds, Jacobi operator, submanifolds.

1. Introduction

Yano and Ishihara called invariant submanifold in a (2n+l)-dimensional differentiable manifold M, carrying an almost contact structure (φ,ξ,η), any submanifold M' such that the tangent space T_xM' is invariant by the linear mapping φ at each point $x \in M'$ i.e. $\varphi_x(T_xM') \subset T_xM'$, [9].
Then they discussed the two only possible cases:
Case I: ξ is never tangent to M', where M' is necessarily even-dimensional.
Case II: ξ is always tangent to M', where M' is necessarily odd-dimensional.
As it concerns the Case II, they proved that an invariant submanifold M', imbedded in an almost contact manifold M in such a way that the vector field ξ is always tangent to M', is an almost contact manifold with the induced almost contact structure. If the almost contact structure of M is normal, then the induced almost contact structure on M' is normal.
Later, in the contact metric (non necessarily normal) case, Blair proved that ξ is always tangent to M' and M' is a minimal submanifold, [1].

This paper is organized as follows. Section 2 is devoted to recall the fundamental data on (almost) contact metric and (almost) paracontact metric manifolds. Section 3 deals with general properties of invariant submanifolds of contact metric manifolds. In Section 4 we consider manifolds whose Jacobi operator anticommutes with the (1,1)-tensor field φ, and the associated paracontact metric structures (h,ξ,η,g_1) and $(\varphi h,\xi,\eta,g_2)$, described in [6], where $h = \frac{1}{2}\mathcal{L}_\xi\varphi$. We study the properties of invariant submanifolds of the paracontact manifolds (M,h,ξ,η,g_1) and $(M,\varphi h,\xi,\eta,g_2)$.
Section 5 focus on submanifolds of non-Sasakian contact metric (κ,μ)-spaces.
Manifolds are always assumed to be smooth and connected. For the exterior differentiation and the curvature tensor we adopt the notation in [7].

2. (Almost) contact and (almost) paracontact manifolds

An *almost contact manifold* is a $(2n+1)$-dimensional manifold M carrying a structure (φ,ξ,η), where φ is a $(1,1)$-tensor field, ξ a vector field and η a 1-form, such that $\varphi^2 = -I + \eta \otimes \xi$, $\eta(\xi) = 1$. It follows that $\varphi(\xi) = 0$, $\eta \circ \varphi = 0$ and φ has rank $2n$. Such a manifold admits a so-called compatible Riemannian metric g, such that $g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y)$, for all $X,Y \in \Gamma(TM)$ and hence $\eta = g(\cdot,\xi)$. The manifold (M,φ,ξ,η,g) is called an *almost contact metric manifold*. It is said to be a *contact metric manifold* if $d\eta(X,Y) = g(X,\varphi Y)$ for all vector fields X,Y. It means that η is a *contact form* i.e. $\eta \wedge (d\eta)^n \neq 0$. One can define the $(1,1)$-tensor field $h = \frac{1}{2}\mathcal{L}_\xi \varphi$. This operator satisfies $h(\xi) = 0$ and it is symmetric with respect to g, so that $\eta \circ h = 0$. Furthermore, it anticommutes with φ and satisfies $\nabla_X \xi = -\varphi X - \varphi h X$, where ∇ is the Levi-Civita connection of g.

An *almost paracontact structure*, [11], on a $(2n+1)$-dimensional manifold M is given by a $(1,1)$-tensor field $\tilde{\varphi}$, a vector field $\tilde{\xi}$ and a 1-form $\tilde{\eta}$, verifying the conditions $\tilde{\varphi}^2 = I - \tilde{\eta} \otimes \tilde{\xi}$, $\tilde{\eta}(\tilde{\xi}) = 1$, and $\tilde{\varphi}$ induces an almost paracomplex structure on each fibre of the distribution $\mathcal{D} = \mathrm{Ker}(\tilde{\eta})$, i.e. the eigendistributions corresponding to the eigenvalues $+1$ and -1 of $\tilde{\varphi}|_{\mathcal{D}}$ have dimension n.
It follows that $\tilde{\varphi}(\tilde{\xi}) = 0$, $\tilde{\eta} \circ \tilde{\varphi} = 0$ and $\tilde{\varphi}$ has constant rank $2n$. Any almost paracontact manifold M admits a semi-Riemannian metric $\tilde{g}$ satisfying $\tilde{g}(\tilde{\varphi}X, \tilde{\varphi}Y) = -\tilde{g}(X,Y) + \tilde{\eta}(X)\tilde{\eta}(Y)$, for all $X,Y \in \Gamma(TM)$, which necessarily has signature $(n+1,n)$. Then $(M,\tilde{\varphi},\tilde{\xi},\tilde{\eta},\tilde{g})$ is called an *almost paracontact metric manifold*. If $d\tilde{\eta}(X,Y) = \tilde{g}(X,\tilde{\varphi}Y)$ for all $X,Y \in \Gamma(TM)$, then $(M,\tilde{\varphi},\tilde{\xi},\tilde{\eta},\tilde{g})$ is called a *paracontact metric manifold*, in which case $\tilde{\eta}$ is a contact form.

3. Invariant submanifolds

Given any tensor field ψ of type (1,1) on a manifold M we say that a submanifold M' is ψ-invariant if $\psi_x(T_xM') \subset T_xM'$, for any $x \in M'$.

A φ-invariant (or simply an invariant) submanifold M', imbedded in an almost contact manifold M in such a way that the vector field ξ is always tangent to M', is an almost contact manifold with the induced, by restriction, structure (φ',ξ',η').
We consider the operators $h = \frac{1}{2}\mathcal{L}_\xi \varphi$ on M and $h' = \frac{1}{2}\mathcal{L}_{\xi'} \varphi'$ on M' and we observe that, for each $X \in \Gamma(TM')$,

$$hX = \frac{1}{2}(\mathcal{L}_\xi \varphi)X = \frac{1}{2}([\xi,\varphi X] - \varphi[\xi,X]) \in \Gamma(TM') \tag{1}$$

and we obtain

$$h' = h|_{TM'} : TM' \to TM' \tag{2}$$

since, for each $X \in \Gamma(TM')$, $h'X = \frac{1}{2}(\mathcal{L}_{\xi'}\varphi')X = \frac{1}{2}(\mathcal{L}_\xi \varphi)X = hX$.

REMARK 1. We remark that M' is also an h-invariant and a φh-invariant submanifold of M. Namely we get $h_x(T_xM') \subset T_xM'$ and $(\varphi h)_x(T_xM') \subset T_xM'$, for any $x \in M'$. However, without new hypotheses, M' is far to carry any interesting structure.

From now on, keeping in mind the Blair result, we come to consider submanifolds of contact manifolds.

Let (M,φ,ξ,η,g) be a $(2n+1)$-dimensional contact metric manifold and M' an invariant submanifold of M. Then ξ is tangent to M', M' is minimal and turns out to be a contact metric manifold with the induced structure (φ',ξ',η',g') obtained by restriction. Moreover, the Sasaki 2-form Φ' of the structure induced on M' verifies:

$$\Phi'(X,Y) = d\eta'(X,Y) = g'(X,\varphi'Y) = g(X,\varphi Y) = d\eta(X,Y) = \Phi(X,Y)$$

for $X,Y \in \Gamma(TM')$ and the operators $h = \frac{1}{2}\mathcal{L}_\xi\varphi$ on M and $h' = \frac{1}{2}\mathcal{L}_{\xi'}\varphi'$ on M' verify (1) and (2).

We denote by ∇ and ∇' the Levi-Civita connections of g and g', respectively. We recall the identities:

$$\nabla_X Y = \nabla'_X Y + \beta(X,Y), \quad \nabla_X N = \nabla^{\perp}_X N - A_N X, \tag{3}$$

for any $X,Y \in \Gamma(TM')$ and $N \in \Gamma(TM'^{\perp})$. It is well known that the Weingarten operator A_N and the second fundamental form β, that in our case is trace-free, are related by the identity $g(A_N X,Y) = g(\beta(X,Y),N)$.

Moreover, as M' is a contact metric manifold with its induced structure, for any vector field X on M' we have:

$$\nabla'_X\xi' = -\varphi'X - \varphi'h'X = -\varphi X - \varphi hX. \tag{4}$$

From now on, to simplify the notation, we shall write ξ, h instead of ξ', h'.

LEMMA 1. *Let M' be an invariant submanifold of a contact metric manifold (M,φ,ξ,η,g). Then, for each $X \in \Gamma(TM')$ and $N \in \Gamma(TM'^{\perp})$ one obtains $\beta(X,\xi) = 0$ and $A_N\xi = 0$.*

Proof. (3), (4) imply $\nabla'_X\xi + \beta(X,\xi) = \nabla_X\xi = -\varphi X - \varphi hX = \nabla'_X\xi$ and then $\beta(X,\xi) = 0$. Moreover, $g(A_N\xi,X) = g(N,\beta(X,\xi)) = 0$ and hence $A_N\xi = 0$. □

THEOREM 1. *Let be (M,φ,ξ,η,g) a contact metric manifold and assume that M admits a paracontact metric structure $(\psi,\xi,\eta,\tilde{g})$ such that for any $X,Y \in \Gamma(TM)$, $\tilde{g}(X,Y) = d\eta(X,\psi Y) + \eta(X)\eta(Y)$. Suppose that M' is a submanifold invariant with respect to φ and ψ. Then for any $x \in M'$ one obtains $T_xM = T_xM' \oplus T_xM'^{\perp_{\tilde{g}}}$.*

Proof. Let be $V \in T_xM'^{\perp}$. For any $X \in T_xM'$ we have

$$\tilde{g}(V,X) = d\eta(V,\psi X) = 0.$$

It follows that $T_xM'^{\perp} \subseteq T_xM'^{\perp_{\tilde{g}}}$ and by reason of dimension this end the proof. □

4. Structures involving the Jacobi operator

Now, referring to Remark 1, to get some interesting structures on the invariant submanifold M', a crucial role will be played by the Jacobi operator on (M,g), $\mathcal{J}$, defined by:

$$\mathcal{J} = R(-,\xi)\xi \tag{5}$$

where R denotes the Riemannian curvature of (M,g).

PROPOSITION 1. *If M' is an invariant submanifold of a contact metric manifold (M,φ,ξ,η,g) then, for each $X \in \Gamma(TM')$, $\mathcal{J}'X = \mathcal{J}X$, where $\mathcal{J}' = R'(-,\xi)\xi$.*

Proof. For each $X \in \Gamma(TM')$, being ξ tangent to M', we have

$$\begin{aligned} \mathcal{J}X &= R(X,\xi)\xi \\ &= R'(X,\xi)\xi - A_{\beta(\xi,\xi)}X + A_{\beta(X,\xi)}\xi + (\nabla_X\beta)(\xi,\xi) - (\nabla_\xi\beta)(X,\xi) \\ &= R'(X,\xi)\xi = \mathcal{J}'X. \end{aligned}$$

In fact, by Lemma 1, one has $(\nabla_\xi\beta)(X,\xi) = \nabla^\perp_\xi\beta(X,\xi) - \beta(\nabla_\xi X,\xi) - \beta(X,\nabla_\xi\xi) = 0$ and analogously $(\nabla_X\beta)(\xi,\xi) = 0$. □

REMARK 2. We are interested in the anti-commuting condition of $\mathcal{J}$ with the structure of a contact metric manifold: $\varphi\mathcal{J} + \mathcal{J}\varphi = 0$. Such a condition obviously holds when $\mathcal{J} = 0$ and the existence of contact metric manifolds with vanishing Jacobi operator is largely discussed in [1].
Now we notice that if M' is an invariant submanifold of a contact metric manifold (M,φ,ξ,η,g) such that $\varphi\mathcal{J} + \mathcal{J}\varphi = 0$, then $\varphi'\mathcal{J}' + \mathcal{J}'\varphi' = 0$. Namely, for $X \in \Gamma(TM')$, $(\varphi'\mathcal{J}' + \mathcal{J}'\varphi')X = (\varphi\mathcal{J} + \mathcal{J}\varphi)X = 0$.

We recall the following result given in [6] as Theorem 1.

THEOREM 2. *Let (M,φ,ξ,η,g) be a contact metric manifold. The following conditions are equivalent:*

i) $\varphi\mathcal{J} + \mathcal{J}\varphi = 0$,

ii) (h,ξ,η) *is an almost paracontact structure on M,*

iii) $(\varphi h,\xi,\eta)$ *is an almost paracontact structure on M.*

If any of the above conditions holds, then M is endowed with two paracontact metric structures (h,ξ,η,g_1) and $(\varphi h,\xi,\eta,g_2)$, where for any $X,Y \in \Gamma(TM)$

$$g_1(X,Y) = d\eta(X,hY) + \eta(X)\eta(Y), \tag{6}$$

$$g_2(X,Y) = d\eta(X,\varphi hY) + \eta(X)\eta(Y). \tag{7}$$

REMARK 3. Let us consider a contact metric manifold (M,φ,ξ,η,g) satisfying $\varphi\mathcal{J}+\mathcal{J}\varphi=0$. Suppose that (M',g') is an invariant submanifold. From Remark 2 and Theorem 2, M' admits two paracontact metric structures (h,ξ,η,g'') and $(\varphi h,\xi,\eta,\tilde{g}'')$ with $g''(X,Y)=d\eta(X,hY)+\eta(X)\eta(Y)$ and $\tilde{g}''(X,Y)=d\eta(X,\varphi hY)+\eta(X)\eta(Y)$, respectively. For the restriction of g_1 and g_2 to M', we get $g_1(X,Y)=d\eta(X,hY)+\eta(X)\eta(Y)=g''(X,Y)$ and $g_2(X,Y)=d\eta(X,\varphi hY)+\eta(X)\eta(Y)=\tilde{g}''(X,Y)$ and M' is a paracontact metric submanifold of (M,h,ξ,η,g_1) and of $(M,\varphi h,\xi,\eta,g_2)$.

Finally, from Remark 1 the following result follows.

PROPOSITION 2. *Let (M,φ,ξ,η,g) be a contact metric manifold satisfying the condition $\varphi\mathcal{J}+\mathcal{J}\varphi=0$, with associated paracontact metric structures (h,ξ,η,g_1) and $(\varphi h,\xi,\eta,g_2)$. If M' is an invariant submanifold of M (with respect to φ), then it is also an invariant submanifold of M with respect to h and to φh. Moreover one can apply Theorem 1.*

4.1. Invariant submanifolds of the paracontact manifold (M,h,ξ,η,g_1)

Let (M,φ,ξ,η,g) be a (2n+1)-dimensional contact metric manifold satisfying the condition $\varphi\mathcal{J}+\mathcal{J}\varphi=0$ and consider the associated paracontact metric structure (h,ξ,η,g_1). Let M' be a (2m+1)-dimensional invariant submanifold of M with respect to φ. Applying Theorem 1, for each $x\in M'$ we have $T_xM=T_xM'\oplus T_xM'^{\perp_1}$.
Hence, considering (M',g_1'), where g_1' denotes the metric induced by g_1 on M', we can write down the equations:

$$\nabla^1_X Y = D_X Y+\alpha(X,Y),\quad \nabla^1_X N = D^{\perp}_X N - A'_N X. \tag{8}$$

where $X,Y\in\Gamma(TM')$, $N\in\Gamma(TM'^{\perp})$, D is the Levi-Civita connection of g_1' and α the second fundamental form, [8].
Now we look for the link between (D,α) and (∇',β)

PROPOSITION 3. *Let M' be an invariant submanifold of a contact metric manifold (M,φ,ξ,η,g) such that $\varphi\mathcal{J}+\mathcal{J}\varphi=0$ and consider (M',g') and (M',g_1'). Then for each $X,Y\in\Gamma(TM')$ we have:*

$$\begin{aligned} D_X Y &= \frac{1}{2}\nabla'_X Y-\eta(Y)hX-\eta(X)hY+\frac{1}{2}\varphi h(\nabla'_X\varphi Y)\\ &- \frac{1}{2}h(\nabla'_X\varphi Y)+\frac{1}{2}\varphi h(\nabla'_X\varphi hY)-\frac{1}{2}\varphi hR'(\xi,X)Y\\ &+ \frac{1}{2}\varphi hA_{\beta(X,Y)}\xi-\{\eta(X)\eta(Y)-\frac{1}{2}g(\mathcal{J}X,Y)-\frac{1}{2}\eta(\nabla'_X Y)\\ &+ g(X+hX-\varphi hX,Y)\}\xi \end{aligned} \tag{9}$$

$$\begin{aligned} \alpha(X,Y) &= \frac{1}{2}\{\beta(X,Y)+\varphi h\beta(X,\varphi Y)-h\beta(X,Y)+\varphi h\beta(X,\varphi hY)\\ &- \varphi h\nabla^{\perp}_{\xi}(\beta(X,Y))+\varphi h\beta(\nabla'_{\xi}X,Y)+\varphi h\beta(X,\nabla'_{\xi}Y)\\ &+ \varphi h\beta(\varphi X,Y)+\varphi h\beta(\varphi hX,Y)\}. \end{aligned} \tag{10}$$

Proof. From Theorem 2 in [6] we know that

$$\begin{aligned}\nabla^1_X Y &= \nabla_X Y - \eta(Y)hX - \eta(X)h(Y) - \eta(X)\eta(Y)\xi - \frac{1}{2}g(\mathcal{J}X,Y)\xi\\ &+\frac{1}{2}\varphi h((\nabla_X\varphi)Y + (\nabla_X\varphi h)Y - R(\xi,X)Y)\\ &+g(X+hX-\varphi hX,Y)\xi.\end{aligned}$$

Using the link between the curvature of ∇ and ∇', [10] p.68, we get

$$\begin{aligned}\nabla^1_X Y &= \nabla'_X Y + \beta(X,Y) - \eta(Y)hX - \eta(X)hY - \eta(X)\eta(Y)\xi\\ &- \frac{1}{2}g(\mathcal{J}X,Y)\xi + \frac{1}{2}\varphi h\{\nabla_X\varphi Y - \varphi\nabla_X Y + \nabla_X\varphi hY - \varphi h\nabla_X Y\\ &- R'(\xi,X)Y + A_{\beta(X,Y)}\xi - A_{\beta(\xi,Y)}X - (\nabla_\xi\beta)(X,Y) + (\nabla_X\beta)(\xi,Y)\}\\ &+ g(X+hX-\varphi hX,Y)\xi\\ &= \nabla'_X Y + \beta(X,Y) - \eta(Y)hX - \eta(X)hY - \eta(X)\eta(Y)\xi\\ &- \frac{1}{2}g(\mathcal{J}X,Y)\xi\\ &+ \frac{1}{2}\varphi h\{\nabla'_X\varphi Y + \beta(X,\varphi Y) - \varphi(\nabla'_X Y) - \varphi(\beta(X,Y))\\ &+ \nabla'_X\varphi hY + \beta(X,\varphi hY) - \varphi h(\nabla'_X Y) - \varphi h(\beta(X,Y))\\ &- R'(\xi,X)Y + A_{\beta(X,Y)}\xi - (\nabla^\perp_\xi(\beta(X,Y)) + \beta(\nabla'_\xi X,Y) + \beta(X,\nabla'_\xi Y)\\ &+ \nabla^\perp_X(\beta(\xi,Y)) - \beta(\nabla'_X\xi,Y) - \beta(\xi,\nabla'_X Y))\}\\ &= \frac{1}{2}\nabla'_X Y - \eta(Y)hX - \eta(X)hY + \frac{1}{2}\varphi h(\nabla'_X\varphi Y) - \frac{1}{2}h(\nabla'_X Y)\\ &+ \frac{1}{2}\varphi h(\nabla'_X\varphi hY) - \frac{1}{2}\varphi hR'(\xi,X)Y + \frac{1}{2}\varphi hA_{\beta(X,Y)}\xi\\ &- \{\eta(X)\eta(Y) - \frac{1}{2}g(\mathcal{J}X,Y) - \frac{1}{2}\eta(\nabla'_X Y) + g(X+hX-\varphi hX,Y)\}\xi\\ &+ \frac{1}{2}\{\beta(X,Y) + \varphi h\beta(X,\varphi Y) - h\beta(X,Y) + \varphi h\beta(X,\varphi hY)\\ &- \varphi h\nabla^\perp_\xi(\beta(X,Y)) + \varphi h\beta(\nabla'_\xi X,Y) + \varphi h\beta(X,\nabla'_\xi Y)\\ &+ \varphi h\beta(\varphi X + \varphi hX,Y)\}.\end{aligned}$$

Comparing with (8) we get (9) and (10). □

COROLLARY 1. *Let M' be an invariant submanifold of a contact metric manifold (M,φ,ξ,η,g) such that $\varphi\mathcal{J}+\mathcal{J}\varphi=0$. Then for each $X\in\Gamma(TM')$, $N\in\Gamma(TM'^\perp)$ we have $\alpha(\xi,X)=0$ and $A'_N\xi=0$.*

Proof. The result follows immediately from Lemma 1, (8) and (10). □

COROLLARY 2. *Let M' be an invariant submanifold of a contact metric manifold (M,φ,ξ,η,g) such that $\varphi\mathcal{J}+\mathcal{J}\varphi=0$. If M' is totally geodesic in (M,g), then it is totally geodesic in (M,g_1).*

Proof. Let M' be totally geodesic with respect to g. Then $\beta = 0$, (10) imply $\alpha = 0$. □

We compute the mean curvature vector field related to the decomposition (8).
In the contact metric manifold (M,φ,ξ,η,g) such that $\varphi\mathcal{J}+\mathcal{J}\varphi = 0$, we consider a local orthonormal frame of type $\{\xi, e_i, \varphi e_i\}$, $1 \leqslant i \leqslant n$, such that $he_i = e_i$. Then, setting $u_i = e_i + \varphi e_i$ and $v_i = e_i - \varphi e_i$, one easily verifies that $\{\xi, u_i, v_i\}$, $i = 1,\dots,n$, is a local orthogonal frame with respect to g_1, such that each u_i is space-like and each v_i is time-like. Indeed, being $\varphi h u_i = u_i$, we have $g_1(u_i,u_i) = g(u_i,\varphi h u_i) = g(u_i,u_i) = 2$. Analogously, $\varphi h v_i = -v_i$ implies that $g_1(v_i,v_i) = -2$.
Then $H = \frac{1}{2m+1}\sum_{i=1}^{m}(\alpha(e_i+\varphi e_i, e_i+\varphi e_i) - \alpha(e_i-\varphi e_i, e_i-\varphi e_i))$ and since each term reduces to $4\alpha(e_i,\varphi e_i)$ we get $H = \frac{4}{2m+1}\sum_{i=1}^{m}\alpha(\varphi e_i, e_i)$.

4.2. Invariant submanifolds of the paracontact manifold $(M,\varphi h,\xi,\eta,g_2)$

Let (M,φ,ξ,η,g) be a (2n+1)-dimensional contact metric manifold satisfying the condition $\varphi\mathcal{J}+\mathcal{J}\varphi = 0$ and $(M,\varphi h,\xi,\eta,g_2)$ the associated paracontact metric manifold. Let M' be a (2m+1)-dimensional invariant submanifold of M. Again by Theorem 1, for each $x \in M'$ we have $T_xM = T_xM' \oplus T_xM'^{\perp_2}$.
From [6] we recall the link between the Levi-Civita connections ∇^2 and ∇ of g_2 and g, respectively.

THEOREM 3. *For every $X,Y,Z \in \Gamma(TM)$ the Levi-Civita connections ∇^2 and ∇ satisfy:*

$$
\begin{aligned}
g(\nabla^2_X Y, Z) &= g(\nabla_X Y, Z) - \eta(X)g(\varphi h Y, Z) - \eta(Y)g(\varphi h X, Z)\\
&\quad - \frac{1}{2}g((\nabla_{hZ}h)Y, X) + \frac{1}{2}g((\nabla_X h)Y, hZ)\\
&\quad + \frac{1}{2}g((\nabla_Y h)X, hZ\big) - \frac{1}{2}\eta(Z)g(\varphi\mathcal{J}X, Y).
\end{aligned} \tag{11}
$$

Hence, considering (M', g_2'), where g_2' denotes the metric induced by g_2 on M', we can write down the equations:

$$
\nabla^2_X Y = \tilde{D}_X Y + \tilde{\alpha}(X,Y), \quad \nabla^2_X N = \tilde{D}^{\perp}_X N - \tilde{A}_N X. \tag{12}
$$

where $X,Y \in \Gamma(TM')$, $N \in \Gamma(TM'^{\perp})$, $\tilde{D}$ is the Levi-Civita connection of g_2' and $\tilde{\alpha}$ the second fundamental form, [8].

PROPOSITION 4. *For each $X,Y \in \Gamma(TM')$, $N \in \Gamma(TM'^{\perp})$ we have:*

$$
\begin{aligned}
g(\tilde{\alpha}(X,Y),N) &= \frac{1}{2}\big\{g\big(h\beta(X,hY) + h\beta(Y,hX), N\big)\\
&\quad + g(A_{hN}hY - [hN,hY] - hA_{hN}Y + h[hN,Y], X)\big\}
\end{aligned} \tag{13}
$$

Proof.

$$\begin{aligned}
g(\nabla^2_X Y, N) &= g(\nabla_X Y, N) - \eta(X) g(\varphi h Y, N) - \eta(Y) g(\varphi h X, N) \\
&- \frac{1}{2} g((\nabla_{hN} h) Y, X) + \frac{1}{2} g\big((\nabla_X h) Y + (\nabla_Y h) X, hN\big) \\
&- \frac{1}{2} \eta(N) g(\varphi l X, Y) \\
&= g(\nabla'_X Y + \beta(X,Y), N) - \frac{1}{2} g(\nabla_{hN} h Y - h \nabla_{hN} Y, X) \\
&+ \frac{1}{2} g\big(\nabla_X h Y - h \nabla_X Y + \nabla_Y h X - h \nabla_Y X, hN\big) \\
&= g(\beta(X,Y), N) \\
&- \frac{1}{2} g(\nabla_{hY} hN + [hN, hY] - h \nabla_Y hN - h[hN, Y], X) \\
&+ \frac{1}{2} g\big(\beta(X, hY) - h\beta(X,Y) + \beta(Y, hX) - h\beta(X,Y), hN\big) \\
&= g(\beta(X,Y), N) \\
&- \frac{1}{2} g(-A_{hN} h Y + [hN, hY] + h A_{hN} Y - h[hN, Y], X) \\
&- g(h\beta(X,Y), hN) + \frac{1}{2} g\big(\beta(X, hY) + \beta(Y, hX), hN\big) \\
&= \frac{1}{2} \big\{ g\big(h\beta(X, hY) + h\beta(Y, hX), N\big) \\
&+ g(A_{hN} h Y - [hN, hY] - h A_{hN} Y + h[hN, Y], X) \big\}
\end{aligned}$$

Since $g(\nabla^2_X Y, N) = g(\tilde{D}_X Y + \tilde{\alpha}(X,Y), N) = g(\tilde{\alpha}(X,Y), N)$, we get (13). □

COROLLARY 3. *Let M' be a (2m+1)-dimensional invariant submanifold of a (2n+1)-dimensional contact metric manifold $(M, \varphi, \xi, \eta, g)$ such that $\varphi \mathcal{J} + \mathcal{J} \varphi = 0$. If M' is totally geodesic with respect to g and for each $X \in \Gamma(TM')$, $N \in \Gamma(TM'^{\perp})$ the tangential part of $[hN, hX] - h[hN, X]$ vanishes, then it is totally geodesic with respect to g_2.*

5. Submanifolds of non-Sasakian contact metric (κ, μ)-spaces

We recall from [2] that a contact metric manifold M is called a contact metric (k, μ)-manifold if its structure (φ, ξ, η, g) satisfies

$$R(X,Y)\xi = (\kappa I + \mu h)(\eta(Y) X - \eta(X) Y) \tag{14}$$

for all $X, Y \in \Gamma(TM)$, where $2h = \mathcal{L}_\xi \varphi$ and κ, μ are real constants, with necessarily $k \leqslant 1$. In the non-Sasakian case (that is for $\kappa \neq 1$), (14) determines the curvature completely. Finally, a complete classification of these manifolds is given in [3] based on the invariant $I_M = \frac{1 - \frac{\mu}{2}}{\sqrt{1-k}}$.

In [5] the authors showed that a non-Sasakian contact metric (κ,μ)-space (M,φ,ξ,η,g) admits a canonical paracontact structure $(\tilde{\varphi},\xi,\eta,\tilde{g})$, where

$$\tilde{\varphi}=\frac{1}{\sqrt{1-\kappa}}h,\quad \tilde{g}=\frac{1}{\sqrt{1-\kappa}}d\eta(\cdot,h\cdot)+\eta\otimes\eta. \tag{15}$$

$$\frac{1}{2}\mathcal{L}_{\xi}\tilde{\varphi}=\tilde{h}=\frac{1}{2\sqrt{1-k}}\left((2-\mu)\varphi\circ h+2(1-\kappa)\varphi\right). \tag{16}$$

In the same paper, the following results have been established.
The Levi-Civita connections ∇, $\tilde{\nabla}$ of g and $\tilde{g}$ respectively are linked by

$$\begin{aligned}\tilde{\nabla}_X Y &= \nabla_X Y+\frac{\mu}{2}\big(\eta(X)\varphi Y+\eta(Y)\varphi X\big)-\frac{1}{\sqrt{1-\kappa}}\big(\eta(X)hY+\eta(Y)hX\big)\\ &+\frac{1}{2}\Big(\frac{2-\mu}{\sqrt{1-\kappa}}g(hX,Y)-2\sqrt{1-\kappa}g(\varphi^2X,Y)\\ &-2g(X,\varphi Y)+2X(\eta(Y))-\eta(\nabla_X Y)\Big)\xi.\end{aligned} \tag{17}$$

and the curvature tensor field of the Levi-Civita connection of $(M,\tilde{g})$ verifies

$$\tilde{R}_{XY}\xi=\tilde{\kappa}(\eta(Y)X-\eta(X)Y)+\tilde{\mu}(\eta(Y)\tilde{h}X-\eta(X)\tilde{h}Y),$$

for all $X,Y\in\Gamma(TM)$, where $\tilde{\kappa}=\kappa-2+\left(1-\frac{\mu}{2}\right)^2$, $\tilde{\mu}=2$.

Let now M' be a $(2m+1)$-dimensional invariant submanifold of a non-Sasakian contact metric $(2n+1)$-dimensional (κ,μ)-space (M,φ,ξ,η,g). Then M' is invariant also with respect to $\tilde{\varphi}$, as it is proportional to h. Applying Theorem 1 we get, for each $x\in M'$, $T_xM=T_xM'\oplus T_xM'^{\perp_{\tilde{g}}}$.

PROPOSITION 5. *Let M' be an invariant submanifold of a non-Sasakian contact metric (κ,μ)-space (M,φ,ξ,η,g). Then the second fundamental forms $\tilde{\beta}$ and β of $\tilde{\nabla}$ and ∇ coincide.*

Proof. For each $X,Y\in\Gamma(TM')$, we can write $\tilde{\nabla}_XY=\tilde{D}'_XY+\tilde{\beta}(X,Y)$. Then (17) implies

$$\tilde{\beta}(X,Y)=norm(\tilde{\nabla}_XY)=norm(\nabla_XY)=\beta(X,Y)$$

since each addendum in the remaining part of (17) is tangent to M'. □

PROPOSITION 6. *Let M' be an invariant submanifold of a non-Sasakian contact metric (κ,μ)-manifold. Then M' is totally geodesic with respect to $\tilde{g}$.*

Proof. In [4] the authors proved that every invariant submanifold of a non-Sasakian contact metric (κ,μ)-manifold is totally geodesic. From Proposition 5 we get that M' is totally geodesic with respect to g if and only if it is totally geodesic with respect to $\tilde{g}$. This completes the proof. □

Now we state two results obtained trough the invariant $I_M = \frac{1-\frac{\mu}{2}}{\sqrt{1-k}}$.

THEOREM 4. *Let (M,φ,ξ,η,g) be a non-Sasakian contact metric (κ,μ)-manifold such that $|I_M| < 1$. Then the paracontact metric structure $(\tilde{\varphi},\xi,\eta,\tilde{g})$ induces on M a canonical contact metric (κ_1,μ_1)-structure (φ_1,ξ,η,g_1) and any invariant submanifold M' of M is totally geodesic with respect to g_1.*

Proof. As in [5], putting $\rho = \sqrt{(1-\kappa)(1-k-(1-\frac{\mu}{2})^2)}$, we have

$$\varphi_1 = \frac{\sqrt{1-\kappa}}{\rho}\tilde{h} = \frac{1}{2\rho}((2-\mu)\varphi h + 2(1-\kappa)\varphi),$$

$$g_1(X,Y) = -d\eta(X,\varphi_1 Y) + \eta(X)\eta(Y), \tag{18}$$

$$\kappa_1 = \kappa + \left(1-\frac{\mu}{2}\right)^2, \ \ \mu_1 = 2.$$

If M' is an invariant submanifold of (M,φ,ξ,η,g), then certainly M' is invariant with respect to φ_1, since for each $X \in \Gamma(TM')$ we have

$$\varphi_1 X = \frac{1}{2\rho}\big((2-\mu)\varphi hX + 2(1-\kappa)\varphi X\big) \in \Gamma(TM').$$

From Theorem 3.1 of [4] it follows that M' is totally geodesic with respect to g_1. □

THEOREM 5. *Let (M,φ,ξ,η,g) be a non-Sasakian contact metric (κ,μ)-manifold such that $|I_M| > 1$. Then the paracontact metric structure $(\tilde{\varphi},\xi,\eta,\tilde{g})$ induces on M a canonical paracontact metric $(\tilde{\kappa}_1,\tilde{\mu}_1)$-structure $(\tilde{\varphi}_1,\xi,\eta,\tilde{g}_1)$ and any invariant submanifold M' of M is totally geodesic with respect to $\tilde{g}_1$.*

Proof. As in [5], putting $\tilde{\rho} = \sqrt{(1-\kappa)\left(\left(1-\frac{\mu}{2}\right)^2-(1-\kappa)\right)}$, we have

$$\tilde{\varphi}_1 = \frac{\sqrt{1-\kappa}}{\tilde{\rho}}\tilde{h} = \frac{1}{2\tilde{\rho}}((2-\mu)\varphi h + 2(1-\kappa)\varphi),$$

$$\tilde{g}_1(X,Y) = d\eta(X,\tilde{\varphi}_1 Y) + \eta(X)\eta(Y), \tag{19}$$

$$\tilde{\kappa}_1 = \kappa - 2 + \left(1-\frac{\mu}{2}\right)^2, \ \ \tilde{\mu}_1 = 2.$$

Let M' be an invariant submanifold of (M,φ,ξ,η,g). Then M' is invariant with respect to $\tilde{\varphi}_1$, since for each $X \in \Gamma(TM')$ we have

$$\tilde{\varphi}_1 X = \frac{1}{2\tilde{\rho}}\big((2-\mu)\varphi hX + 2(1-\kappa)\varphi X\big) \in \Gamma(TM').$$

Again we can apply Theorem 1 so that $T_xM'^{\perp_{\tilde{g}_1}} = T_xM'^{\perp}$ for each $x \in M'$. Hence we can write the identities:

$$\tilde{\nabla}^1_X Y = \tilde{D}^1_X Y + \tilde{\alpha}^1(X,Y); \quad \tilde{\nabla}^1_X N = \tilde{D}^{1\perp}_X N + \tilde{A}^1_N X,$$

for each $X,Y \in \Gamma(TM')$, $N \in \Gamma(TM'^{\perp_{\tilde{g}_1}})$.
From the identity (5.17) of [5], we know that:

$$\begin{aligned}\tilde{\nabla}^1_X Y = \tilde{\nabla}_X Y + \eta(X)\left(\tilde{\varphi}Y - \frac{\sqrt{1-\kappa}}{\tilde{\rho}}\tilde{h}Y\right) + \eta(Y)\left(\tilde{\varphi}X - \frac{\sqrt{1-\kappa}}{\tilde{\rho}}\tilde{h}X\right)\\ + \left(\frac{\tilde{\rho}}{\sqrt{1-\kappa}}\big(\tilde{g}(X,Y) - \eta(X)\eta(Y)\big) + \tilde{g}(X,\tilde{\varphi}\tilde{h}Y)\right)\xi.\end{aligned} \tag{20}$$

Since M' is invariant with respect to $\tilde{\varphi}$ and, by Proposition 6, totally geodesic with respect to $\tilde{g}$, then $\tilde{\nabla}_X Y$ is tangent to M'; moreover all the other terms of $\tilde{\nabla}^1_X Y$ are tangent to M', since as stated in [5]

$$\tilde{h} = \frac{1}{2\sqrt{1-k}}\left((2-\mu)\,\varphi\circ h + 2(1-\kappa)\,\varphi\right)$$

and this completes the proof. □

Again from [5] we get:

THEOREM 6. *Let (M,φ,ξ,η,g) be a contact (κ,μ)-manifold such that $|I_M| < 1$. Then M admits a sequence of $(1,1)$-tensor fields $(\phi_n)_{n\in\mathbb{N}}$ and a sequence of metric tensor fields $(G_n)_{n\in\mathbb{N}}$, defined by*

$$\phi_0 = \varphi, \quad \phi_1 = \frac{1}{2\sqrt{1-\kappa}}\mathcal{L}_\xi \phi_0, \tag{21}$$

$$\phi_n = \frac{1}{2\sqrt{1-\kappa-\left(1-\frac{\mu}{2}\right)^2}}\mathcal{L}_\xi \phi_{n-1}, \tag{22}$$

$$G_{2n} = -d\eta(\cdot,\phi_{2n}) + \eta\otimes\eta, \quad G_{2n+1} = d\eta(\cdot,\phi_{2n+1}) + \eta\otimes\eta, \tag{23}$$

such that, for each $n \in \mathbb{N}$, $(\phi_{2n}, \xi, \eta, G_{2n})$ is a contact metric (κ_{2n}, μ_{2n})-structure, while $(\phi_{2n+1},\xi,\eta,G_{2n+1})$ is a paracontact metric $(\kappa_{2n+1},\mu_{2n+1})$-structure, where

$$\kappa_0 = \kappa, \quad \kappa_{2n} = \kappa + \left(1-\frac{\mu}{2}\right)^2, \quad \mu_{2n} = 2, \tag{24}$$

$$\kappa_{2n+1} = \kappa - 2 + \left(1-\frac{\mu}{2}\right)^2, \quad \mu_{2n+1} = 2. \tag{25}$$

Moreover, for each $n \in \mathbb{N}$, $(\phi_{2n+1},\xi,\eta,G_{2n+1})$ is the canonical paracontact metric structure induced by $(\phi_{2n},\xi,\eta,G_{2n})$ according to (15)

Hence we have the following result.

THEOREM 7. *Let (M,φ,ξ,η,g) be a contact metric (κ,μ)-manifold such that $|I_M|<1$, and M' an invariant submanifold. Then, for each $n\in\mathbb{N}$, M' is totally geodesic with respect to G_{2n} and G_{2n+1} defined in* (23).

Proof. Under the given hypotheses we can easily verify by induction that M' is invariant with respect to ϕ_n. In fact, M' is invariant with respect to ϕ_1, since it coincides with $\tilde{\varphi}$ in (15), and for each $n\in\mathbb{N}$, $n\geqslant 2$, $X\in\Gamma(TM')$

$$\phi_n(X)=\frac{\sqrt{1-\kappa}}{2\rho}\mathcal{L}_\xi\phi_{n-1}(X)=\frac{\sqrt{1-\kappa}}{2\rho}\big([\xi,\phi_{n-1}X]-\phi_{n-1}[\xi,X]\big)\in\Gamma(TM').$$

If we fix an even number $2n$, we obtain that M' is an invariant submanifold of the contact metric (κ_{2n},μ_{2n})-manifold $(M,\phi_{2n},\xi,\eta,G_{2n})$ and then M' is totally geodesic with respect to G_{2n}. For the next number $2n+1$, we get that M' is an invariant submanifold of the paracontact metric $(\kappa_{2n+1},\mu_{2n+1})$-manifold $(M,\phi_{2n+1},\xi,\eta,G_{2n+1})$. Moreover $(\phi_{2n+1},\xi,\eta,G_{2n+1})$ is the canonical paracontact metric structure induced by $(\phi_{2n},\xi,\eta,G_{2n})$ and hence, from Theorem 4, we know that M' is totally geodesic with respect to G_{2n+1}. □

References

[1] Blair, D.E., *Riemannian geometry of contact and symplectic manifolds. Second Edition*. Progress in Mathematics **203**, Birkhäuser, Boston 2010.

[2] Blair, D.E., Koufogiorgos,T. and Papantoniou, B.J., Contact metric manifolds satisfyng a nullity condition, Israel J. Math. **91** (2005), 189–214.

[3] Boeckx, E., A full classification of contact metric (κ,μ)-spaces, Illinois J. Math. **44** (2000), 212–219.

[4] Cappelletti Montano, B., Di Terlizzi, L. and Tripathi, M.M., Invariant submanifolds of contact (κ,μ)-manifolds, Glasgow Math J. **50** (2008), 499-507.

[5] Cappelletti Montano, B. and Di Terlizzi, L., Geometric structures associated with a contact metric (κ,μ)-space, Pacific J. of Mathemathics, Vol 246, No.2 (2010), 257-291.

[6] Di Terlizzi, L. and Dileo, G., Some paracontact metric structures on contact metric manifolds, Rend. Sem. Mat. Univ. Pol. Torino, Geom. Struc. on Riem. Man-Bari Vol 73/1 (2015), 89-100.

[7] Kobayashi, S. and Nomizu, K., *Foundations of differential geometry*, Vol.I, II, Interscience Publishers, New York, 1963, 1969.

[8] O'Neill, B., *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, London, 1983.

[9] Yano, K. and Ishihara, S., Invariant submanifolds of an almost contact manifold, Kodai Math. Sem. Rep. 21 (1969), 350–364.

[10] Yano, K. and Kon, M., *Structures on manifolds*, Series in Pure Mathematics, World Scientific, 1984.

[11] Zamkovoy, S., Canonical connections on paracontact manifolds, Ann. Glob. Anal. Geo., 36 (2009), 37-60.

AMS Subject Classification: 53C15, 53C50, 53D10, 53C25.

Luigia DI TERLIZZI, Anna Maria PASTORE
Dipartimento di Matematica, Università degli Studi di Bari Aldo Moro
Via E. Orabona 4, 70125 Bari, ITALY
e-mail: luigia.diterlizzi@uniba.it, annamariapastore.uniba@gmail.com

Lavoro pervenuto in redazione il 10-5-18.