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A NEW CLASS OF NONCONFORMING FINITE ELEMENTS FOR ARBITRARY ORDER

Abstract.

We introduce and analyze a simple nonconforming two dimensional finite element of arbitrary order and based on any convex polygonal element by appropriate enrichment functions based on Legendre polynomials. This generalizes the results obtained in [2], [4], [7] for the first order case.

1. Introduction

Crouzeix and Raviart [8] (1973) first considered the nonconforming linear *triangular* element with three nodes located at midpoints of edges. This simplest nonconforming P_1-P_0 element (piecewise linear elements for velocity and piecewise constants for pressure), which is called C-R-triangular element today, has been successfully utilized for solving the stationary Stokes equations. Han (1984) proposed a nonconforming *rectangular* element for solving the stationary Stokes equations [10] and Navier-Stokes equations [11]. For the Stokes equations, Rannacher and Turek [15] (1992) introduced the so-called 'rotated' Q_1-nonconforming finite element on arbitrarily convex quadrilaterals. The corresponding local finite element spaces are obtained by rotating the mixed term of the bilinear element, and the local degrees of freedom are either the average of the function over the edge or its value at the midpoint of the edge. Nonconforming finite elements enjoy better stability properties compared to the conforming finite elements. They have attracted the attention of many mathematicians, see, e. g, [5], [6], [9], [12], [13],[14], [16]. In [1] we introduce a simple nonconforming quadrilateral finite element and then we derive the optimal convergence result for arbitrary regular quadrilaterals. In [2], [4], we establish a new class of nonconforming finite elements by including additional enriched functions (not necessary polynomials) to the standard $Q_1(K)$ element on convex polytope. In [7], the authors provide a simple condition, which is both necessary and sufficient, that guarantees the existence of an enriched Crouzeix-Raviart element for a general d-simplex case. In [3], an extended family of nonconforming quasi-Wilson elements type is introduced for solving linear planar elastic problem. This class of nonconforming finite elements has been constructed by using the approach of [4] for enrichment of the standard bilinear finite element.

In this note, we propose a methodology to construct a simple nonconforming finite element for arbitrary order and for any convex polygonal element, by a special enrichment functions based on Legendre polynomials. This result is a generalization of [2], [4], [7], and gives a large family of nonconforming finite elements.

2. Problem setting

Let K be a convex polygonal element of $\mathbb{R}^2$ with vertices $\{a_i\}_{1\leqslant 1\leqslant n}$, with ($n \geqslant 3$), we set $e_i = [a_i, a_{i+1}]$ its edges (with convention $a_{n+1} = a_1$). First we denote by l_k ($k \in \mathbb{N}^*$) the Legendre polynomial of degree k,

$$l_k(x) = \frac{1}{2^k k!}\frac{d^k}{dx^k}((x^2-1)^k), \tag{2.1}$$

verifying

$$\text{if} \quad \varepsilon \in \{-1,1\}, \ \ l_k(\varepsilon) = (\varepsilon)^k,$$

and

$$\forall n,m \in \mathbb{N} \ \ \int_{-1}^{1} l_n(x) l_m(x) dx = \frac{2}{2n+1}\delta_n^m,$$

where δ_n^m stands for the Kronecker delta function.

We denote by $P_k(-1,1)$ the space of polynomials of degree less than or equal to k on $[-1,1]$. Then for $v \in P_n(-1,1)$ such that

$$\forall p \in P_{n-1}(-1,1), \ \int_{-1}^{1} p(t)v(t)dt = 0, \tag{2.2}$$

we get

$$v(t) = Cl_n(t), \tag{2.3}$$

with C a constant in $\mathbb{R}$.

Using (2.3) we deduce that

$$\forall v \in P_n(-1,1), \ \int_{-1}^{1} l_n'(t)v(t)dt = [l_n(t)v(t)]_{-1}^{1} - \int_{-1}^{1} l_n(t)v'(t)dt = v(1) - (-1)^n v(-1).$$

For $e = [a,b]$ and $v \in L^2(e)$, we set

$$\hat{v}(t) = v(t\frac{b-a}{2} + \frac{b+a}{2}), \tag{2.4}$$

then

$$\forall v,w \in L^2(e), \ \ \int_e vwd\sigma = \frac{meas(e)}{2}\int_{-1}^{1}\hat{v}(t)\hat{w}(t)dt. \tag{2.5}$$

We define the following spaces

$$S_k(\partial K) = \{v \in L^2(\partial K), \ \ v_{|e_i} \in P_k(e_i), \ i = 1,\cdots,n\},$$

and

$$P_k(\partial K) = S_k(\partial K) \cap C^0(\partial K).$$

To simplify the notations, for $v \in P_k(\partial K)$, we set $v^i = v_{|e_i}$ and $v_i = v(a_i)$.

To establish some unisolvence properties, we consider the following vector space

$$A_k = \{v \in P_k(\partial K), \quad \forall \mu \in S_{k-1}(\partial K), \quad \int_{\partial K} \mu v d\sigma = 0\}.$$

We have the following result with respect to the dimension of the space A_k.

LEMMA 1. *If $k \times n$ is odd, then $dim(A_k) = 0$, otherwise $dim(A_k) = 1$.*

Proof. For $v \in P_k(\partial K)$, it is clear that $\hat{v}^i(t) = c_i l_k(t)$, and so $v_{i+1} = (-1)^k v_i$. By seeing that $v_{i+1} = c_{i+1}$, we deduce that :

$$\forall i \in \{1, ..., n\}, \quad v_i = (-1)^{k(i-1)} v_1 \quad \text{and} \quad c_i = (-1)^{k(i-1)} c_1,$$

so, $v_1 = (-1)^{kn} v_1$.

If $k \times n$ is odd, then $v_i = 0$ and we deduce that $v = 0$ (unisolvence property). The second case is trivial, we have just to choose $c_i = c \neq 0$ and to use the fact that $\hat{v}^i = c(-1)^{k(i-1)} l_k(t)$. □

Let $\mathcal{B}_{k-1}(\partial K)$ be a canonical basis of $S_{k-1}(\partial K)$. We put

$$\Sigma_{\partial K} = \{\forall \mu \in \mathcal{B}_{k-1}(\partial K), \ v \longmapsto \int_{\partial K} \mu v d\sigma\},$$

the set of linear forms defined on $\mathcal{C}^0(\partial K)$. Since

$$dim(P_k(\partial K)) = Card(\Sigma_{\partial K}) = n \times k,$$

we have then the following lemma

LEMMA 2. *If $n \times k$ is odd, the triplet $(\partial K, P_k(\partial K), \Sigma_{\partial K})$ is a finite element.*

For the second case $n \times k$ even, we consider the following linear form

$$\mathcal{L}_0(v) := \int_{\partial K} \mu_{\partial K} v d\sigma, \tag{2.6}$$

with

$$\hat{\mu}^i_{\partial K}(t) = \frac{(-1)^{k \times i}}{meas(e_i)} \times l'_k(t). \tag{2.7}$$

We have also the following result

LEMMA 3. *If $k \times n$ is even,*

$$\forall v \in P_k(\partial K), \quad \mathcal{L}_0(v) := \int_{\partial K} \mu_{\partial K} v d\sigma = 0.$$

Proof. For $v \in P_k(\partial K)$ and $e_i \subset \partial K$ (edge of ∂K), we have

$$\int_{e_i} \mu_{\partial K} v d\sigma = \frac{(-1)^{k\times i}}{2}(v_i + (-1)^{k+1} v_{i+1}),$$

which implies

$$\forall v \in P_k(\partial K) \quad \int_{\partial K} \mu_{\partial K} v d\sigma = \frac{v_1}{2}((-1)^{k\times n} - 1) = 0.$$

For $v_1 \neq 0$, we have then that the triplet $(\partial K, P_k(\partial K), \Sigma_{\partial K})$ is not necessary a finite element. ☐

3. New enriched nonconforming finite elements for arbitrary order

Let us consider the following linear form on $C^0(\partial K)$:

$$v \longrightarrow \mathcal{L}_1(v) = \sum_{i=1}^{n} (-1)^{i\times k} v_i. \tag{3.8}$$

We can deduce the following unisolvence result

LEMMA 4. *If $v \in A_k$ such that $\mathcal{L}_1(v) = 0$, then we have $v = 0$.*

Proof. Let $v \in A_k$, we have : $v_i = (-1)^{k(i-1)} v_1$, consequently $\mathcal{L}_1(v) = n(-1)^k v_1 = 0$, and so $v_i = 0$. With the same arguments as in Lemma 1, we have $v = 0$. ☐

For $\mu_{\partial K}$ defined in (2.7), let w_K be a function $L^2(\partial K)$ such that

$$\mathcal{L}_0(w_K) = \int_{\partial K} \mu_{\partial K} w_K d\sigma \neq 0. \tag{3.9}$$

Let us remark that using Lemma 3, the function $w_K \notin P_k(\partial(K))$.

We can define a new enrichment non conforming finite element by the following theorem.

THEOREM 1. *The triplet $(\partial K, P_k(\partial K) \oplus Span(w_K), \Sigma_{\partial K} \cup (\mathcal{L}_1))$ is a finite element.*

Proof. For the proof of Theorem 1, we denote by $R_k = P_k(\partial K) \oplus Span(w_K)$ and

$$\Sigma_{\partial K} = \{\forall \mu \in \mathcal{B}_{k-1}(\partial K), \quad v \longrightarrow \int_{\partial K} \mu v d\sigma, \mathcal{L}_1(v)\}$$

the degrees of freedom of an element $v \in \mathcal{C}^0(\partial K)$. We have

$$dim(R_k) = card(\Sigma_{\partial K}). \tag{3.10}$$

Let $v \in R_k$ cancel the degrees of freedom, $v = p + \alpha w_K$, with $p \in P_k(\partial K)$ and $\alpha \in \mathbb{R}$. By using the Lemma 3, we can see that

$$0 = \mathcal{L}_0(v) = \mathcal{L}_0(p) + \alpha \mathcal{L}_0(w_K) = \alpha \mathcal{L}_0(w_K),$$

since $\mathcal{L}_0(w_K) \neq 0$ and $\mathcal{L}_0 \in \Sigma_{\partial K}$, we deduce that $\alpha = 0$, consequently $v \in A_k$, and so $v = 0$. We conclude that R_k is $\Sigma_{\partial K}-$unisolvent.
This completes the proof of Theorem 1. ☐

Finally, concerning the construction of the function w_K, let $w_K \in L^2(\partial K)$ be defined by :

$$\hat{w}_K^i(t) = c_i(-1)^{k \times i}(1-t^2)l_k'(t),$$

with $\sum_{i=1}^{n} c_i \neq 0$. Since $w_K^i \in P_{k+1}(e_i)$ and $w_K(a_i) = 0$, we obtain $w_K \in P_{k+1}(\partial K)$. Furthermore

$$\mathcal{L}_0(w_K) = \int_{\partial K} \mu_{\partial k} w_K d\sigma = \frac{\sum_{i=1}^n c_i}{2} \int_{-1}^{1} (1-t^2)(l_k')^2(t)dt \neq 0.$$

Let us remark that we can also choose the function w_K defined by

$$\hat{w}_K^i(t) = c_i(-1)^{k \times i}\big((1+t)(1-t)^k + (-1)^k(1-t)(1+t)^k\big).$$

4. Application

Let K be a triangle (or convex quadrilateral element) and $(K, R_k(K), \Sigma_K)$ the Lagrange finite element of order $k \in \mathbb{N}^*$. We put $R_k^0(K) = R_k(K) \cap H_0^1(K)$, it exists $\Sigma_K^0 \subset \Sigma_K$ such as $(K, R_k^0(K), \Sigma_K^0)$ is a finite element. Let $(\lambda_i) \in R_1(K)$ checking $\lambda_i(a_j) = \delta_i^j$.
If $k \times n$ is odd (K is a triangle), we put

$$\Sigma_K = \{\forall \mu \in \mathcal{B}_{k-1}(\partial K),\ v \longrightarrow \int_{\partial K} \mu v d\sigma\} \cup \Sigma_K^0 \ \ ; \ \ D_k(K) = P_k(K).$$

If not

$$\Sigma_K = \{\forall \mu \in \mathcal{B}_{k-1}(\partial K),\ v \longrightarrow \int_{\partial K} \mu v d\sigma, \mathcal{L}_1(v)\} \cup \Sigma_K^0 \ \ ; \ \ D_k(K) = R_k(K) \oplus Span(w_K),$$

with (for example)

$$w_K = \lambda_1 \times \lambda_2^{k-1} + (-1)^k \lambda_2 \times \lambda_1^{k-1}.$$

Finally, we can deduce the following new result:

THEOREM 2. *The triplet $(K, \Sigma_K, D_k(K)$ is a finite element. Furthermore $P_k(K) \subset R_k(K)$.*

5. Conclusion

In this paper, we establish a new and large bi-dimensional family of nonconforming finite element, based on specific enrichment of some classical spaces for arbitrary order and for a general convex polygonal element. This idea can be extended on higher dimensional spaces and will be subject of future works.

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