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DIMENSION OF TCHEBYCHEFFIAN SPLINE SPACES OVER PLANAR T-MESHES: THE CONFORMALITY METHOD

Abstract. We present the conformality method for studying the dimension of Tchebycheffian spline spaces over T-meshes. This is a Tchebycheffian extension of the smoothing cofactor-conformality method elaborated in the literature for polynomial spline spaces. We apply this method to obtain a dimension formula which is equivalent to the one recently obtained by the so-called homological approach.

1. Introduction

Univariate Tchebycheffian splines are smooth piecewise functions where the different pieces are drawn from extended Tchebycheff (ET-)spaces [20, 23]. They are a natural generalization of univariate polynomial splines but offer a more flexible framework, due to the wide variety of ET-spaces. Multivariate extensions of Tchebycheffian splines can be obtained in a simple and elegant way from their univariate counterpart by means of tensor-product structures. As a consequence, they are attractive in several application areas ranging from geometric modeling (see, e.g., [8, 11, 16, 27]) to numerical simulation (see, e.g., [2, 18, 19]).

Tchebycheffian spline spaces over T-meshes [4, 5] have been introduced as a generalization of polynomial spline spaces over T-meshes [9, 24] aiming to combine the richness of ET-spaces with the possibility of supporting local refinement, without deviating too much from the simple tensor-product structure. As in the polynomial case, a complete understanding of these spline spaces requires the knowledge of the dimension of the space defined on a prescribed T-mesh for a given smoothness. It is important to understand when the dimension only depends on combinatorial quantities of the T-mesh (such as number of vertices, edges and faces), on the given smoothness, and on the componentwise dimensions of the underlying ET-spaces because otherwise the considered spline space is not robust for practical use.

The dimension problem of polynomial spline spaces on a prescribed T-mesh for a given componentwise degree and smoothness turns out to be a challenging task. It has been addressed by three different kinds of techniques:

(i) homological techniques [10, 21],

(ii) Bernstein-Bézier representations and minimal determining sets [24, 25],

(iii) smoothing cofactors and conformality conditions [9, 13, 14, 15].

Each of them can be seen as a version, finetuned for T-meshes, of the corresponding approach developed in the eighties to face the dimension problem of spline spaces over triangulations or general rectilinear partitions [1, 7, 12, 17, 26]. They all have strong and weak points, and the sharpest results can be obtained by a combined application.

The dimension of Tchebycheffian spline spaces over planar T-meshes has been investigated so far by using a suitable extension of (i) for ET-spaces [3, 4, 5] and a generalization of (ii) for particular subspaces [6]. In this paper we extend the smoothing cofactor-conformality method to Tchebycheffian spline spaces over T-meshes and we use it for studying their dimension. The application of this method allows us to obtain a dimension formula which is completely in line with the dimension results in [5] obtained by the homological approach and can be used to sharpen them.

2. Tchebycheffian spline spaces over T-meshes

In this section we discuss Tchebycheffian spline spaces over T-meshes. To this end, we recall the definition and few properties of ET-spaces and T-meshes.

2.1. Extended Tchebycheff spaces

We start by defining ET-spaces on a certain interval [23].

DEFINITION 1 (Extended Tchebycheff space). *Given an integer* $p \geqslant 0$ *and an interval* J*, a space* $\mathbb{T}_p(J) \subset C^p(J)$ *of dimension* $p+1$ *is an extended Tchebycheff space on* J *if any Hermite interpolation problem with* $p+1$ *data on* J *has a unique solution in* $\mathbb{T}_p(J)$*.*

From the definition it follows that any non-trivial element in $\mathbb{T}_p(J)$ has at most p roots in J counting multiplicities.

EXAMPLE 1. The nullspace $\mathbb{L}_p$ of differential operator $L_p := D_x^{p+1} + \sum_{i=0}^{p} c_i D_x^i$ with real coefficients is an ET-space of dimension $p+1$. If the characteristic polynomial has only real roots then $\mathbb{L}_p$ is an ET-space on the real line. If some of the roots are not real then $\mathbb{L}_p$ is an ET-space on a suitable interval J. The space $\mathbb{P}_p := \langle 1, x, \dots, x^p \rangle$ of algebraic polynomials of degree less than or equal to p is a special case with $L_p = D_x^{p+1}$.

EXAMPLE 2. The space $\langle 1, x, \dots, x^{p-2}, U(x), V(x) \rangle$ is an ET-space of dimension $p+1$ on J provided that $\langle D_x^{p-1} U(x), D_x^{p-1} V(x) \rangle$ is an ET-space on J; see [3, 8].

ET-spaces are a natural extension of the space of algebraic polynomials, because they enjoy the same structural properties as polynomial spaces; see [5, 23] for more details. In particular, from Definition 1 it follows that for any $\bar{x} \in J$ the ET-space $\mathbb{T}_p(J)$ admits a Taylor-like basis $\{\psi_{\bar{x},i}\}_{i=0}^{p}$ such that

$$D_x^j \psi_{\bar{x},i}(\bar{x}) = \delta_{ij}, \quad i = 0, \dots, p, \quad j = 0, \dots, p, \tag{2.1}$$

where δ_{ij} stands for the classical Kronecker delta.

EXAMPLE 3. For $\mathbb{T}_p(J) = \mathbb{P}_p$ we have $\psi_{\bar{x},i}(x) = \frac{(x-\bar{x})^i}{i!}$, $i = 0,\ldots,p$.

2.2. T-meshes

We recall the concepts and definitions related to T-meshes, using the notation given in [4, 21]. We consider a domain $\Omega \subset \mathbb{R}^2$ which is a finite union of closed axis-aligned rectangles, called *cells*, whose interiors are pairwise disjoint. This domain Ω is assumed to be simply connected and its interior Ω^o is connected. We denote by $[a_h, b_h] \times [a_v, b_v]$ the smallest rectangle containing Ω.

DEFINITION 2 (T-mesh). *A T-mesh $\mathcal{T} := (\mathcal{T}_2, \mathcal{T}_1, \mathcal{T}_0)$ on Ω is defined as:*

- *$\mathcal{T}_2$ is the collection of cells in Ω;*
- *$\mathcal{T}_1 = \mathcal{T}_1^h \cup \mathcal{T}_1^v$ is a finite set of closed axis-aligned horizontal and vertical segments in $\bigcup_{\sigma\in\mathcal{T}_2} \partial\sigma$, called edges;*
- *$\mathcal{T}_0 := \bigcup_{\tau\in\mathcal{T}_1} \partial\tau$ is a finite set of points, called vertices;*

such that

- *for each $\sigma \in \mathcal{T}_2$, $\partial\sigma$ is a finite union of elements of $\mathcal{T}_1$;*
- *for $\sigma, \sigma' \in \mathcal{T}_2$ with $\sigma \neq \sigma'$, $\sigma \cap \sigma' = \partial\sigma \cap \partial\sigma'$ is a finite union of elements of $\mathcal{T}_1 \cup \mathcal{T}_0$;*
- *for $\tau, \tau' \in \mathcal{T}_1$ with $\tau \neq \tau'$, $\tau \cap \tau' = \partial\tau \cap \partial\tau' \subset \mathcal{T}_0$;*
- *for each $\gamma \in \mathcal{T}_0$, $\gamma = \tau_h \cap \tau_v$ where τ_h is a horizontal edge and τ_v is a vertical edge.*

We denote by $\mathcal{T}_1^o$ the set of *interior edges*, i.e., the edges intersecting Ω^o. Analogously, $\mathcal{T}_0^o$ represents the set of *interior vertices*, i.e., the vertices in Ω^o. Moreover, $\mathcal{T}_1^{o,h}$ and $\mathcal{T}_1^{o,v}$ indicate the sets of the horizontal and vertical interior edges of $\mathcal{T}$, respectively, and we set $\mathcal{T}_1^o := \mathcal{T}_1^{o,h} \cup \mathcal{T}_1^{o,v}$. A *segment* of $\mathcal{T}$ is a connected union of edges of $\mathcal{T}$ belonging to the same straight line. Given any $\tau \in \mathcal{T}_1^o$, we denote by $\rho(\tau)$ the maximal segment composed of edges of $\mathcal{T}_1^o$ containing τ. A maximal segment is

- a *cross-cut* if both its endpoints belong to $\partial\Omega$,
- a *ray* if only one of its endpoints belongs to $\partial\Omega$,
- a *maximal interior segment (MIS)* if both its endpoints belong to Ω^o.

These concepts are illustrated in Figure 1. We denote by $\textsc{mis}(\mathcal{T})$ the set of all MISs. The set of all horizontal (resp., vertical) MISs is denoted by $\textsc{mis}_h(\mathcal{T})$ (resp., $\textsc{mis}_v(\mathcal{T})$).

We now formalize the concept of smoothness in the context of T-meshes.

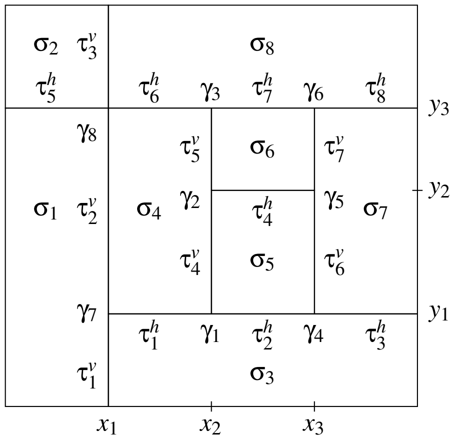

Figure 1: Example of a T-mesh. Cross-cuts: $\tau_5^h \cup \tau_6^h \cup \tau_7^h \cup \tau_8^h$ and $\tau_1^v \cup \tau_2^v \cup \tau_3^v$; ray: $\tau_1^h \cup \tau_2^h \cup \tau_3^h$; MISs: τ_4^h, $\tau_4^v \cup \tau_5^v$, and $\tau_6^v \cup \tau_7^v$.

DEFINITION 3 (Smoothness). *With each edge* $\tau \in \mathcal{T}_1^o$, *we associate an integer* $r(\tau) \geqslant 0$. *We say that* $f \in C^{r(\tau)}(\tau)$ *if the partial derivatives of* f *up to order* $r(\tau)$ *are continuous across the edge* τ. *We assume that* $r(\tau) = r(\tau')$ *for all* τ, τ' *lying on the same straight line. A smoothness distribution on* $\mathcal{T}$ *is defined as*

$$\mathbf{r} := \{r(\tau), \forall \tau \in \mathcal{T}_1^o\},$$

and leads to the following class of smooth functions on Ω*:*

$$C^{\mathbf{r}}(\mathcal{T}) := \{f : \Omega \to \mathbb{R} : f \in C^{r(\tau)}(\tau), \forall \tau \in \mathcal{T}_1^o\}.$$

Given a smoothness distribution $\mathbf{r}$ on $\mathcal{T}$, with each vertex $\gamma \in \mathcal{T}_0^o$, we associate two integers $r_h(\gamma), r_v(\gamma)$, where $r_h(\gamma) := r(\tau_v)$ and $r_v(\gamma) := r(\tau_h)$ such that $\gamma = \tau_h \cap \tau_v$ and $\tau_h \in \mathcal{T}_1^{o,h}$, $\tau_v \in \mathcal{T}_1^{o,v}$.

In the following, we denote by ℓ either h or v. Let $p_\ell \in \mathbb{N}$, and let $\mathbb{T}_{p_\ell}^\ell([a_\ell, b_\ell])$ be an ET-space of dimension $p_\ell + 1$ on $J_\ell := [a_\ell, b_\ell]$. Then, we define the tensor-product space

$$\mathbb{P}_{\mathbf{p}}^{\mathbf{T}} := \mathbb{T}_{p_h}^h([a_h, b_h]) \otimes \mathbb{T}_{p_v}^v([a_v, b_v]), \tag{2.2}$$

where $\mathbf{p} := (p_h, p_v)$ and $\mathbf{T} := (\mathbb{T}_{p_h}^h, \mathbb{T}_{p_v}^v)$. If the space (2.2) is the space of bivariate algebraic polynomials of bi-degree $\mathbf{p}$, then it is denoted by $\mathbb{P}_{\mathbf{p}}$.

DEFINITION 4 (Tchebycheffian spline space over a T-mesh). *Let* $\mathcal{T}$ *be a T-mesh with a smoothness distribution* $\mathbf{r}$, *and let* $p_h, p_v \in \mathbb{N}$. *The Tchebycheffian spline space over the T-mesh* $\mathcal{T}$, *denoted by* $\mathbb{S}_{\mathbf{p}}^{\mathbf{T},\mathbf{r}}(\mathcal{T})$, *is defined as the space of functions in* $C^{\mathbf{r}}(\mathcal{T})$ *such that, restricted to any cell* $\sigma \in \mathcal{T}_2$, *they belong to* $\mathbb{P}_{\mathbf{p}}^{\mathbf{T}}$, *i.e.,*

$$\mathbb{S}_{\mathbf{p}}^{\mathbf{T},\mathbf{r}}(\mathcal{T}) := \left\{s \in C^{\mathbf{r}}(\mathcal{T}) : s_{|\sigma} \in \mathbb{P}_{\mathbf{p}}^{\mathbf{T}}, \sigma \in \mathcal{T}_2\right\}.$$

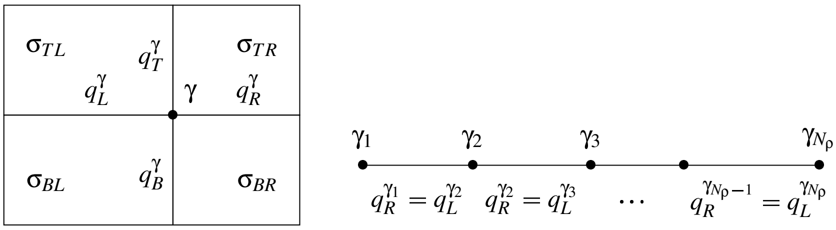

Figure 2: Left: smoothing terms around an interior vertex γ that is not a T-vertex (one smoothing term should be dropped for a T-vertex). Right: smoothing terms along a maximal segment ρ with vertices $\gamma_1, \dots, \gamma_{N_\rho}$.

In the following, we will assume the usual condition on the smoothness:

$$r(\tau_v) < p_h, \quad \forall \tau_v \in \mathcal{T}_1^{o,v}, \quad r(\tau_h) < p_v, \quad \forall \tau_h \in \mathcal{T}_1^{o,h}.$$

3. Smoothing terms and conformality conditions for $\mathbb{S}_{\mathbf{p}}^{\mathbf{T},\mathbf{r}}(\mathcal{T})$

In this section we develop the Tchebycheffian extension of the method of smoothing cofactors and conformality conditions proposed in [7, 26] for investigating the dimension of spline spaces over general rectilinear partitions of planar domains and used by several authors to address the dimension of polynomial spline spaces over planar T-meshes; see [9, 13, 14, 15] and references therein. Our presentation follows the one provided in [15] for the polynomial case.

A spline $s \in \mathbb{S}_{\mathbf{p}}^{\mathbf{T},\mathbf{r}}(\mathcal{T})$ is identified by its expression on the various cells of $\mathcal{T}$. Starting from any cell in $\mathcal{T}_2$ we can reach any other cell in $\mathcal{T}_2$ by following a path that crosses interior edges of $\mathcal{T}$; see [26]. Of course, the expressions of s on adjacent cells must satisfy certain smoothness conditions across the common edge. We now describe these smoothness conditions. Let $\gamma := (\bar{x}, \bar{y})$ be an interior vertex of $\mathcal{T}$. Using the related Taylor-like functions in (2.1), for any $s \in \mathbb{S}_{\mathbf{p}}^{\mathbf{T},\mathbf{r}}(\mathcal{T})$ we have (see Figure 2, left)

$$q_L^\gamma := s_{|\sigma_{TL}} - s_{|\sigma_{BL}} = \sum_{j=0}^{p_h} \sum_{k=r_v(\gamma)+1}^{p_v} a_{j,k}^{\gamma,L} \psi_{\bar{x},j}^h \otimes \psi_{\bar{y},k}^v,$$

$$q_R^\gamma := s_{|\sigma_{TR}} - s_{|\sigma_{BR}} = \sum_{j=0}^{p_h} \sum_{k=r_v(\gamma)+1}^{p_v} a_{j,k}^{\gamma,R} \psi_{\bar{x},j}^h \otimes \psi_{\bar{y},k}^v,$$

$$q_B^\gamma := s_{|\sigma_{BR}} - s_{|\sigma_{BL}} = \sum_{j=r_h(\gamma)+1}^{p_h} \sum_{k=0}^{p_v} b_{j,k}^{\gamma,B} \psi_{\bar{x},j}^h \otimes \psi_{\bar{y},k}^v,$$

$$q_T^\gamma := s_{|\sigma_{TR}} - s_{|\sigma_{TL}} = \sum_{j=r_h(\gamma)+1}^{p_h} \sum_{k=0}^{p_v} b_{j,k}^{\gamma,T} \psi_{\bar{x},j}^h \otimes \psi_{\bar{y},k}^v.$$

The functions q_L^γ, q_R^γ and q_B^γ, q_T^γ will be referred to as *smoothing terms* along the corresponding horizontal and vertical edges, respectively. We deduce

$$q_R^\gamma - q_L^\gamma = s_{|\sigma_{TR}} - s_{|\sigma_{BR}} - s_{|\sigma_{TL}} + s_{|\sigma_{BL}} = \sum_{j=0}^{p_h} \sum_{k=r_v(\gamma)+1}^{p_v} c_{j,k}^{\gamma,h} \psi_{\bar{x},j}^h \otimes \psi_{\bar{y},k}^v,$$

$$q_B^\gamma - q_T^\gamma = s_{|\sigma_{BR}} - s_{|\sigma_{BL}} - s_{|\sigma_{TR}} + s_{|\sigma_{TL}} = \sum_{j=r_h(\gamma)+1}^{p_h} \sum_{k=0}^{p_v} c_{j,k}^{\gamma,v} \psi_{\bar{x},j}^h \otimes \psi_{\bar{y},k}^v.$$

Moreover,

$$0 \equiv q_R^\gamma - q_L^\gamma + q_B^\gamma - q_T^\gamma = \sum_{j=0}^{p_h} \sum_{k=r_v(\gamma)+1}^{p_v} c_{j,k}^{\gamma,h} \psi_{\bar{x},j}^h \otimes \psi_{\bar{y},k}^v + \sum_{j=r_h(\gamma)+1}^{p_h} \sum_{k=0}^{p_v} c_{j,k}^{\gamma,v} \psi_{\bar{x},j}^h \otimes \psi_{\bar{y},k}^v,$$

and from (2.1) we get for $m = 0, \dots, r_h(\gamma)$ and $n = 0, \dots, r_v(\gamma)$,

$$0 \equiv D_x^m (q_R^\gamma - q_L^\gamma + q_B^\gamma - q_T^\gamma)_{|(x=\bar{x},y)} = \sum_{k=r_v(\gamma)+1}^{p_v} c_{m,k}^{\gamma,h} \psi_{\bar{y},k}^v(y),$$

$$0 \equiv D_y^n (q_R^\gamma - q_L^\gamma + q_B^\gamma - q_T^\gamma)_{|(x,y=\bar{y})} = \sum_{j=r_h(\gamma)+1}^{p_h} c_{j,n}^{\gamma,v} \psi_{\bar{x},j}^h(x).$$

By the linear independence of the Taylor-like functions, the above equalities imply

$$c_{m,k}^{\gamma,h} = 0, \quad k = r_v(\gamma)+1, \dots, p_v, \quad m = 0, \dots, r_h(\gamma),$$

$$c_{j,n}^{\gamma,v} = 0, \quad j = r_h(\gamma)+1, \dots, p_h, \quad n = 0, \dots, r_v(\gamma).$$

Hence,

$$q_R^\gamma - q_L^\gamma = \sum_{j=r_h(\gamma)+1}^{p_h} \sum_{k=r_v(\gamma)+1}^{p_v} c_{j,k}^{\gamma,h} \psi_{\bar{x},j}^h \otimes \psi_{\bar{y},k}^v,$$

$$q_B^\gamma - q_T^\gamma = \sum_{j=r_h(\gamma)+1}^{p_h} \sum_{k=r_v(\gamma)+1}^{p_v} c_{j,k}^{\gamma,v} \psi_{\bar{x},j}^h \otimes \psi_{\bar{y},k}^v,$$

and

$$0 \equiv \sum_{j=r_h(\gamma)+1}^{p_h} \sum_{k=r_v(\gamma)+1}^{p_v} (c_{j,k}^{\gamma,h} + c_{j,k}^{\gamma,v}) \psi_{\bar{x},j}^h \otimes \psi_{\bar{y},k}^v.$$

Using again the linear independence of the Taylor-like functions, we obtain

$$c_{j,k}^{\gamma,h} = -c_{j,k}^{\gamma,v}, \quad j = r_h(\gamma)+1, \dots, p_h, \quad k = r_v(\gamma)+1, \dots, p_v.$$

In conclusion, we have

$$q_R^\gamma - q_L^\gamma = d^\gamma, \quad q_B^\gamma - q_T^\gamma = -d^\gamma,$$

where

$$d^{\gamma}(x,y) := \sum_{j=r_h(\gamma)+1}^{p_h} \sum_{k=r_v(\gamma)+1}^{p_v} c_{j,k}^{\gamma,h} \psi_{\bar{x},j}^{h}(x)\, \psi_{\bar{y},k}^{v}(y). \tag{3.3}$$

Consider first a horizontal MIS $\rho \in \mathrm{MIS}_h(\mathcal{T})$. Let $\gamma_1, \ldots, \gamma_{N_\rho}$ be the vertices of ρ (numbered consecutively), and let $q_R^{\gamma_i} = q_L^{\gamma_{i+1}}$, $i = 1, \ldots, N_\rho - 1$ be the smoothing terms associated with the edges of ρ; see Figure 2, right. From the above discussion we obtain the relations

$$q_R^{\gamma_1} = d^{\gamma_1}, \; q_R^{\gamma_2} - q_R^{\gamma_1} = d^{\gamma_2}, \; \ldots, \; q_R^{\gamma_{N_\rho-1}} - q_R^{\gamma_{N_\rho-2}} = d^{\gamma_{N_\rho-1}}, \; -q_R^{\gamma_{N_\rho-1}} = d^{\gamma_{N_\rho}}.$$

This means that the functions d^{γ_i} associated with the vertices γ_i of ρ completely determine all the smoothing terms associated with the edges of ρ. Moreover, we have an additional constraint, called the *conformality condition* along ρ,

$$\sum_{i=1}^{N_\rho} d^{\gamma_i} \equiv 0. \tag{3.4}$$

Of course, we get a similar constraint for each vertical MIS $\rho \in \mathrm{MIS}_v(\mathcal{T})$.

Consider now a horizontal cross-cut or a (right-boundary) horizontal ray as in Figure 1. Then, using again the notation from Figure 2, right, we obtain the relations

$$\begin{gathered} q_R^{\gamma_2} - q_R^{\gamma_1} = d^{\gamma_2}, \; \ldots, \; q_R^{\gamma_{N_\rho-1}} - q_R^{\gamma_{N_\rho-2}} = d^{\gamma_{N_\rho-1}}, \quad \text{or} \\ q_R^{\gamma_1} = d^{\gamma_1}, \; q_R^{\gamma_2} - q_R^{\gamma_1} = d^{\gamma_2}, \; \ldots, \; q_R^{\gamma_{N_\rho-1}} - q_R^{\gamma_{N_\rho-2}} = d^{\gamma_{N_\rho-1}}, \end{gathered}$$

respectively. Note that these relations only involve d^{γ_i} associated with the interior vertices γ_i of the cross-cut or ray. They determine all the smoothing terms associated with the edges in the ray, and all the smoothing terms but one associated with the edges in the cross-cut. We say that a cross-cut has a *free smoothing term*. A similar reasoning holds for other kinds of cross-cuts or rays.

REMARK 1. Recalling Example 3, if $\mathbb{P}_{\mathbf{p}}^{\mathbf{T}} = \mathbb{P}_{\mathbf{p}}$ then

$$\begin{aligned} q_L^{\gamma}(x,y) &= (y-\bar{y})^{r_v(\gamma)+1}\, \bar{q}_L^{\gamma}(x,y), \quad & q_R^{\gamma}(x,y) &= (y-\bar{y})^{r_v(\gamma)+1}\, \bar{q}_R^{\gamma}(x,y), \\ q_B^{\gamma}(x,y) &= (x-\bar{x})^{r_h(\gamma)+1}\, \bar{q}_B^{\gamma}(x,y), \quad & q_T^{\gamma}(x,y) &= (x-\bar{x})^{r_h(\gamma)+1}\, \bar{q}_T^{\gamma}(x,y), \end{aligned}$$

for some polynomials $\bar{q}_L^{\gamma}, \bar{q}_R^{\gamma} \in \mathbb{P}_{p_h, p_v - r_v(\gamma)-1}$ and $\bar{q}_B^{\gamma}, \bar{q}_T^{\gamma} \in \mathbb{P}_{p_h - r_h(\gamma)-1, p_v}$, which are called *smoothing cofactors*; see [15, Section 3]. Furthermore, (3.3) becomes

$$d^{\gamma}(x,y) = (x-\bar{x})^{r_h(\gamma)+1}(y-\bar{y})^{r_v(\gamma)+1}\, \bar{d}^{\gamma}(x,y), \quad \bar{d}^{\gamma} \in \mathbb{P}_{p_h - r_v(\gamma)-1, p_v - r_v(\gamma)-1},$$

and the conformality condition (3.4) reduces to the one considered in [15, Section 3] up to the factorization of the powers $(x-\bar{x})^{r_h(\gamma)+1}$, $(y-\bar{y})^{r_v(\gamma)+1}$.

From here on, for notational convenience, we assume that the smoothness distribution is constant across all the vertical (resp., horizontal) edges, and we set

$$r_h := r(\tau_v), \quad \forall \tau_v \in \mathcal{T}_1^{o,v}, \quad r_v := r(\tau_h), \quad \forall \tau_h \in \mathcal{T}_1^{o,h}.$$

We are now ready for our dimension formula. The result follows from the above discussion, taking into account that each element in $\mathbb{S}_{\mathbf{p}}^{\mathbf{T},\mathbf{r}}(\mathcal{T})$ is identified by its expression on a single cell and by a sequence of smoothing terms associated with the interior edges of $\mathcal{T}$. Recalling that the dimension of the space $\mathbb{P}_{\mathbf{p}}^{\mathbf{T}}$ is $(p_h+1)(p_v+1)$, that each cross-cut has a free smoothing term, and that the other smoothing terms can be determined by the functions (3.3) associated with the interior vertices of $\mathcal{T}$, we obtain the following theorem in complete analogy with the polynomial spline case [15, 26].

THEOREM 1. *Let $\mathbb{S}_{\mathbf{p}}^{\mathbf{T},\mathbf{r}}(\mathcal{T})$ be a Tchebycheffian spline space over a T-mesh $\mathcal{T}$. Let C_h (resp., C_v) be the number of horizontal (resp., vertical) cross-cuts in $\mathcal{T}$. Then,*

$$\begin{aligned}\dim\bigl(\mathbb{S}_{\mathbf{p}}^{\mathbf{T},\mathbf{r}}(\mathcal{T})\bigr) &= (p_h+1)(p_v+1) + C_h(p_h+1)(p_v-r_v) + C_v(p_h-r_h)(p_v+1)\\ &\quad + |\mathcal{T}_0^o|(p_h-r_h)(p_v-r_v) - K_0,\end{aligned} \tag{3.5}$$

where K_0 is the rank of the linear system obtained by collecting the conformality conditions (3.4) for all $\rho \in \mathrm{MIS}(\mathcal{T})$.

Let $\rho \in \mathrm{MIS}_h(\mathcal{T})$ belong to the line $y=\bar{y}$, and let $\gamma_1 = (\bar{x}_1,\bar{y}),\ldots,\gamma_{N_\rho} = (\bar{x}_{N_\rho},\bar{y})$ be the vertices of ρ. Then, by combining (3.3) and (3.4) we get

$$\sum_{k=r_v+1}^{p_v} \psi^v_{\bar{y},k}(y) \sum_{i=1}^{N_\rho} \sum_{j=r_h+1}^{p_h} c^{\gamma_i,h}_{j,k}\, \psi^h_{\bar{x}_i,j}(x) \equiv 0,$$

or, equivalently,

$$\sum_{i=1}^{N_\rho} \sum_{j=r_h+1}^{p_h} c^{\gamma_i,h}_{j,k}\, \psi^h_{\bar{x}_i,j}(x) \equiv 0, \quad k = r_v+1,\ldots,p_v.$$

Representing this w.r.t. any basis of $\mathbb{T}^h_{p_h}$ leads to at most $(p_h+1)(p_v-r_v)$ equations. Likewise, for any $\rho \in \mathrm{MIS}_v(\mathcal{T})$, (3.4) amounts to at most $(p_h-r_h)(p_v+1)$ equations. So,

$$K_0 \leqslant |\mathrm{MIS}_h(\mathcal{T})|(p_h+1)(p_v-r_v) + |\mathrm{MIS}_v(\mathcal{T})|(p_h-r_h)(p_v+1). \tag{3.6}$$

Recalling the Euler's formula, $|\mathcal{T}_2| - |\mathcal{T}_1^o| + |\mathcal{T}_0^o| = 1$, and noticing that

$$|\mathcal{T}_1^{o,h}| = C_h + |\mathcal{T}_0^o| - |\mathrm{MIS}_h(\mathcal{T})|, \quad |\mathcal{T}_1^{o,v}| = C_v + |\mathcal{T}_0^o| - |\mathrm{MIS}_v(\mathcal{T})|,$$

we deduce from (3.5) with an elementary calculation that

$$\begin{aligned}\dim\bigl(\mathbb{S}_{\mathbf{p}}^{\mathbf{T},\mathbf{r}}(\mathcal{T})\bigr) &= |\mathcal{T}_2|(p_h+1)(p_v+1) - |\mathcal{T}_1^{o,h}|(p_h+1)(r_v+1)\\ &\quad - |\mathcal{T}_1^{o,v}|(r_h+1)(p_v+1) + |\mathcal{T}_0^o|(r_h+1)(r_v+1)\\ &\quad + |\mathrm{MIS}_h(\mathcal{T})|(p_h+1)(p_v-r_v) + |\mathrm{MIS}_v(\mathcal{T})|(p_h-r_h)(p_v+1) - K_0.\end{aligned}$$

Taking into account (3.6), this shows that the dimension formula (3.5) is in agreement with the dimension formula in [5, Theorem 4.1] obtained by homological techniques.

For T-meshes with no MISs, Theorem 1 provides an explicit expression for the dimension because $K_0 = 0$. In general, finding the value of K_0 in (3.5) is a difficult task, completely equivalent to finding the value of the homology term in [5, Theorem 4.1]. In the polynomial case, a key ingredient for computing K_0 is determining the number of solutions of (3.4) for the various MISs in $\mathcal{T}$. With the same line of arguments as in [15, Lemma 2] for the polynomial case, and using [5, Theorem 2.4], we have the following result. It is remarkable that such a dimension is independent of the particular ET-spaces we are dealing with.

PROPOSITION 1. *Let* $u_+ := \max(0,u)$. *The dimension of the solution space of* (3.4) *equals*

$$\begin{aligned} &(p_v - r_v)(N_\rho(p_h - r_h) - (p_h + 1))_+, && \text{if } \rho \in \text{MIS}_h(\mathcal{T})\ , \\ &(p_h - r_h)(N_\rho(p_v - r_v) - (p_v + 1))_+, && \text{if } \rho \in \text{MIS}_v(\mathcal{T})\ . \end{aligned}$$

4. Conclusion

We have presented the conformality method for studying the dimension (and constructing a basis) of Tchebycheffian spline spaces over T-meshes. This is a Tchebycheffian extension of the results elaborated in the last decade for polynomial spline spaces.

The obtained results mimic the corresponding ones for the polynomial case, and give new insights into the dimension problem for Tchebycheffian spline spaces over T-meshes. In particular, Proposition 1 paves the path for the Tchebycheffian extension of the sharpened dimension results recently obtained for polynomial spline spaces (see [14] and references therein), and provides an important step towards the identification of families of stable Tchebycheffian spline spaces, i.e., spaces whose dimension does not depend on the exact geometry of the T-mesh.

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