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A DATA COMPLETION METHOD FOR CAUCHY PROBLEM BASED ON AN EXPLICIT ERROR ESTIMATES

Abstract. In this paper, we propose a new iterative method for solving Cauchy problem of Laplace equation by using a prediction-correction technique based on an explicit a posteriori error estimate. The a posteriori error estimate is designed to regularize and improve the approximate solution and to accelerate the rate of convergence. We show the convergence results and we give algorithms of resolution. Numerical experiments are provided to illustrate the efficiency of the proposed methods.

1. Introduction

The Cauchy problem for Laplace equation has been known since Hadamard to be ill-posed [7, 8, 12]. This problem arises in many branches of science and engineering such as non-destructive testing, steady-state inverse heat conduction, electro-cardiology and Lamé operator in elasticity etc. [9, 16]. Let Ω be a simply connected domain in $\mathbb{R}^d$, $d = 2,3$, with Lipschitz continuous boundary, ν is the unit normal to the boundary $\partial\Omega$, oriented outward. We assume that $\partial\Omega$ is partitioned into two open connected portions Γ_C and Γ_I such that $\partial\Omega = \overline{\Gamma}_I \cup \overline{\Gamma}_C$. Each of them is of a non-vanishing measure. Consider the Cauchy problem for the Laplace equation

$$(1.1) \qquad \begin{cases} \Delta u = 0 \text{ in } \Omega, \\ u = f \text{ on } \Gamma_C, \\ \dfrac{\partial u}{\partial \nu} = \phi \text{ on } \Gamma_C. \end{cases}$$

We suppose that $f \in H^{\frac{1}{2}}(\Gamma_c)$ and $\phi \in H^{-\frac{1}{2}}(\Gamma_c)$ are compatibles, see [1, 2], the goal is to approach numerically the pair (φ, ψ) such that there exists a function u solution of (1.1) with $\varphi = u$ and $\psi = \dfrac{\partial u}{\partial \nu}$ on Γ_I. Let (u_δ, w_δ) be an approximation of $(u, \dfrac{\partial u}{\partial \nu})$ on Γ_I, we define two error functions on Γ_I as follows: $e_1 = u_{|\Gamma_I} - u_\delta$ and $e_2 = \dfrac{\partial u}{\partial \nu}_{|\Gamma_I} - w_\delta$. There is a stream of literature in the field of Cauchy problem and various numerical methods have been proposed [2, 4, 6, 9, 10, 13, 14].

In this paper, we present a new approach of data completion method based on an explicit error estimate. In the first step, we specify an initial solution guess u^0 and we construct a sequence $(u^k)_{k\in\mathbb{N}}$ as follows : for a given u^k we compute an explicit error

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estimates p_n^k of $e_1^k = u_{|\Gamma_I} - u^k$ by using a moment method [3, 5, 11, 15] and we put u^{k+1} the corrected one. We apply the same techniques to construct an approximation for $\frac{\partial u}{\partial \nu}_{|\Gamma_I}$. In the second step, we use the precedent method as a regularization technique for any iterative method for data completion for problem (1.1). Suppose that we use an iterative method to construct a sequence $(v^k)_{k\in\mathbb{N}}$ that converges to $u|_{\Gamma_I}$, we apply on each iteration a correction operator defined by the explicit error estimate in the first step. The convergence results of these methods are established. The goal of this method is to accelerate the convergence and regularize the approximate solution. To illustrate the proposed approach, we give some numerical experiments.

The paper is organised as follows: In section 2, we define the data completion methods based on explicit error estimates. We give some convergence results, and we define a regularisation technique for some data completion methods. In section 3, we give the numerical algorithms for the methods described in section 2. We also give some numerical experiments which illustrate the proposed approach.

2. Data completion methods

We define new iterative methods to solve Cauchy problem (1.1), the algorithms work by using a prediction-correction technique, that allows to accelerate the convergence of the approximate solution and is used like a regularization technique for numerical approximation of the problem of data completion.

2.1. Completion by using an a posteriori error estimates

We denote by $W(\Omega)$, $W_1(\Omega)$ and $W_2(\Omega)$ the functional spaces defined by

$$W(\Omega) = \left\{ v \in H^1(\Omega) : \Delta v = 0 \text{ in } \Omega \right\},$$

$$W_1(\Omega) = \{ v \in W(\Omega) :\ v = 0 \text{ on } \Gamma_I \} \text{ and } W_2(\Omega) = \left\{ v \in W(\Omega) :\ \frac{\partial v}{\partial \nu} = 0 \text{ on } \Gamma_I \right\}.$$

Let u be a solution of the problem defined by (1.1) and applying Green's formula, for any $v \in W(\Omega)$, we get

$$\int_{\Gamma_I} \left(\frac{\partial u}{\partial \nu} v - \frac{\partial v}{\partial \nu} u \right) ds = \int_{\Gamma_C} \left(\frac{\partial v}{\partial \nu} u - \frac{\partial u}{\partial \nu} v \right) ds. \tag{2.2}$$

Let us denote $l(v) = \int_{\Gamma_C} (\frac{\partial v}{\partial \nu} f - \phi v) ds$. In the first part of this section, we propose a new method to approximate u on Γ_I. We have the following result given in [5].

PROPOSITION 1. *If the Cauchy problem* (1.1) *has a solution u such that $u_{|\Gamma_I} \in L^2(\Gamma_I)$, then $\sigma = u_{|\Gamma_I}$ satisfies the following moment problem*

$$-\int_{\Gamma_I} \frac{\partial v}{\partial \nu} \sigma ds = l(v), \textit{ for all } v \in W_1(\Omega). \tag{2.3}$$

Conversely if $\sigma \in L^2(\Gamma_I)$ *satisfies* (2.3)*, then there exists a solution* u *of the Cauchy problem* (1.1) *such that* $u_{|\Gamma_I} \in L^2(\Gamma_I)$*.*

Let $\mathcal{U}_0$ be a space of functions defined on Γ_I where we specify an initial boundary temperature guess u^0 on Γ_I, for example we can take the trace for functions belonging to $W(\Omega)$ on the part of boundary Γ_I. The aim of this paragraph is to construct a sequence of approximate solutions $(u^k)_{k\in\mathbb{N}}$ to u on Γ_I. Let $\{v_j, j \in \mathbb{N}\}$ be a sequence of orthonormal functions, for the inner product of $L^2(\Gamma_I)$, such that

$$v_j \in W(\Omega),\ \forall j \in \mathbb{N} \text{ and } \overline{Span\{v_{j|\Gamma_I}\}_{j=0}^{\infty}} = L^2(\Gamma_I). \tag{2.4}$$

The following theorem gives the definition and convergence of this sequence.

THEOREM 1. *Suppose that the following problem*

$$\Delta Q_j = 0 \ \text{in } \Omega, \qquad Q_j = 0 \ \text{on } \Gamma_I, \qquad \frac{\partial Q_j}{\partial \nu} = v_j \ \text{on } \Gamma_I, \tag{2.5}$$

has a sufficiently smooth solution for any $j \in \mathbb{N}$ *and the Cauchy problem* (1.1) *has a solution* u *such that* $u_{|\Gamma_I} \in L^2(\Gamma_I)$*. Let* u^0 *be an initial boundary temperature guess in* $\mathcal{U}_0$*. We construct the sequence* $(u^k)_{k\geqslant 0}$ *as follow, let* u^k *be an approximate solution of* u *on* Γ_I*, for an integer* n *in* $\mathbb{N}$ *big enough, we put*

$$p_n^k = \sum_{j=0}^{n} m_j^k v_j, \tag{2.6}$$

with

$$m_j^k = -\int_{\Gamma_I} v_j u^k ds - \int_{\Gamma_C} \left(\frac{\partial Q_j}{\partial \nu} f - \phi Q_j\right) ds, \forall j \in \mathbb{N}, \tag{2.7}$$

and we define u^{k+1} *by :* $u^{k+1} = p_n^k + u^k$*. Then for* n *big enough* u^k *converge to* $u_{|\Gamma_I}$ *on* $L^2(\Gamma_I)$ *as* k *goes to* $+\infty$*.*

Proof. By proposition 1, we obtain the following moment problem

$$\int_{\Gamma_I} \frac{\partial v}{\partial \nu} u ds = -\int_{\Gamma_C} \left(\frac{\partial v}{\partial \nu} f - \phi v\right) ds, \tag{2.8}$$

where $v \in W_1(\Omega)$. Taking $v = Q_j$, $j \in \mathbb{N}$ in equation (2.8), where Q_j is the solution of (2.5), we obtain

$$\int_{\Gamma_I} v_j (u - u^k) ds = -\int_{\Gamma_I} Q_j u^k ds - \int_{\Gamma_C} \left(\frac{\partial Q_j}{\partial \nu} f - \phi Q_j\right) ds, \tag{2.9}$$

then we find the moments $m_j^k = \langle e_1^k, v_j\rangle = \int_{\Gamma_I} e_1^k v_j ds$, given by (2.7), with $e_1^k = u_{|\Gamma_I} - u^k$ and $\langle .,. \rangle$ is the inner product of $L^2(\Gamma_I)$. By construction u^k is in $\mathcal{U}_0 + V_n$ for all $k \in \mathbb{N}$ with

$$V_n = Span\{v_0, v_1, ..., v_n\}.$$

From [1], p_n^k defined by (2.6) is an explicit a posteriori error estimate for e_1^k for every k in $\mathbb{N}$ and $p_n^0 \longrightarrow e_1^0$ in $L^2(\Gamma_I)$ as $n \to +\infty$. We have $u^1 = p_n^0 + u^0 = p_n^0 + u_{|\Gamma_I} - e_1^0$, then

$$\|u^1 - u_{|\Gamma_I}\|^2_{L^2(\Gamma_I)} = \|p_n^0 - e_1^0\|^2_{L^2(\Gamma_I)} = \|\sum_{j=m+1}^{\infty} m_j^0 v_j\|^2_{L^2(\Gamma_I)} = \sum_{j=m+1}^{\infty} (m_j^0)^2.$$

☐

We propose a numerical method to approach $\frac{\partial u}{\partial \nu}$ on $L^2(\Gamma_I)$. In the first, we need the following result, for the proof we refer to [10].

PROPOSITION 2. *If the Cauchy problem* (1.1) *has a solution* u *such that* $\frac{\partial u}{\partial \nu}|_{\Gamma_I} \in L^2(\Gamma_I)$*, then* $\beta = \frac{\partial u}{\partial \nu}|_{\Gamma_I}$ *satisfies the following moment problem*

$$\int_{\Gamma_I} v\beta ds = l(v), \tag{2.10}$$

where $v \in W_2(\Omega)$. *Conversely, if* $\beta \in L^2(\Gamma_I)$ *satisfies (2.10), then there exists a solution* u *of the Cauchy problem* (1.1) *such that* $\frac{\partial u}{\partial \nu}|_{\Gamma_I} \in L^2(\Gamma_I)$.

Let $\mathcal{U}_1$ be a space of functions defined on Γ_I where we specify an initial boundary flux of temperature guess w^0 on Γ_I, The objective is to construct a sequence of approximate solutions for $\frac{\partial u}{\partial \nu}|_{\Gamma_I}$ on Γ_I.

THEOREM 2. *suppose that the following problem*

$$\Delta D_j = 0 \text{ in } \Omega, D_j = v_j \text{ on } \Gamma_I, \frac{\partial D_j}{\partial \nu} = 0 \text{ on } \Gamma_I, \tag{2.11}$$

has a sufficiently smooth solution for any $j \in \mathbb{N}$ *and the Cauchy problem* (1.1) *has a solution such that* $\frac{\partial u}{\partial \nu}|_{\Gamma_I} \in L^2(\Gamma_I)$. *Let* w^0 *be an initial boundary flux of temperature guess in* $\mathcal{U}_1$. *We construct a sequence* $(w^k)_{k\geqslant 0}$ *as follow, let* w^k *be an approximation of* $\frac{\partial u}{\partial \nu}$ *on* Γ_I, *for a fixed* n *in* $\mathbb{N}$ *we put*

$$q_n^k = \sum_{j=0}^{n} \mu_j^k v_j. \tag{2.12}$$

with

$$\mu_j^k = \int_{\Gamma_C} \left(\frac{\partial D_j}{\partial \nu} f - \phi D_j \right) ds - \int_{\Gamma_I} w^k v_j ds, \forall j \in \mathbb{N}. \tag{2.13}$$

and define w^{k+1} *by* $w^{k+1} = w^k + q_n^k$. *Then for* m *big enough* w^k *converges to* $\frac{\partial u}{\partial \nu}|_{\Gamma_I}$ *on* $L^2(\Gamma_I)$ *as* k *goes to* $+\infty$.

Proof. By proposition 2, $\frac{\partial u}{\partial \nu}|_{\Gamma_I}$ satisfies the following moment problem

$$\int_{\Gamma_I} v \frac{\partial u}{\partial \nu}|_{\Gamma_I} ds - \int_{\Gamma_I} w^k v ds = l(v) - \int_{\Gamma_I} w^k v ds, \tag{2.14}$$

where $v \in W_2(\Omega)$. Taking $v = D_j$, $j \in \mathbb{N}$ in equation (2.14), where D_j is the solution of the (2.11), we obtain the moments of e_2

$$\int_{\Gamma_I} v_j (\frac{\partial u}{\partial \nu}|_{\Gamma_I} - w^k) ds = l(D_j) - \int_{\Gamma_I} w^k v_j ds,$$

$\mu_j^k = \langle e_2^k, v_j \rangle = \int_{\Gamma_I} e_2^k v_j ds$, given by (2.13). By construction w^k is in $\mathcal{U}_1 + V_n$ for all $k \in \mathbb{N}$. From [1], q_n^k defined by (2.12) is an explicit a posteriori error estimate for e_2^k for every k in $\mathbb{N}$ and $q_n^k \longrightarrow e_2^k$ in $L^2(\Gamma_I)$ as $n \to \infty$. We have $w^1 = q_n^0 + w^0 = q_n^0 + \frac{\partial u}{\partial \nu}|_{\Gamma_I} - e_2^0$, then $\|w^1 - \frac{\partial u}{\partial \nu}|_{\Gamma_I}\|^2_{L^2(\Gamma_I)} = \|q_n^0 - e_1^0\|^2_{L^2(\Gamma_I)} = \|\sum_{j=m+1}^{\infty} \mu_j^0 v_j\|^2_{L^2(\Gamma_I)} = \sum_{j=m+1}^{\infty} (\mu_j^0)^2$.

□

2.2. Regularization of iterative methods

For any iterative method for data completion method, we present a correction technique based on the explicit error estimate. Each iteration of our new method consists of a prediction and a correction, the prediction method is obtained with the explicit error estimate given in [1]. Without loss of generality, we apply our technique to the method of Kozlov et al. [14]. We consider the two following mixed well-posed problems

$$(P1) \begin{cases} \Delta u = 0 \text{ in } \Omega, \\ u = f \text{ on } \Gamma_C, \\ \frac{\partial u}{\partial \nu} = g \text{ on } \Gamma_I; \end{cases}$$

and

$$(P2) \begin{cases} \Delta u = 0 \text{ in } \Omega, \\ \frac{\partial u}{\partial \nu} = \phi \text{ on } \Gamma_C, \\ u = \psi \text{ on } \Gamma_I; \end{cases}$$

where f and ϕ are the original Cauchy data as seen in (1.1). The Kozlov iterative procedure for solving the problem (1.1) is as follows

1. *The first approximation u_0 of the solution u of (1.1) is obtained by solving $(P1)$, where g is an arbitrary initial approximation of the normal derivative on Γ_I.*
2. *Having constructed u_{2k}, we find u_{2k+1} by solving $(P2)$ with $\psi = u_{2k}$ on Γ_I.*
3. *We then find u_{2k+2} by solving $(P1)$ with $g = \frac{\partial u_{2k+1}}{\partial \nu}$ on Γ_I.*

4. *Repeat steps 2. and 3. until the following stopping criterion is satisfied*

$$\|u_{2k+2}-u_{2k}\|_{H^1(\Omega)}+\|u_{2k+1}-u_{2k-1}\|_{H^1(\Omega)}+\|u_{2k+2}-u_{2k+1}\|_{H^1(\Omega)}\leqslant \eta_T,$$

where η_T *is a small prescribed positive quantity.*

Our methods correct the approximate solutions $u_{2k|\Gamma_I}$ and $\frac{\partial u_{2k+1}}{\partial \nu}|_{\Gamma_I}$ respectively on step 2 and step 3, then we obtain the following algorithm.

1. The first approximation u_0 of the solution u of (1.1) is obtained by solving $(P1)$, where g is an arbitrary initial approximation of the normal derivative on Γ_I.
2. We approach $e_1 = u_{|\Gamma_I} - u_{2k|\Gamma_I}$ by p_n^{2k} defined in theorem 1 and given by

$$p_n^{2k} = \sum_{j=0}^{n} m_j^{2k} v_j, \tag{2.15}$$

for a fixed n in $\mathbb{N}$, with

$$m_j^{2k} = -\int_{\Gamma_I} v_j u_{2k} ds - \int_{\Gamma_C}\left(\frac{\partial Q_j}{\partial \nu} f - \phi Q_j\right) ds, \forall j \in \{0,1,\cdots,n\} \tag{2.16}$$

where u_{2k} is the approached solution obtained by solving problem $(P1)$, then we find u_{2k+1} by solving $(P2)$ with $\psi = u_{2k} + p_n^{2k}$ on Γ_I.

3. We approach $e_2 = \frac{\partial u}{\partial \nu}|_{\Gamma_I} - \frac{\partial u_{2k+1}}{\partial \nu}|_{\Gamma_I}$ by p_n^{2k+1} defined in theorem 2 and given by $p_n^{2k+1} = \sum_{j=0}^{n} \mu_j^{2k+1} v_j$. for a fixed n in $\mathbb{N}$, with

$$\mu_j^{2k+1} = \int_{\Gamma_C}\left(\frac{\partial D_j}{\partial \nu} u - \frac{\partial u}{\partial \nu} D_j\right) ds - \int_{\Gamma_I} u_{2k+1} v_j ds, \forall j \in \{0,1,\cdots,n\}. \tag{2.17}$$

where u_{2k+1} is the approached solution obtained by solving problem $(P2)$, then we find u_{2k+2} by solving $(P1)$ with $\psi = \frac{\partial u_{2k+1}}{\partial \nu}|_{\Gamma_I} + p_n^{2k+1}$ on Γ_I.

4. Repeat steps 2. and 3. until the following stopping criterion is satisfied

$$\|u_{2k+2}-u_{2k}\|_{H^1(\Omega)}+\|u_{2k+1}-u_{2k-1}\|_{H^1(\Omega)}+\|u_{2k+2}-u_{2k+1}\|_{H^1(\Omega)}\leqslant \eta_T,$$

where η_T is a small prescribed positive quantity.

3. Numerical implementation

3.1. Algorithms of resolution

In this section, our consideration is restricted to the case where $\Omega \subset \mathbb{R}^2$, we suppose that Γ_I is the segment of the x-axis $\Gamma_I = \{(x,y) \in \mathbb{R}^2 | y = 0, 0 < x < 1\}$ and Γ_C is a

Lipschitz curve in $\{(x,y) \in \mathbb{R}^2 | y > 0\}$ which connects the two points $(0,0)$ and $(1,0)$, such that $\overline{\Gamma}_I \cup \overline{\Gamma}_C = \partial\Omega$. where it is known that in two dimensions any simply connected domain can be transformed by a conformal mapping to the domain which we consider. Note that a conformal mapping does not change Laplace's equation.

Let $\{b_j\}_{j \in \mathbb{N}}$ be an orthonormal basis of $L^2(]0,1[)$. The following result gives p_n^k, $k \in \mathbb{N}$ explicitly on a Hilbert space $L^2(\Gamma_I)$. Consider the sequence $\{q_j\}_{j \in \mathbb{N}}$ such that $q_j(x) = \int_1^x b_j(t)dt$. For $z = (x,y) \in \mathbb{R}^2$, we define the sequence of functions $Q_j(x,y)$ by $Q_j(x,y) = Im(q_j(x+iy))$, $j \in \mathbb{N}$, where Im is the imaginary part and i is the complex number such that $i^2 = -1$. It is easy to verify that for every $j \in \mathbb{N}$, we have $Q_j \in W(\Omega)$, $\dfrac{\partial Q_j}{\partial \nu}(x,y)_{|\Gamma_I} = b_j(x)$ and $Q_j(x,y)_{|\Gamma_I} = 0$. Then, the moments m_j^k with $0 \leqslant j \leqslant n$ and $k \in \mathbb{N}$ in equation (2.7) are given by

$$m_j^k = \int_{\Gamma_I} b_j u^k ds - \int_{\Gamma_C} \left(\frac{\partial Q_j}{\partial \nu} f - \phi Q_j \right) ds.$$

If we are interested in approaching $\dfrac{\partial u}{\partial \nu}$ on Γ_I. The following result gives q_n^k $k \in \mathbb{N}$ explicitly on a Hilbert basis of $L^2(\Gamma_I)$. Let us define the sequence $\{D_j\}_{j \in \mathbb{N}}$ such that $D_j(x,y) = Re(b_j(x+iy))$, $j \in \mathbb{N}$, where Re represents the real part. It is easy to verify that for all $j \in \mathbb{N}$, we have

$$D_j \in W(\Omega), \frac{\partial D_j}{\partial \nu}(x,y)_{|\Gamma_I} = 0 \text{ and } D_j(x,y)_{|\Gamma_I} = b_j(x).$$

Then, the moments μ_j^k with $0 \leqslant j \leqslant n$ and $k \in \mathbb{N}$ in equation (2.13) are given by

$$\mu_j^k = l(D_j) - \int_{\Gamma_I} w^k b_j ds = \int_{\Gamma_I} e_2 b_j ds, \forall j \in \mathbb{N}.$$

To construct the sequence of approximate solutions, $(u^k)_{k \geqslant 0}$ of u in Γ_I, defined in Theorem 1, we consider the following algorithm

```
INPUT numbers $n \in \mathbb{N}$, $\varepsilon$; functions $b_j$ and $Q_j$ for $j = 0, \ldots, n$.
Step 1 Set $k = 0$ and give an initial boundary temperature guess $u^0$ on $\Gamma_I$ in $\mathcal{U}_0$.
Step 2 While $\|u^{k+1} - u^k\|^2_{L^2(\Gamma_I)} > \varepsilon$ do steps 3-5.
    Step 3 Get $p_n^k$ by using the algorithm in [1] .
    Step 4 Set $u^{k+1} = u^k + p_n^k$.
    Step 5 Set $k = k+1$.
Step 6 OUTPUT $(u^{k+1})$;
    STOP.
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To obtain the algorithm of construction of the sequence $(w^k)_{k \geqslant 0}$, we replace Q_j, m_j^k and p_n^k respectively by D_j, μ_j^k and q_n^k in the precedent algorithm.

3.2. Numerical results

In this section, a numerical example is devised to verify the validity of the algorithms given in the precedent section. We always construct an approximation of the unknown

Cauchy data $u\,|_{\Gamma_I}$ and $\dfrac{\partial u}{\partial \nu}\,|_{\Gamma_I}$ using the given data f and ϕ on Γ_C. It is more convenient to consider Ω as a simply bounded Lipschitz connected domain of $\mathbb{R}^2$ defined by $\Omega = \{(x,y) \in \mathbb{R}^2 : 0 < x < 1, 0 < y < 1\}$ where $\Gamma_I = \{(x,y) \in \mathbb{R}^2 : y = 0, 0 < x < 1\}$, and Γ_C is the Lipschitz curve in $\{(x,y) \in \mathbb{R}^2 : y \geqslant 0\}$ which connects the two points $(0,0)$ and $(1,0)$ such that $\left\{(x,y) \in \mathbb{R}^2 : y = 0, 0 < x < 1\right\} \cup \Gamma_I = \partial\Omega$. Let $(L_n)_n$ be a sequence of shifted Legendre polynomials normalized. These polynomials are defined for all $j = 0, 1, \ldots$ by

$$L_j(x) = \sum_{k=0}^{j} C_{jk} x^k, \tag{3.18}$$

where $C_{00} = 1$, $C_{j,0} = (2j+1)^{\frac{1}{2}}$, $C_{j,k} = -C_{j,k-1}(\frac{j}{k}+1)(\frac{j+1}{k}-1)$ for $j = 0, 1, 2, 3,$ and $k = 1, 2, 3, \ldots, j$. Then

$$(A)\left\{\begin{array}{l} \overline{Span\{L_j\}_{j=0}^{\infty}} = L^2(\Gamma_I). \\ \{L_j\}_{j\in\mathbb{N}} \text{ is an orthonormal basis in } L^2(\Gamma_I). \end{array}\right.$$

We choose the following function as a test example

$$u_1(x,y) = x^3 - 3xy + e^{2y} sin(2x) - e^y cos(x).$$

Firstly, we give an approximation to $u\,|_{\Gamma_I}$ and $\dfrac{\partial u}{\partial \nu}\,|_{\Gamma_I}$ for a given u^k and w^k as is mentioned in figure 1 by using the first method. Secondly, we apply Kozlov method to obtain others numerical solutions, figure 2. Finally we apply the modified Kozlov method to obtain solutions in figure 3. We compare the rate of convergence between Kozlov method and the modified Kozlov method in table 3.1.

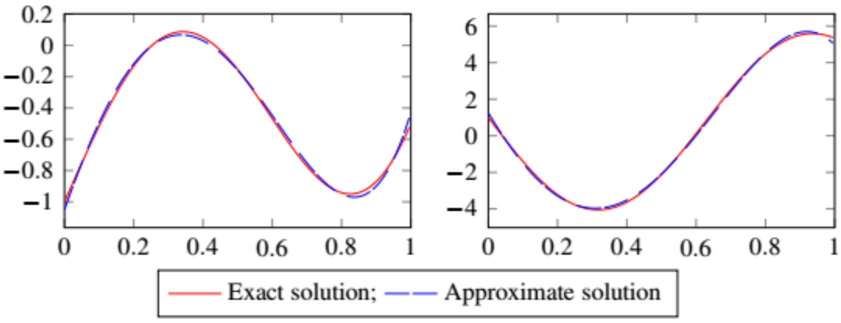

Figure 1: The exact solution $u(x,0)$ and the approximate one $u^k(x,0)$ with $k = 4$ on the left. On the right the exact solution $\dfrac{\partial u}{\partial \nu}(x,0)$ and the approximate one $w^k(x,0)$ with $k = 7$.

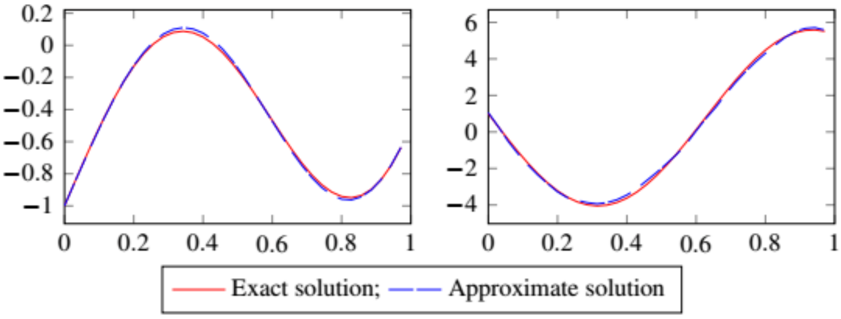

Figure 2: The exact solution $u(x,0)$ on the left and $\dfrac{\partial u}{\partial \nu}$ on the right and the approximate one by using Kozlov method after 200 iteration.

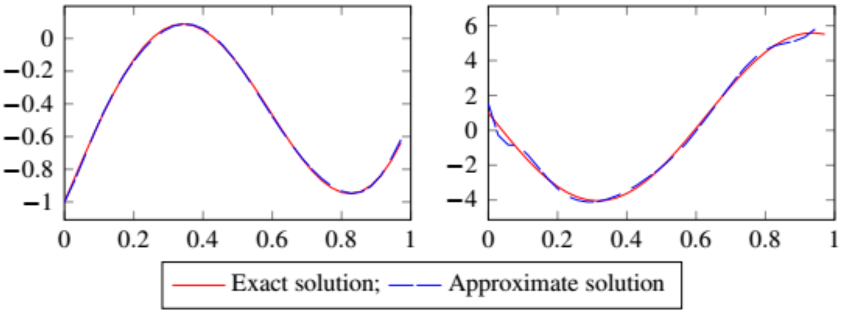

Figure 3: The exact solution $u(x,0)$ on the left and $\dfrac{\partial u}{\partial \nu}$ on the right and the approximate one by using Modified Kozlov method after one iteration.

Table 3.1: Error based on the number of iterations

Stopping criterion $\eta_0 = \|u_1 - u_2\|_2^2$		10^{-5}	$5\,10^{-6}$	10^{-6}	$5\,10^{-7}$	10^{-7}
number of iterations	Kozlov	41	47	67	79	> 900
	Kozlov Modified	1	1	1	2	2

4. Conclusion

Two iterative methods to solve the Cauchy problem for Laplace equation have been presented. The methods use an explicit a posteriori error estimate defined in [1]. The methods have considerably a better rate of convergence, they are used to improve the approximate solution and to regularize some numerical methods of data completion.

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