Rendiconti Sem. Mat. Univ. Pol. Torino
Vol. 76, 2 (2018), 151 – 163

G. Monegato*

SINGULAR INTEGRAL EQUATIONS IN THE DESIGN OF INNOVATIVE AIRPLANE WING CONFIGURATIONS

Abstract. We present a short survey on the main mathematical results we have recently obtained on the optimization of airplane Truss-Braced Wing configurations. These results include a definition of the corresponding induced drag constrained minimization problem, whence of the associated Euler-Lagrange system of Cauchy singular integral equations. For the latter, we have proved the existence and uniqueness of its solution in proper weighted Sobolev type spaces. Moreover, we have defined a simple and efficient numerical approach for the system solution; an error estimate has also been derived.

1. Introduction

Traditional monoplane aircrafts have been used for many decades. Promising improvements are represented by C-Wing, Box-Wing and Truss-Braced Wing (TBW) configurations. See the figures reported below, that have been chosen from the many images one can find in internet. Studies showed potential advantages in terms of fuel savings and overall efficiency. These investigations have been carried out in relatively complex multidisciplinary design and optimization computational frameworks and, thus, some theoretical questions regarding the optimal conditions under which the induced drag is minimized, need to be specifically addressed.

Figure 1: C-Wing aircraft [10]

*Dipartimento di Scienze Mathematiche, Dipartimento di Eccellenza 2018-2022, Politecnico di Torino.

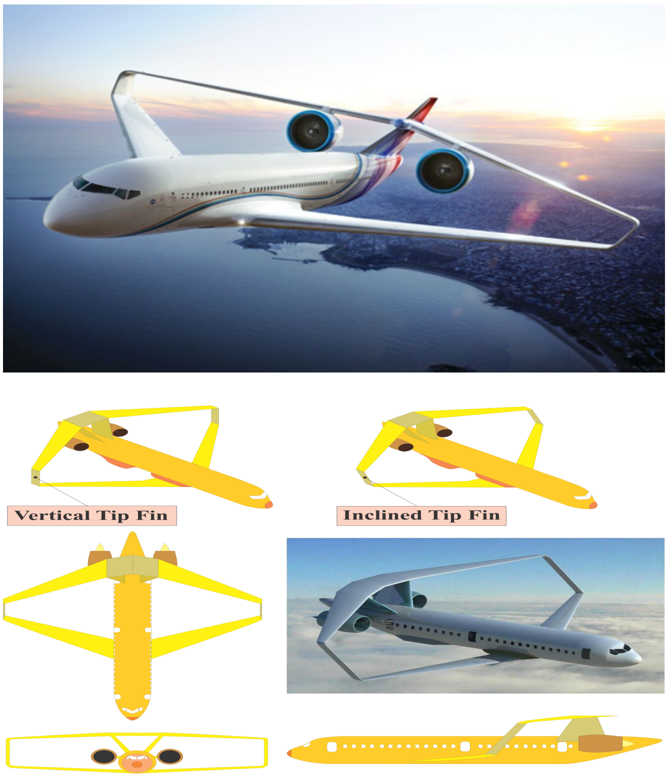

Figure 2: Box-Wing aircrafts [11],[2]

A few years ago Luciano Demasi started a project addressing open theoretical questions on minimum induced drag conditions of Box-Wings (see Figure 2). The work was motivated by the renovated aeronautical interest on the configuration initially proposed by Prandtl a century ago and named by him Best Wing System for its superior aerodynamic performance in terms of induced drag. The Box-Wing presented several theoretical challenges and mathematical difficulties related to the closed path of the lifting surface.

We recall that in the case of cruise speed, a three dimensional wing can be replaced by a curve in the $x-z$ plane. Under this assumption, first in [3], and then in [4] (see also [9]), it was then shown that the Box-Wing, and in general closed systems, could be analyzed as limit case of a C-Wing (i.e. open wing; see Figure 1) whose endpoints are brought close to each other in a limit process. It was also verified that the optimal condition for a closed system can also be asymptotically obtained from a couple of open disjoint wings when the two wings are brought close to each other until they identify the path of the given closed wing.

The mathematical procedures finalized to the numerical solution of the problem

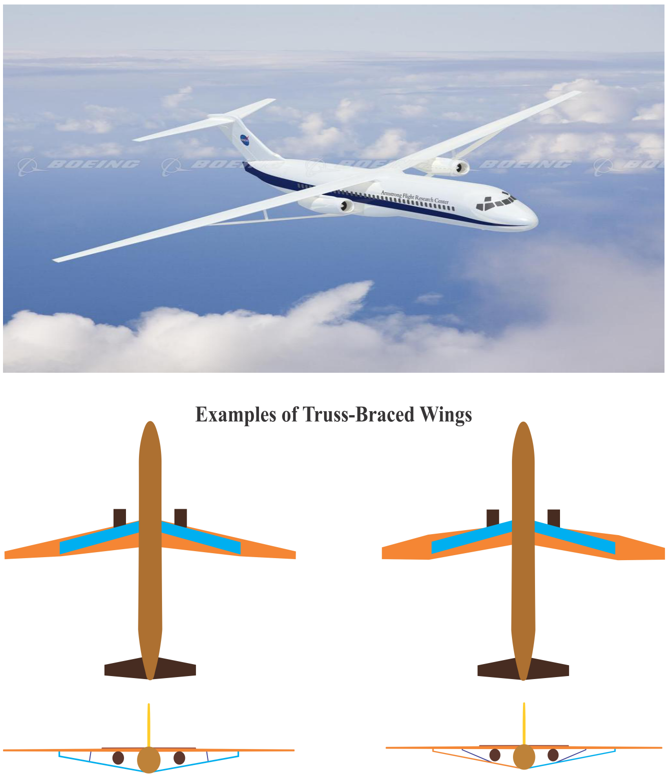

Figure 3: Truss-Braced Wing aircrafts [12],[2]

Euler-Lagrange equations where then extended in [4, 5, 6] to more complex systems such as multiwings. In this case, the Euler-Lagrange equations are reduced to a system of Cauchy singular integral equations. This has been further examined in [13], where the existence and uniqueness of its solution in proper weighted Sobolev type spaces have been proved. Moreover a simple and efficient discrete collocation method for the solution approximation has been defined and examined. In particular, its stability and convergence have been proved, and an error estimate derived.

Since for more complex configurations, coinciding with the boundaries of multiply connected regions, such as the Truss-Braced Wings (see Figure 3), we are not aware of a procedure for deriving the corresponding Euler-Lagrange equations, in [7, 8, 13] the authors have proposed a possible approach.

In this paper we present a short survey on the main mathematical and numerical results that have been recently obtained for TBW configurations (see, in particular, [13]). These results include the proof of existence and uniqueness of the solution of the associated induced drag constrained minimization problem for a system of open disjoint wings, the construction of an efficient numerical approach for the approximation

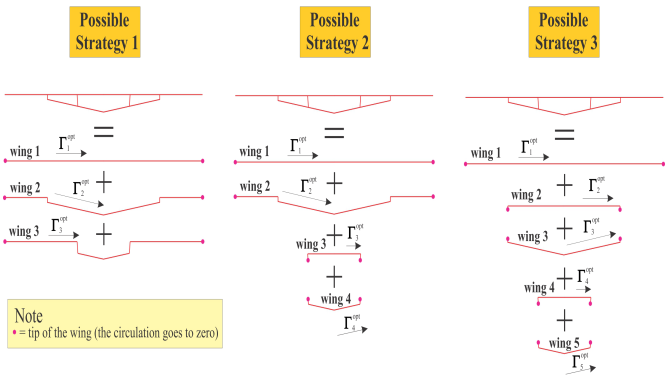

Figure 4: TBW decomposition examples

of its solution, for which an error estimate has been derived, and a final definition of the solution of the original TBW minimization problem.

2. The Drag Minimization Problem

As shown in [4], the drag minimization problem for a symmetric (with respect to the z-axis) Box-Wing has a (optimal) solution, which is uniquely defined up to an arbitrary constant. However, the contribution of the latter to the drag and to the associated lift is null. In that same paper, it has also been shown by an intensive numerical testing, but unfortunately not yet by a mathematical proof, that a symmetric Box-Wing configuration can be considered as the "closure" of a symmetric C-Wing, in the sense that when we "close", at the north or south poles, the open wing we obtain the same minimum drag. The corresponding optimal solutions differ by a constant, since in the case of the open wing the zero endpoint condition is imposed. Note that the latter condition implies that the corresponding optimal solution will then vanish at the wing closure point.

In the above mentioned paper it has also been shown that a similar property holds when a closed wing is defined as the closure of two symmetric (with respect to the z-axis) open curves. In this case, being the problem solution symmetric with respect to the z-axis, it will necessarily vanish at the two symmetric closure points.

Thus, a Box-Wing problem can always be approximated by that defined either on a single (symmetric) open curve with its two endpoint close to each other, or on a couple of (symmetric) open curves with their corresponding endpoints close to each other. We recall that it is known that the solution behavior near each curve endpoints is given by the square root of the distance of a curve point from the endpoint.

The TBW case has a more complex geometry, which does not allow to examine the associated minimization problem directly on it. To overcome this difficulty, the above properties have suggested the authors of [7, 8] to decompose a (symmetric) TBW configuration into N symmetric open wings, close to each other, letting then their distances tending to zero. Of course there are infinite many way of decomposing a TBW into a finite number of open (symmetric) curves. For a sample of them, see for example the case reported in Figure 4. Since a TBW does not have sharp edges, these being smoothly rounded, the open curve defined by the chosen decomposition are smooth. The intensive numerical testing performed in the above two papers seem to confirm that the drag minimum does not depend on the chosen decomposition.

Because of the above remarks, here and in the following we consider the multiwing case, that is, the case of a system of $N \geqslant 1$ symmetric disjoint open wings, each defined by an open lifting line ℓ_k, $k = 1,\dots,N$, in the Cartesian y-z plane. The value $N = 1$ identifies a single C-Wing, a case discussed in [3, 4, 9]. We assume that each curve ℓ_k has a parametric representation $\psi_k(t) = \begin{bmatrix} \psi_{1k}(t) & \psi_{2k}(t) \end{bmatrix}^T$, with $\psi_{ik} \in \mathbf{C}^m[-1,1]$ for some $m \geqslant 2$, and $|\psi_k'(t)| \neq 0$. We further assume that the functions $\psi_{1k}(t), k = 1,\dots,N$, are not all constant on $[-1,1]$, that is, the lifting lines are not all vertical segments, a nonsense case for an airplane multiwing system. Each curve has it own arc length abscissas, denoted by

$$\eta_k(t) = \int_0^t \left|\psi_k'(s)\right| \mathrm{d}s, \tag{2.1}$$

running from $\eta_k(-1) = -a_k$ to $\eta_k(1) = a_k$ for some positive real number a_k.

A point on ℓ_k, where the aerodynamic forces are calculated, is denoted by $\mathbf{r}_k = \begin{bmatrix} y_k & z_k \end{bmatrix}^T = \begin{bmatrix} y_k(\eta_k) & z_k(\eta_k) \end{bmatrix}^T$. The *total lift* L and the *induced drag* D_{ind}, given in terms of the (unknown) *circulations* Γ_k on ℓ_k, are defined by:

$$L = L(\Gamma) = \sum_{k=1}^{N} L(\Gamma_k) \quad \text{with} \quad L(\Gamma_k) = -\rho_\infty V_\infty \int_{-a_k}^{a_k} \tau_{yk}(\eta_k)\Gamma_k(\eta_k)\,\mathrm{d}\eta_k \tag{2.2}$$

and

$$D_{\text{ind}} = D_{\text{ind}}(\Gamma) = -\rho_\infty \sum_{j=1}^{N} \int_{-a_j}^{a_j} v_{nj}(\eta_j)\Gamma_j(\eta_j)\mathrm{d}\eta_j, \tag{2.3}$$

where $\Gamma = \begin{bmatrix} \Gamma_j \end{bmatrix}_{j=1}^N$; ρ_∞ (density) and V_∞ (free stream velocity) are given positive constants.

The quantity $\tau_{yk}(\eta_k) = y_k'(\eta_k)$ is the projection on the y-axis of the unit vector tangent to the lifting line ℓ_k, while

$$v_{nj}(\eta_j) = \frac{1}{4\pi} \sum_{k=1}^{N} \int_{-a_k}^{a_k} \Gamma_k'(\xi_k)\, Y_{jk}(\eta_j,\xi_k)\,\mathrm{d}\xi_k, \quad -a_j < \eta_j < a_j, \tag{2.4}$$

where

$$Y_{jk}(\eta_j,\xi_k) = -\frac{\mathrm{d}}{\mathrm{d}\eta_j} \ln\left|\mathbf{r}_k(\xi_k) - \mathbf{r}_j(\eta_j)\right|. \tag{2.5}$$

is the so-called *normalwash* associated with ℓ_j.

The (constrained) wing minimization problem we have to solve then takes the following form: minimize, in a suitable functional space, the functional $D_{\text{ind}}(\Gamma)$, subject to the prescribed lift constraint

$$\sum_{k=1}^{N} L(\Gamma_k) = L_{\text{pres}}. \tag{2.6}$$

To solve this problem, we set

$$\Gamma_{0k}(t) := \Gamma_k(\eta_k(t)), \quad \mathbf{r}_{0k}(t) := \mathbf{r}_k(\eta_k(t)) = \psi_k(t),$$

$$Y_{0jk}(t,s) := -\frac{\mathrm{d}}{\mathrm{d}t} \ln |\mathbf{r}_{0k}(s) - \mathbf{r}_{0j}(t)|, \, t,s \in [-1,1] \tag{2.7}$$

and

$$\Gamma'_{0k}(t) = \Gamma'(\eta_k(t))\eta'_k(t), \, \psi'_{1k}(t) = y'_k(\eta_k(t))\eta'_k(t), \, Y_{0jk}(t,s) = Y_{jk}(\eta_j(t),\eta_k(s))\eta'_j(t).$$

Condition (2.6) then take the new form

$$\sum_{k=1}^{N} \int_{-1}^{1} \psi'_{1k}(t)\Gamma_{0k}(t)\,\mathrm{d}t = \gamma := -\frac{L_{\text{pres}}}{\rho_\infty V_\infty}. \tag{2.8}$$

Moreover, we have

$$D_{\text{ind}} = D_{\text{ind}}(\Gamma_0) = -\frac{\rho_\infty}{4\pi} \sum_{j=1}^{N} \int_{-1}^{1} \sum_{k=1}^{N} \int_{-1}^{1} Y_{0jk}(t,s)\Gamma'_{0k}(s)\,\mathrm{d}s\,\Gamma_{0j}(t)\,\mathrm{d}t, \tag{2.9}$$

where $\Gamma_0 = \left[\ \Gamma_{0k}\ \right]_{k=1}^{N}$ is the new unknown of our (constrained) minimization problem.

In the following, a brief survey of the main theoretical results obtained in [13] is presented.

3. Functional spaces and problem reformulation

Let $\varphi(t) := \sqrt{1-t^2}$ and $\mathbf{L}^2_\varphi$ be the real Hilbert space of all (classes of) quadratic summable functions w.r.t. the weight $\varphi(t)$, equipped with the inner product

$$\langle f,g\rangle_\varphi := \int_{-1}^{1} f(t)g(t)\varphi(t)\,\mathrm{d}t$$

and the norm $\|f\|_\varphi = \sqrt{\langle f,f\rangle_\varphi}$. By $\left\{p_n^\varphi : n \in \mathbb{N}_0\right\}$ we denote the system of orthonormal, w.r.t. $\varphi(t)$, polynomials $p_n^\varphi(t)$ of degree n with positive leading coefficient. Note that for the above weight function, these are the well-known normalized Chebyshev polynomials of the second kind $U_n(t)$.

Then we define the family of Sobolev-type spaces (see [1])

$$\mathbf{L}^{2,r}_{\varphi} = \mathbf{L}^{2,r}_{\varphi}(-1,1)\,, r \geqslant 0, \quad \mathbf{L}^{2,0}_{\varphi} = \mathbf{L}^{2}_{\varphi}$$

$$\mathbf{L}^{2,r}_{\varphi} := \left\{ f \in \mathbf{L}^2_{\varphi} : \sum_{n=0}^{\infty} (1+n)^{2r} \left| \langle f, p^{\varphi}_n \rangle_{\varphi} \right|^2 < \infty \right\},$$

with

$$\langle f, g \rangle_{\varphi,r} = \sum_{n=0}^{\infty} (1+n)^{2r} \langle f, p^{\varphi}_n \rangle_{\varphi} \langle g, p^{\varphi}_n \rangle_{\varphi}$$

and the norm $\|f\|_{\varphi,r} := \sqrt{\langle f, f \rangle_{\varphi,r}}$. The set $\mathbf{L}^{2,r}_{\varphi}$ is a Hilbert space. We further define

$$\mathbf{V} := \left\{ f = \varphi u : u \in \mathbf{L}^{2,1}_{\varphi} \right\},$$

$$\langle f, g \rangle_{\mathbf{V}} := \langle \varphi^{-1} f, \varphi^{-1} g \rangle_{\varphi,1} \quad \text{and} \quad \|f\|_{\mathbf{V}} := \left\| \varphi^{-1} f \right\|_{\varphi,1}.$$

LEMMA 1 (see [9]). *For $f \in \mathbf{V}$, we have $f \in \mathbf{C}[-1,1]$ with $f(\pm 1) = 0$.*

Setting $\left[\ f_k\ \right]_{k=1}^{N}$ to identify the vector $\left[\ f_1 \quad \dots \quad f_N\ \right]^T$, the problem we aim to solve can be written as follows, where, here and in the following, $\langle \cdot, \cdot \rangle$ denotes the (standard) unweighted $\mathbf{L}^2$ inner product (i.e. $\langle \cdot, \cdot \rangle_{\varphi}$ with $\varphi \equiv 1$).

(P) *Find a function $\Gamma_0 = \left[\ \Gamma_{0k}\ \right]_{k=1}^{N} \in \mathbf{V}^N$, which minimizes the functional (cf. (2.9))*

$$F(\Gamma_0) := -\sum_{j=1}^{N} \int_{-1}^{1} \sum_{k=1}^{N} \int_{-1}^{1} Y_{0jk}(t,s) \Gamma'_{0k}(s)\, \mathrm{d}s\, \Gamma_{0j}(t)\, \mathrm{d}t$$

subject to (cf. (2.8)) $\displaystyle\sum_{k=1}^{N} \left\langle \psi'_{1k}, \Gamma_{0k} \right\rangle = \gamma,$

If we define

$$\langle \mathbf{f}, \mathbf{g} \rangle_N := \sum_{k=1}^{N} \langle f_k, g_k \rangle\,, \quad \mathbf{f} = \left[\ f_k\ \right]_{k=1}^{N}, \mathbf{g} = \left[\ g_k\ \right]_{k=1}^{N}$$

and

$$(\mathcal{A}\mathbf{f})(t) := \left[\ -\frac{1}{\pi} \sum_{k=1}^{N} \int_{-1}^{1} Y_{0jk}(t,s) f'_k(s)\, \mathrm{d}s\ \right]_{j=1}^{N}, \quad -1 < t < 1, \tag{3.10}$$

then the problem can be reformulated as follows:

(P) *Find a function $\Gamma_0 = \left[\ \Gamma_{0k}\ \right]_{k=1}^{N} \in \mathbf{V}^N$ minimizing the functional $F(\Gamma_0) := \langle \mathcal{A}\Gamma_0, \Gamma_0 \rangle_N$ on $\mathbf{V}^N$ subject to $< \psi'_1, \Gamma_0 >_N = \gamma$, with $\psi_1 = \left[\ \psi_{1k}\ \right]_{k=1}^{N}$.*

4. The Euler-Lagrange Equation and its Properties

First we note that it can be easily shown (see [9], Lemma 3) that the kernel $Y_{0jj}(t,s)$ has the representation

$$Y_{0jj}(t,s) = \frac{1}{s-t} + K_j(t,s), \tag{4.11}$$

where the function $K_j : [-1,1]^2 \longrightarrow \mathbb{R}$ is continuous together with its partial derivatives $\dfrac{\partial^{i+\ell} K_j(t,s)}{\partial t^i \partial s^\ell}$, $i,\ell \in \mathbb{N}_0$, $i+\ell \leqslant m-2$. The next lemma and theorem then follow.

LEMMA 2. *The operator* $\mathcal{A} : \mathbf{V}^N \longrightarrow (\mathbf{L}^2_\varphi)^N$ *is a linear and bounded one and, consequently,* $\langle \mathcal{A}\mathbf{f}, \mathbf{f} \rangle_N$ *is well defined for all* $\mathbf{f} \in \mathbf{V}^N$. *Furthermore, for all* $\mathbf{f} \in \mathbf{V}^N$, *the relation*

$$\mathcal{A}\mathbf{f} = \mathcal{D}\mathcal{B}\mathbf{f} \tag{4.12}$$

holds true, where

$$(\mathcal{B}\mathbf{f})(t) := \left[\sum_{k=1}^{N} \frac{1}{\pi} \int_{-1}^{1} \ln|\mathbf{r}_{0k}(s) - \mathbf{r}_{0j}(t)|\, f_k'(s)\, \mathrm{d}s \right]_{j=1}^{N} \tag{4.13}$$

and where $\mathcal{D}$ *is the operator of generalized differentiation.*

The operator $\mathcal{D} : \mathbf{V}^N \longrightarrow (\mathbf{L}^2_\varphi)^N$ defined by $\mathcal{D}\mathbf{f} := \left[\ \mathcal{D}f_k\ \right]_{k=1}^N$ is an isometrical isomorphism, where $\|\mathbf{f}\|_{\mathbf{V}^N} = \left(\sum_{k=1}^N \|f_k\|^2_{\mathbf{V}} \right)^{\frac{1}{2}}$, $\|\mathbf{f}\|_{(\mathbf{L}^2_\varphi)^N} = \sqrt{\langle \mathbf{f}, \mathbf{f} \rangle_{\varphi,N}}$, and $\langle \mathbf{f}, \mathbf{g} \rangle_{\varphi,N} = \sum_{k=1}^N \langle f_k, g_k \rangle_\varphi$.

In the following, the symbol Θ will denote the trivial element of the linear space under consideration.

THEOREM 1. *The operator* $\mathcal{A} : \mathbf{V}^N \longrightarrow (\mathbf{L}^2_\varphi)^N$ *is symmetric and positive definite, i.e.* $\forall\, \mathbf{f}, \mathbf{g} \in \mathbf{V}^N$, $\langle \mathcal{A}\mathbf{f}, \mathbf{g} \rangle_N = \langle \mathbf{f}, \mathcal{A}\mathbf{g} \rangle_N$ *and,* $\forall\, \mathbf{f} \in \mathbf{V}^N \backslash \{\Theta\}$, $\langle \mathcal{A}\mathbf{f}, \mathbf{f} \rangle_N > 0$.

For the given $\gamma \in \mathbb{R}$, define the corresponding (affine) manifold

$$\mathbf{V}^N_\gamma := \left\{ \mathbf{f} = \left[\ f_k\ \right]_{k=1}^N \in \mathbf{V}^N : \sum_{k=1}^N \langle f_k, \psi'_{1k} \rangle = \gamma \right\}.$$

If we set $\Psi_1 = \left[\ \psi_{1k}\ \right]_{k=1}^N$ and $\Psi_1' = \left[\ \psi'_{1k}\ \right]_{k=1}^N$, then the following result holds.

THEOREM 2. *The element* $\Gamma_0^* \in \mathbf{V}_\gamma^N$ *is a solution of Problem* (P) *if and only if there is a* $\beta \in \mathbb{R}$ *such that*

$$\mathcal{A}\Gamma_0^* = \beta \Psi_1' . \tag{4.14}$$

This solution is unique, if it exists.

LEMMA 3. *Equation* (4.14) *can be written equivalently as*

$$\mathcal{B}\Gamma_0^* = \beta\Psi_1 + \delta, \quad \Gamma_0^* \in \mathbf{V}_\gamma^N, \ (\beta \in \mathbb{R}, \delta \in \mathbb{R}^N). \tag{4.15}$$

Moreover, we have

$$\mathcal{B}\mathbf{f} = \mathcal{A}_0\mathbf{f} \quad \forall \mathbf{f} \in \mathbf{V}^N, \tag{4.16}$$

where

$$(\mathcal{A}_0\mathbf{f})(t) := \left[\ \sum_{k=1}^N \frac{1}{\pi} \int_{-1}^1 Y_{0kj}(s,t) f_k(s)\,\mathrm{d}s \ \right]_{j=1}^N = -(\mathcal{S}\mathbf{f})(t) + (\mathcal{K}_0\mathbf{f})(t) \tag{4.17}$$

with

$$(\mathcal{S}\mathbf{f})(t) := \left[\ \frac{1}{\pi} ⨍_{-1}^1 \frac{f_j(s)\,\mathrm{d}s}{s-t} \ \right]_{j=1}^N, \quad -1 < t < 1$$

$$(\mathcal{K}_0\mathbf{f})(t) := \left[\ \sum_{k=1}^N \frac{1}{\pi} \int_{-1}^1 K_{kj}(s,t) f_k(s)\,\mathrm{d}s \ \right]_{j=1}^N \tag{4.18}$$

and

$$K_{kj}(s,t) := \begin{cases} K_j(s,t) & : \ k = j, \\ Y_{0kj}(s,t) & : \ k \neq j. \end{cases} \tag{4.19}$$

Above and in the following, the symbol $⨍$ means that the integral is defined in the Cauchy principal value sense.

Using the above results, the following theorem has then be proved.

THEOREM 3. *Assume that* $\psi_{ik} \in \mathbf{C}^3[-1,1]$, $i = 1,2$, $k = 1,\ldots,N$. *Then,*

(a) *the operator* $\mathcal{A}_0 : \left(\mathbf{L}^2_{\varphi^{-1}}\right)^N \longrightarrow \left(\mathbf{L}^2_{\varphi^{-1}}\right)^N$ *has a trivial null space, i.e.,*

$$N(\mathcal{A}_0) = \left\{ \mathbf{f} \in \left(\mathbf{L}^2_{\varphi^{-1}}\right)^N : \mathcal{A}_0\mathbf{f} = \Theta \right\} = \{\Theta\};$$

If furthermore, the vector valued functions $\left[\ \psi_{1k}(t) \ \right]_{k=1}^N$, *are not constant on* $[-1,1]$, *then:*

(b) *equation* (4.15) *possesses a unique solution* $(\Gamma_0^*, \beta, \delta) \in \mathbf{V}_\gamma^N \times \mathbb{R} \times \mathbb{R}^N$*;*

(c) *Problem* (P) *is uniquely solvable.*

5. A Collocation-Quadrature Method

The previous Lemma 3 allows us to reduce the minimization problem we have to solve to a system of N Cauchy integral equations (see (4.15)). Indeed, taking identity (4.16) into account, system (4.15) can be rewritten in the new form

$$\mathcal{A}_0\mathbf{f} = \beta\Psi_1 + \delta, \quad (\mathbf{f},\beta,\delta) \in \mathbf{V}_\gamma^N \times \mathbb{R} \times \mathbb{R}^N \tag{5.20}$$

with (see (4.17))

$$\mathcal{A}_0 = -\mathcal{S} + \mathcal{K}_0 : \left(\mathbf{L}^2_{\varphi^{-1}}\right)^N \longrightarrow \left(\mathbf{L}^2_{\varphi^{-1}}\right)^N.$$

Its unique solution is denoted by $(\mathbf{f}^*,\beta^*,\delta^*)$.

To construct a numerical method for its solution, first we collocate our Cauchy singular integral equations at the zeros of the $(n+1)$-degree Chebyshev polynomial of the first kind $T_{n+1}(t)$

$$t_{\ell n} = \cos\frac{(2\ell-1)\pi}{2n+2}, \ell = 1,\dots,n+1.$$

Then we consider the image space $R(\mathcal{P}_n)$ of the orthoprojection $\mathcal{P}_n : \left(\mathbf{L}^2_{\varphi^{-1}}\right)^N \longrightarrow \left(\mathbf{L}^2_{\varphi^{-1}}\right)^N$ defined by

$$\mathcal{P}_n\mathbf{f} = \left[\ \sum_{k=0}^{n-1}\langle f_j, U_k\rangle\varphi U_k\ \right]_{j=1}^N.$$

For $\mathbf{f}^n = \left[\ f_k^n\ \right]_{k=1}^N \in R(\mathcal{P}_n)$ we have (see [9, Section 5]):

$$(\mathcal{S}\mathbf{f}^n)(t_{\ell n}) = \left[\ \sum_{i=1}^{n}\frac{\varphi(s_{in})}{n+1}\frac{f_j^n(s_{in})}{s_{in}-t_{\ell n}}\ \right]_{j=1}^N. \tag{5.21}$$

The discretization of the integrals defining $(\mathcal{K}_0\mathbf{f}^n)(t_{\ell n})$ (see (4.18)) and the constraint inner product $\langle f_k, \psi'_{1k}\rangle$, is performed by multiplying and dividing their integrand functions by $\varphi(s)$, and then applying the n-point Gauss-Chebyshev rule of the second kind, whose nodes and coefficients are given by $s_{ni} = \cos\frac{i\pi}{n+1}$ and $\frac{\pi}{n+1}[\varphi(s_{ni})]^2$, $i = 1,\dots,n$, respectively. After this, we obtain the following approximations of $(\mathcal{K}_0\mathbf{f}^n)(t_{\ell n})$:

$$(\mathcal{K}_n^0\mathbf{f}^n)(t_{\ell n}) := \left[\ \frac{1}{n+1}\sum_{k=1}^{N}\sum_{i=1}^{n}\varphi(s_{in})K_{kj}(s_{in},t_{\ell n})f_k^n(s_{in})\ \right]_{j=1}^N \tag{5.22}$$

and of the associated constraint:

$$\frac{\pi}{n+1}\sum_{k=1}^{N}\sum_{i=1}^{n}\varphi(s_{in})\psi'_{1k}(s_{in})f_k^n(s_{in}) = \gamma. \tag{5.23}$$

Further, recalling definition (4.19) and identity (4.11), and taking into account representations (5.21) and (5.22), we obtain the following discretization of $(\mathcal{A}_0\mathbf{f}^n)(t_{\ell n})$ defined by (4.17):

$$-\left(\mathcal{S}\mathbf{f}^n\right)(t_{\ell n})+\left(\mathcal{K}_n^0\mathbf{f}^n\right)(t_{\ell n})$$
$$=\left[\frac{1}{n+1}\sum_{k=1}^{N}\sum_{i=1}^{n}\varphi(s_{in})Y_{0kj}(s_{in},t_{\ell n})f_k^n(s_{in})\right]_{j=1}^{N},\ell=1,\cdots,n+1.$$

Thus, for every integer $n\geqslant 1$, we have to find $(\mathbf{f}^n,\beta^n,\delta^n)\in R(\mathcal{P}_n)\times\mathbb{R}\times\mathbb{R}^N$ such that the equations

(5.24) $$-(\mathcal{S}\mathbf{f}^n)(t_{\ell n})+(\mathcal{K}_n^0\mathbf{f}^n)(t_{\ell n})=\beta^n\Psi_1(t_{\ell n})+\delta^n,\quad \ell=1,\ldots,n+1,$$

are satisfied together with condition (5.23).

Finally, by introducing the Lagrange interpolation operators

$$\mathcal{A}_n=-\mathcal{S}_n+\mathcal{K}_n\qquad \mathcal{S}_n=\mathcal{L}_n^1\mathcal{S}\mathcal{P}_n\qquad \mathcal{K}_n=\mathcal{L}_n^1\mathcal{K}_n^0\mathcal{P}_n$$

$$(\mathcal{L}_n^1\mathbf{g})(t)=\sum_{\ell=1}^{n+1}\frac{T_{n+1}(t)}{(t-t_{\ell n})T'_{n+1}(t_{\ell n})}\mathbf{g}(t_{\ell n})$$

$$(\mathcal{L}_n^2\mathbf{g})(t)=\sum_{i=1}^{n}\frac{U_n(t)}{(t-s_{in})U'_n(s_{in})}\mathbf{g}(s_{in}).$$

the previous system can be written in the form

(5.25) $$\mathcal{A}_n\mathbf{f}^n=\beta^n\mathcal{L}_n^1\Psi_1+\delta^n,\quad (\mathbf{f}^n,\beta^n,\delta^n)\in R(\mathcal{P}_n)\times\mathbb{R}\times\mathbb{R}^N$$

together with the constraint

(5.26) $$\sum_{k=1}^{N}\left\langle\mathcal{L}_n^2\psi'_{1k},f_k^n\right\rangle=\gamma.$$

Once the system unknowns $\mathbf{f}^n(s_{in}), i=1,\ldots,n$, together with β^n and δ^n, have been determined, we define the continuous approximant of $\mathbf{f}^*$

$$\mathbf{f}^{n*}(t)=\sum_{i=1}^{n}\frac{\varphi(t)}{\varphi(s_{in})}\frac{U_n(t)}{(t-s_{in})U'_n(s_{in})}\mathbf{f}^n(s_{in}),\quad -1\leqslant t\leqslant 1.$$

For this collocation-quadrature method, the following convergence estimate is a particular case of the more general one proved in [13].

THEOREM 4. *Assume $\psi_{ik} \in \mathbf{C}^r[-1,1]$, $i = 1,2$, $k = 1,\dots,N$, for some integer $r \geqslant 3$, $\gamma \neq 0$. Then, for all sufficiently large n (say $n \geqslant n_0$), there exists a unique solution $(\mathbf{f}^{n*}, \beta^{n*}, \delta^{n*}) \in R(\mathcal{P}_n) \times \mathbb{R} \times \mathbb{R}^N$ of (5.25), (5.26). Moreover, $\mathbf{f}^* \in \left(\varphi \mathbf{L}_{\varphi}^{2,r-2}\right)^N$ and*

$$\left(\sum_{k=1}^{N} \|f_k^{n*} - f_k^*\|_{\varphi^{-1}}^2 + |\beta^{n*} - \beta^*|^2 + \sum_{k=1}^{N} |\delta_k^{n*} - \delta_k^*|^2 \right)^{\frac{1}{2}} \leqslant c\, n^{2-r}$$

with the constant $c > 0$ independent of n.

For the corresponding algorithm and some numerical tests see [13].

References

[1] BERTHOLD D., HOPPE W., AND SILBERMANN B., *A fast algorithm for solving the generalized airfoil equation*, J. Comput. Appl. Math., **43** 1-2 (1992), 185–219.

[2] CAVALLARO R., DEMASI L., *Challenges, Ideas, and Innovations of Joined-Wing Configurations: A Concept from the Past, an Opportunity for the Future*, Progress in Aerospace Sciences **87** (2016), 1–93.

[3] DEMASI L., DIPACE A., MONEGATO G., AND CAVALLARO C., *Invariant formulation for the minimum induced drag conditions of non-planar wing systems*, AIAA Journal **52** 10 (2014), 2223–2240.

[4] DEMASI L., MONEGATO G., DIPACE A., AND CAVALLARO R., *Minimum induced drag theorems for joined wings, closed systems, and generic biwings: theory*, J. Optim. Theory Appl. **169** 1 (2016), 200–235.

[5] DEMASI L., MONEGATO G., DIPACE A., AND CAVALLARO R., *Minimum induced drag theorems for joined wings, closed systems, and generic biwings: applications*, J. Optim. Theory Appl. **169** 1 (2016), 236–261.

[6] DEMASI L., MONEGATO G., AND CAVALLARO R., *Minimum induced drag theorems for multiwing systems*, AIAA Journal **55** 10 (2017), 3266 – 3287.

[7] DEMASI L., MONEGATO G., CAVALLARO R., AND RYBARCZYK R., *Minimum Induced Drag Conditions for Truss-Braced Wings*, AIAA SciTech Forum. Aerospace Sciences Meeting, January 2018, Kissimmee, Florida, USA. AIAA 2018-1790, 1–27.

[8] DEMASI L., MONEGATO G., CAVALLARO R., AND RYBARCZYK R., *Minimum Induced Drag Conditions for Truss-Braced Wings*, AIAA Journal, (2018), accessed October 10, 2018. doi: http://arc.aiaa.org/doi/abs/10.2514/1.J057225

[9] JUNGHANNS P., MONEGATO G., AND DEMASI L., *Properties and numerical solution of an integral equation to minimize airplane drag*. In: Contemporary Computational Mathematics - a celebration of the 80th birthday of Ian Sloan (J. Dick, F.Y. Kuo, H. Wožniakowski, eds.), Springer International Publishing, 2018.

[10] PHOTO: BAUHAUS-LUFTFAHRT https://www.businessinsider.com.au/how-the-electric-ce-liner-plane-works-2012-9

[11] The Jets of the Future, Popular Science.

[12] https://secure.boeingimages.com/archive/Boeing-Transonic-Truss-Braced-Wing-Concept-2JRSXLJ2FN5W.html

[13] JUNGHANNS P., MONEGATO G., AND DEMASI L., *Properties and Numerical Solution of an Integral Equation System to Minimize Airplane Drag for a Multiwing System*, submitted.

AMS Subject Classification: 49K30, 49K21, 45E05, 65R20

Giovanni Monegato,
Department of Mathematical Sciences, Politecnico di Torino
Corso Duca degli Abruzzi, 24, 10129 Torino, Italy
Email: `giovanni.monegato@polito.it`

Lavoro pervenuto in redazione il 16-5-19