Rendiconti Sem. Mat. Univ. Pol. Torino
Vol. 76, 2 (2018), 87 – 96

J. Chen* **- C.-J. Li**

THE SPLINE FINITE ELEMENT METHOD

Abstract. The finite element method (FEM) plays a very important part in numerical analysis. One common practice to construct compatible quadrilateral element is isoparametric transformation. The Serendipity isoparametric elements are broadly used since their interpolate nodes are located on the boundary of the quadrilateral element. The Serendipity elements have a disadvantage of accuracy loss when elements are distorted. Because the serendipity elements possess the low order completeness in the Cartesian coordites though they have higher order terms of the isoparametric coordinates. The improvement of the Serendipity elements has attracted many researchers. Splines are piecewise polynomials satisfying certain continuity conditions, which are applied widely in the finite element method. The shape functions can be treated as splines. In this talk, we briefly introduce some progress on the spline finite element method presented in [5, 6, 7, 8, 9, 10, 11]. The key is to construct the interpolation bases (or the shape functions) by the spline functions and the B-net method for different cases. The spline finite elements can have good accuracy even for distorted meshes. The main results are as follows:

1. For plane problems, we constructed four quadrilateral elements with 4, 8, 12 and 17 nodes, which possess completeness of orders 1, 2, 3 and 4 in the Cartesian coordinates, respectively. The two diagonals are connected in a convex quadrangle, and the quadrangle is divided into four subtriangles. The triangular area coordinates and the B-net method are used in each triangle. The corresponding spline interpolation bases can be constructed in the B-net form. Some numerical examples showed that, in comparison with other known elements from the literatures, the new spline family elements can be competitive in both regular and distorted meshes.
2. For plate problems, we constructed two spline elements for thin plate and thick plate based on the discrete Kirchhoff theory and the Mindlin/Reissner theory respectively. In these cases, two sets of cubic spline Hermite interpolation bases are used. Numerical examples were discussed to show that the plate elements combined with the spline interpolation bases can possess good accuracy.
3. This idea for construction of spline element was also extend to 3D problem. For 3D problems, we constructed three spline elements for pyramid, hexahedral and triangular prism, which possess the second order of completeness of the Cartesian coordinates. By dividing the pyramid, hexahedral and triangular prism elements into two, six and three sub-tetrahedrons respectively, we constructed the spline interpolation bases by the tetrahedral volume coordinates and the B-net method. These three elements can exactly model the quadratic fields.

In conclusion, the spline finite element method has some good properties:

1. The spline interpolation bases are expressed into the B-net form. The computation on products, integrals and derivatives of the shape functions can be simplified greatly by using this representation.
2. No mapping or coordinate transformation is required and thus no Jacobian matrix and its inverse are evaluated. Hence, the spline elements are less sensitive to mesh distortions compared with the corresponding Serendipity isoparametric elements.
3. The spline elements, which are constructed from the corresponding spline interpolation bases, are conforming and can possess good accuracy.

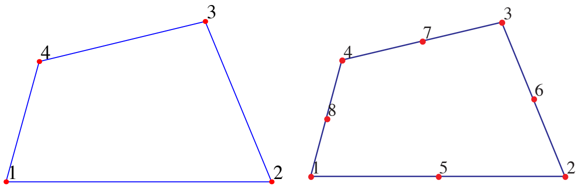
---

[*] The author was supported by the National Natural Science Foundation of China (No 11572081), and the Program for Liaoning Innovative Talent in University (LR2017054).

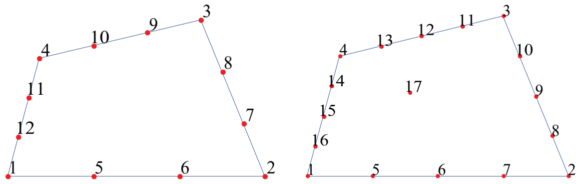
1. The family of quadrilateral spline elements

In order to overcome the disadvantage of accuracy loss for the isoparametric Serendipity elements when quadrilateral elements are distorted [1, 2], we considered to construct the quadrilateral elements directly by the spline method without isoparametric transformation. Spline functions are piecewise polynomials satisfying certain continuity conditions, which have many good approximation properties and have been widely applied in numerical analysis and computer aided geometric design etc [3, 4].

In this section, we introduce the family of quadrilateral spline elements constructed by the spline interpolation bases presented in [5, 6]. There are four spline elements L4, L8, L12, L17, corresponding to the 4, 8, 12, 17 nodes respectively, as shown in Fig. 1.



(a) L4 element (b) L8 element



(c) L12 element (d) L17 element

Figure 1: The quadrilateral elements with 4, 8, 12, 17 nodes respectively.

The main idea is to subdivide the quadrilateral element into triangulation, and construct the spline (piecewise polynomials) bases by the continuous conditions between the subtriangles. The B-net method is a very powerful tool to represent spline functions on the triangulation [4]. On each subtriangle, the n-th degree Bernstein polynomials of the area coordinates $(\lambda_1, \lambda_2, \lambda_3)$ are defined by

$$\begin{aligned} B^n_{i,j,k}(\lambda_1,\lambda_2,\lambda_3) &= \frac{n!}{i!j!k!}\lambda_1^i\lambda_2^j\lambda_3^k,\ i+j+k=n, \\ &\lambda_1,\lambda_2,\lambda_3 \geqslant 0,\ \lambda_1+\lambda_2+\lambda_3 = 1. \end{aligned} \tag{1.1}$$

Then the polynomial can be represented in the following B-net form:

$$p(x,y) = f(\lambda_1,\lambda_2,\lambda_3) = \sum_{i+j+k=n} b_{i,j,k} B^n_{i,j,k}(\lambda_1,\lambda_2,\lambda_3),$$

where $b_{i,j,k}$ are called its Bézier coefficients corresponding to the Bernstein basis $B^n_{i,j,k}$ and the domain points $\xi_{i,j,k} = (i/n, j/n, k/n)$.

There are some advantages on the computation of polynomials in the B-net form. The computation on products, integrals and derivatives of the functions can be simplified greatly by using their Bézier coefficients and the area coordinates on each triangular cell [6].

For a convex quadrangle D, denote the corner nodes by P_1, P_2, P_3, P_4, and denote the intersection of two diagonals $\overline{P_1P_3}$ and $\overline{P_2P_4}$ by P_0, as shown in Fig. 2. The quadrangle is divided into four subtriangles $\Delta_1, \dots, \Delta_4$.

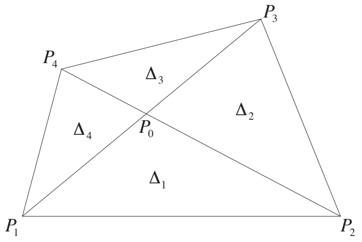


Figure 2: A convex triangulated quadrangle.

Denote Δ by the above triangulation of D, then a spline space on Δ is defined by

$$S^r_d(\Delta) = \{s \in C^r(D) : s\,|_{\Delta_i} \in \mathbb{P}_d, i = 1,2,3,4\}. \tag{1.2}$$

It means that a spline function in $S^r_d(\Delta)$ is a piecewise polynomial of degree d, and C^r continuous on the two diagonals $\overline{P_1P_3}$ and $\overline{P_2P_4}$. In [5, 6], four spline interpolation bases are constructed according to the element nodes with degree of 1, 2, 3, 4, respectively, then the spline elements are obtained consequently.

1.1. The interpolation basis for the 4-node quadrilateral spline element

For splines of degree 1, there are 5 nodes on the quadrangle, whose indexes are shown in Fig. 3. The Bézier coefficients are denoted by $b_1, \dots, b_5$. We obtained 4 linear spline basic functions $N^{(1)}_1(x,y), \dots, N^{(1)}_4(x,y)$ in a subspace of $S^0_1(\Delta)$, with the following Bézier coefficients [6]:

$$\begin{aligned} \{Nb^{(1)}_1\} &= \{1,0,0,0,\tfrac{b}{2}\}^T, \\ \{Nb^{(1)}_2\} &= \{0,1,0,0,\tfrac{a}{2}\}^T, \\ \{Nb^{(1)}_3\} &= \{0,0,1,0,\tfrac{d}{2}\}^T, \\ \{Nb^{(1)}_4\} &= \{0,0,0,1,\tfrac{c}{2}\}^T, \end{aligned} \tag{1.3}$$

where

$$a = \frac{|\overline{P_4P_0}|}{|\overline{P_4P_2}|},\ b = \frac{|\overline{P_3P_0}|}{|\overline{P_3P_1}|},\ c = 1-a,\ d = 1-b. \tag{1.4}$$

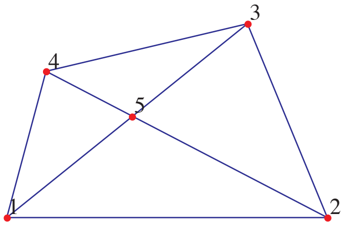


Figure 3: The B-net domain points of degree 1.

1.2. The interpolation basis for the 8-node quadrilateral spline element

In order to construct the 8-node spline element, which possesses the second order completeness in the Cartesian coordinates, the spline space $S_2^1(\Delta)$ was considered.

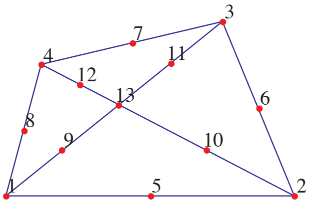


Figure 4: The B-net domain points of degree 2.

By the B-net method, there are 13 domain points on the quadrangle. Their indexes are shown in Fig. 4. The corresponding Bézier coefficients are simply denoted by $b_1, \ldots, b_{13}$. By the C^1 continuous conditions, the Bézier coefficients must satisfy the following linear system:

$$\begin{cases} b\cdot b_5 - b_{10} + d\cdot b_6 = 0, \\ b\cdot b_9 - b_{13} + d\cdot b_{11} = 0, \\ b\cdot b_8 - b_{12} + d\cdot b_7 = 0, \\ c\cdot b_8 - b_9 + a\cdot b_5 = 0, \\ c\cdot b_{12} - b_{13} + a\cdot b_{10} = 0, \\ c\cdot b_7 - b_{11} + a\cdot b_6 = 0. \end{cases} \tag{1.5}$$

By solving the linear system (1.5) with rank of 5, we obtained 8 spline functions denoted by $N_1^{(2)}(x,y), \ldots, N_8^{(2)}(x,y)$ interpolating the 8 nodes shown in Fig. 1(b) with the following Bézier coefficients $\{Nb_1^{(2)}\},\ldots,\{Nb_8^{(2)}\}$ [5]:

$$
\begin{aligned}
\{Nb_1^{(2)}\} &= \{1,0,0,0,-\tfrac{1}{2},0,0,-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{b}{2},0,-\tfrac{b}{2},-\tfrac{b}{2}\}^T,\\
\{Nb_2^{(2)}\} &= \{0,1,0,0,-\tfrac{1}{2},-\tfrac{1}{2},0,0,-\tfrac{a}{2},-\tfrac{1}{2},-\tfrac{a}{2},0,-\tfrac{a}{2}\}^T,\\
\{Nb_3^{(2)}\} &= \{0,0,1,0,0,-\tfrac{1}{2},-\tfrac{1}{2},0,0,-\tfrac{d}{2},-\tfrac{1}{2},-\tfrac{d}{2},-\tfrac{d}{2}\}^T,\\
\{Nb_4^{(2)}\} &= \{0,0,0,1,0,0,-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{c}{2},0,-\tfrac{c}{2},-\tfrac{1}{2},-\tfrac{c}{2}\}^T,\\
\{Nb_5^{(2)}\} &= \{0,0,0,0,2,0,0,0,2a,2b,0,0,2ab\}^T,\\
\{Nb_6^{(2)}\} &= \{0,0,0,0,0,2,0,0,0,2d,2a,0,2ad\}^T,\\
\{Nb_7^{(2)}\} &= \{0,0,0,0,0,0,2,0,0,0,2c,2d,2cd\}^T,\\
\{Nb_8^{(2)}\} &= \{0,0,0,0,0,0,0,2,2c,0,0,2b,2bc\}^T.
\end{aligned}
\tag{1.6}
$$

1.3. The interpolation basis for the 12-node quadrilateral spline element

For the cubic spline space $S_3^1(\Delta)$, there are 25 domain points on the quadrangle, whose indexes are shown in Fig. 5. The corresponding Bézier coefficients are simply denoted by $b_1,\ldots,b_{25}$. The C^1 continuous conditions between two polynomials defined on two adjacent triangles are given in Eq. (1.7).

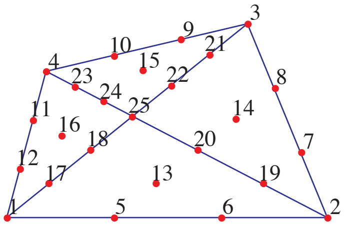


Figure 5: The B-net domain points of degree 3.

$$
\tag{1.7}
\begin{cases}
b\cdot b_{11} - b_{23} + d\cdot b_{10} = 0,\\
b\cdot b_{16} - b_{24} + d\cdot b_{15} = 0,\\
b\cdot b_{18} - b_{25} + d\cdot b_{22} = 0,\\
b\cdot b_{13} - b_{20} + d\cdot b_{14} = 0,\\
b\cdot b_{6} - b_{19} + d\cdot b_{7} = 0,\\
a\cdot b_{5} - b_{17} + c\cdot b_{12} = 0,\\
a\cdot b_{13} - b_{18} + c\cdot b_{16} = 0,\\
a\cdot b_{20} - b_{25} + c\cdot b_{24} = 0,\\
a\cdot b_{14} - b_{22} + c\cdot b_{15} = 0,\\
a\cdot b_{8} - b_{21} + c\cdot b_{9} = 0.
\end{cases}
$$

It is obvious that the dimension of the solution space is 16. In order to construct an element with the 12 nodes shown in Fig. 1(c), we have chosen a subspace with dimension of 12, and the basis satisfying symmetry, partition of unity and reproducing the polynomials of degree 3. Denote by $N_1^{(3)}(x,y),\dots,N_{12}^{(3)}(x,y)$ the 12 cubic spline functions with Bézier coefficients $\{Nb_1^{(3)}\},\dots,\{Nb_{12}^{(3)}\}$ were presented in [6].

1.4. The interpolation basis for the 17-node quadrilateral spline element

The 17-node element was constructed in the spline space $S_4^3(\Delta)$. There are 41 nodes on the quadrangle, whose indexes are shown in Fig. 6.

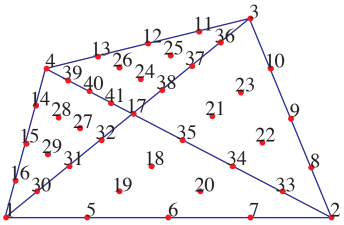


Figure 6: The B-net domain points of degree 4.

By using the same method for the 8 and 12 nodes elements, we obtained 17 quartic spline functions $N_1^{(4)}(x,y),\dots,N_{17}^{(4)}(x,y)$ interpolating the 17 nodes shown in Fig. 1(d), with Bézier coefficients $\{Nb_1^{(4)}\},\dots,\{Nb_{17}^{(4)}\}$ presented in [6].

1.5. The completeness order of the spline family elements

The following theorem shows that the spline family elements L4, L8, L12 and L17 possess completeness of orders 1-4 in the Cartesian coordinates, respectively.

THEOREM 1. *[6] Let D be an arbitrary convex quadrilateral domain $P_1P_2P_3P_4$, the four spline elements L4, L8, L12, L17 and their interpolation bases $N_i^{(k)}(x,y)$ $(k=1,2,3,4)$ are given in the above subsections. We define the following linear interpolants*

$$\begin{aligned}(N^{(1)}f)(x,y) &:= \textstyle\sum_{i=1}^{4} f(P_i^{(1)})N_i^{(1)}(x,y),\\ (N^{(2)}f)(x,y) &:= \textstyle\sum_{i=1}^{8} f(P_i^{(2)})N_i^{(2)}(x,y),\\ (N^{(3)}f)(x,y) &:= \textstyle\sum_{i=1}^{12} f(P_i^{(3)})N_i^{(3)}(x,y),\\ (N^{(4)}f)(x,y) &:= \textstyle\sum_{i=1}^{17} f(P_i^{(4)})N_i^{(4)}(x,y),\end{aligned} \tag{1.8}$$

where $P_i^{(k)}$ $(k=1,2,3,4)$ are the corresponding nodes shown in Fig. 1. Then

$$(N^{(k)}f)(x,y) \equiv f(x,y),\ (x,y)\in D,$$

for all $f(x,y)\in\mathbb{P}_k$, $k=1,2,3,4$, respectively.

2. The cubic spline Hermite interpolation basis for thin plate bending quadrilateral elements

In this section, we introduce two conforming quadrilateral thin plate elements by the cubic spline Hermite interpolation bases presented in [7]. Consider the cubic spline space $S_3^1(\Delta)$ as mentioned in Subsection 1.3. By solving the linear system of the C^1 continuity conditions on the two diagonals given in Eq. (1.7), we obtained 16 linear independent spline functions denote by $S_1(x,y), S_2(x,y), \ldots, S_{16}(x,y)$. The corresponding Bézier coefficients of each function were presented in [7].

2.1. The 16-DOF quadrilateral spline element for plate bending

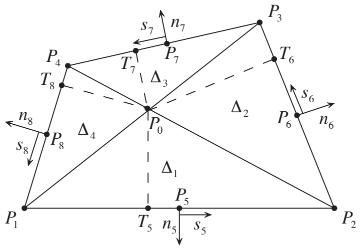


Figure 7: Geometry of the quadrilateral element.

For a given quadrangle $P_1P_2P_3P_4$ and P_0 the intersection of two diagonals $\overline{P_1P_3}$ and $\overline{P_2P_4}$, denote the Cartesian coordinates of the points by $P_i = (x_i, y_i)$, $i = 1,2,\ldots,4$. Let P_5, P_6, P_7, P_8 be the four midpoints, n_i and s_i be the identity normal and tangent vectors of the four sides respectively, as shown in Fig. 7. Let T_5, T_6, T_7, T_8 be the four perpendicular foots from P_0 to the four sides respectively, and h_i be the length of the perpendicular line $\overline{P_0T_i}$, i.e.,

$$h_i = |\overline{P_0T_i}|,\, i = 5,6,7,8.$$

$$\Delta x_5 = x_2 - x_1,\, \Delta x_6 = x_3 - x_2,\, \Delta x_7 = x_4 - x_3,\, \Delta x_8 = x_1 - x_4,$$

$$\Delta y_5 = y_2 - y_1,\, \Delta y_6 = y_3 - y_2,\, \Delta y_7 = y_4 - y_3,\, \Delta y_8 = y_1 - y_4,$$

$$l_i = \sqrt{\Delta {x_i}^2 + \Delta {y_i}^2},\, i = 5,6,7,8.$$

Let the four triangles be $\Delta_1 = \Delta P_0P_1P_2$, $\Delta_2 = \Delta P_0P_2P_3$, $\Delta_3 = \Delta P_0P_3P_4$, $\Delta_4 = \Delta P_0P_4P_1$. Then the area coordinates of P_0 with respect to each triangle Δ_i $(i = 1,2,3,4)$ are same, i.e., $(1,0,0)$. It is easy to know that $(0, \alpha_i, \beta_i)$ are the area coordinates of the perpendicular foot T_i with respect to the triangle Δ_{i-4} $(i = 5,6,7,8)$, where

$$\beta_i = ((x_0 - x_{i-4})\Delta x_i + (y_0 - y_{i-4})\Delta y_i)/l_i^2,\, \alpha_i = 1 - \beta_i.$$

Hence, the unit normal vectors can be expressed in area coordinates by

$$n_i = ((0,\alpha_i,\beta_i)-(1,0,0))/h_i = (-1,\alpha_i,\beta_i)/h_i,\ i=5,6,7,8.$$

By the 16 cubic spline functions $S_1(x,y), S_2(x,y), \ldots, S_{16}(x,y)$, we defined a set of Hermite interpolation functions as follows:

$$\begin{aligned}
H_1 &= S_1+S_{12}+S_5+\beta_8 S_{16}+\alpha_5 S_{13},\\
H_2 &= S_2+S_6+S_7+\beta_5 S_{13}+\alpha_6 S_{14},\\
H_3 &= S_3+S_8+S_9+\beta_6 S_{14}+\alpha_7 S_{15},\\
H_4 &= S_4+S_{10}+S_{11}+\beta_7 S_{15}+\alpha_8 S_{16},\\
H_1^x &= -\tfrac{\Delta x_8}{3}S_{12}+\tfrac{\Delta x_5}{3}S_5-\tfrac{(a+\beta_8)\Delta x_8+a\Delta x_5}{6}S_{16}+\tfrac{c\Delta x_8+(c+\alpha_5)\Delta x_5}{6}S_{13},\\
H_2^x &= -\tfrac{\Delta x_5}{3}S_6+\tfrac{\Delta x_6}{3}S_7-\tfrac{(d+\beta_5)\Delta x_5+d\Delta x_6}{6}S_{13}+\tfrac{b\Delta x_5+(b+\alpha_6)\Delta x_6}{6}S_{14},\\
H_3^x &= -\tfrac{\Delta x_6}{3}S_8+\tfrac{\Delta x_7}{3}S_9-\tfrac{(c+\beta_6)\Delta x_6+c\Delta x_7}{6}S_{14}+\tfrac{a\Delta x_6+(a+\alpha_7)\Delta x_7}{6}S_{15},\\
H_4^x &= -\tfrac{\Delta x_7}{3}S_{10}+\tfrac{\Delta x_8}{3}S_{11}-\tfrac{(b+\beta_7)\Delta x_7+b\Delta x_8}{6}S_{15}+\tfrac{d\Delta x_7+(d+\alpha_8)\Delta x_8}{6}S_{16},\\
H_1^y &= -\tfrac{\Delta y_8}{3}S_{12}+\tfrac{\Delta y_5}{3}S_5-\tfrac{(a+\beta_8)\Delta y_8+a\Delta y_5}{6}S_{16}+\tfrac{c\Delta y_8+(c+\alpha_5)\Delta y_5}{6}S_{13},\\
H_2^y &= -\tfrac{\Delta y_5}{3}S_6+\tfrac{\Delta y_6}{3}S_7-\tfrac{(d+\beta_5)\Delta y_5+d\Delta y_6}{6}S_{13}+\tfrac{b\Delta y_5+(b+\alpha_6)\Delta y_6}{6}S_{14},\\
H_3^y &= -\tfrac{\Delta y_6}{3}S_8+\tfrac{\Delta y_7}{3}S_9-\tfrac{(c+\beta_6)\Delta y_6+c\Delta y_7}{6}S_{14}+\tfrac{a\Delta y_6+(a+\alpha_7)\Delta y_7}{6}S_{15},\\
H_4^y &= -\tfrac{\Delta y_7}{3}S_{10}+\tfrac{\Delta y_8}{3}S_{11}-\tfrac{(b+\beta_7)\Delta y_7+b\Delta y_8}{6}S_{15}+\tfrac{d\Delta y_7+(d+\alpha_8)\Delta y_8}{6}S_{16},\\
H_5^n &= -\tfrac{2}{3}h_5 S_{13},\ H_6^n = -\tfrac{2}{3}h_6 S_{14},\\
H_7^n &= -\tfrac{2}{3}h_7 S_{15},\ H_8^n = -\tfrac{2}{3}h_8 S_{16}.
\end{aligned} \tag{2.9}$$

THEOREM 2. *[7] Let D be an arbitrary convex quadrilateral domain $P_1P_2P_3P_4$, P_5,P_6,P_7,P_8 be the midpoints of the four sides, $H_i(x,y)$, $H_i^x(x,y)$, $H_i^y(x,y)$ $(i=1,2,3,4)$ and $H_i^n(x,y)$ $(i=5,6,7,8)$ be the spline Hermite interpolation functions given in Eq. (2.9). Define the linear interpolant*

$$\begin{aligned}
(Hf)(x,y) := \quad & \sum_{i=1}^{4}\left(f(P_i)H_i(x,y)+f_x(P_i)H_i^x(x,y)+f_y(P_i)H_i^y(x,y)\right)\\
& +\sum_{i=5}^{8} f_n(P_i)H_i^n(x,y),
\end{aligned} \tag{2.10}$$

where $f_x(P_i), f_y(P_i)$ and $f_n(P_i)$ are the partial and normal derivatives of f at P_i. Then

$$(Nf)(x,y) \equiv f(x,y),\ (x,y)\in D,$$

for all $f(x,y)\in \mathbb{P}_3$.

By Theorem 2, we can obtain a 16-DOF element for plate bending problem, denoted by QS16. It can exactly model cubic displacement.

2.2. The 12-DOF quadrilateral spline element for plate bending

In order to eliminate the four coefficients $w_{n5},\ldots,w_{n8}$ at the midpoints, we set the normal derivative w_n to be a linear function on the four sides. This is equivalent to find a subspace of $S_3^1(\Delta)$ of dimension 12.

Firstly, we constructed another set of 12 spline functions from the 16 cubic spline functions $S_1(x,y), S_2(x,y), \ldots, S_{16}(x,y)$:

$$\begin{aligned}
&\tilde{S}_1 = S_1 - \tfrac{\beta_8}{2}S_{16} - \tfrac{\alpha_5}{2}S_{13}, && \tilde{S}_2 = S_2 - \tfrac{\beta_5}{2}S_{13} - \tfrac{\alpha_6}{2}S_{14},\\
&\tilde{S}_3 = S_3 - \tfrac{\beta_6}{2}S_{14} - \tfrac{\alpha_7}{2}S_{15}, && \tilde{S}_4 = S_4 - \tfrac{\beta_7}{2}S_{15} - \tfrac{\alpha_8}{2}S_{16},\\
&\tilde{S}_5 = S_5 + \tfrac{a}{2}S_{16} + \tfrac{a+2\alpha_5-\beta_5}{2}S_{13}, && \tilde{S}_6 = S_6 + \tfrac{b}{2}S_{14} + \tfrac{b-\alpha_5+2\beta_5}{2}S_{13},\\
&\tilde{S}_7 = S_7 + \tfrac{d}{2}S_{13} + \tfrac{d+2\alpha_6-\beta_6}{2}S_{14}, && \tilde{S}_8 = S_8 + \tfrac{a}{2}S_{15} + \tfrac{a-\alpha_6+2\beta_6}{2}S_{14},\\
&\tilde{S}_9 = S_9 + \tfrac{c}{2}S_{14} + \tfrac{c+2\alpha_7-\beta_7}{2}S_{15}, && \tilde{S}_{10} = S_{10} + \tfrac{d}{2}S_{16} + \tfrac{d-\alpha_7+2\beta_7}{2}S_{15},\\
&\tilde{S}_{11} = S_{11} + \tfrac{b}{2}S_{15} + \tfrac{b+2\alpha_8-\beta_8}{2}S_{16}, && \tilde{S}_{12} = S_{12} + \tfrac{c}{2}S_{13} + \tfrac{c-\alpha_8+2\beta_8}{2}S_{16}.
\end{aligned} \tag{2.11}$$

Then we have transformed them to a set of Hermite interpolation functions as follows:

$$\begin{aligned}
&\tilde{H}_1 = \tilde{S}_1 + \tilde{S}_{12} + \tilde{S}_5, && \tilde{H}_2 = \tilde{S}_2 + \tilde{S}_6 + \tilde{S}_7,\\
&\tilde{H}_3 = \tilde{S}_3 + \tilde{S}_8 + \tilde{S}_9, && \tilde{H}_4 = \tilde{S}_4 + \tilde{S}_{10} + \tilde{S}_{11},\\
&\tilde{H}_1^x = -\tfrac{\Delta x_8}{3}\tilde{S}_{12} + \tfrac{\Delta x_5}{3}\tilde{S}_5, && \tilde{H}_2^x = -\tfrac{\Delta x_5}{3}\tilde{S}_6 + \tfrac{\Delta x_6}{3}\tilde{S}_7,\\
&\tilde{H}_3^x = -\tfrac{\Delta x_6}{3}\tilde{S}_8 + \tfrac{\Delta x_7}{3}\tilde{S}_9, && \tilde{H}_4^x = -\tfrac{\Delta x_7}{3}\tilde{S}_{10} + \tfrac{\Delta x_8}{3}\tilde{S}_{11},\\
&\tilde{H}_1^y = -\tfrac{\Delta y_8}{3}\tilde{S}_{12} + \tfrac{\Delta y_5}{3}\tilde{S}_5, && \tilde{H}_2^y = -\tfrac{\Delta y_5}{3}\tilde{S}_6 + \tfrac{\Delta y_6}{3}\tilde{S}_7,\\
&\tilde{H}_3^y = -\tfrac{\Delta y_6}{3}\tilde{S}_8 + \tfrac{\Delta y_7}{3}\tilde{S}_9, && \tilde{H}_4^y = -\tfrac{\Delta y_7}{3}\tilde{S}_{10} + \tfrac{\Delta y_8}{3}\tilde{S}_{11}.
\end{aligned} \tag{2.12}$$

THEOREM 3. *[7] Let D be an arbitrary convex quadrilateral domain $P_1P_2P_3P_4$, and $\tilde{H}_i(x,y)$, $\tilde{H}_i^x(x,y)$, $\tilde{H}_i^y(x,y)$ $(i = 1,2,3,4)$ be the spline Hermite interpolation functions are given in Eq. (2.12). Define the linear interpolant*

$$(\tilde{H}f)(x,y) \quad := \quad \sum_{i=1}^{4}(f(P_i)H_i(x,y) + f_x(P_i)H_i^x(x,y) + f_y(P_i)H_i^y(x,y)), \tag{2.13}$$

where $f_x(P_i), f_y(P_i)$ are the partial derivatives at P_i. Then

$$(\tilde{H}f)(x,y) \equiv f(x,y),\ (x,y) \in D,$$

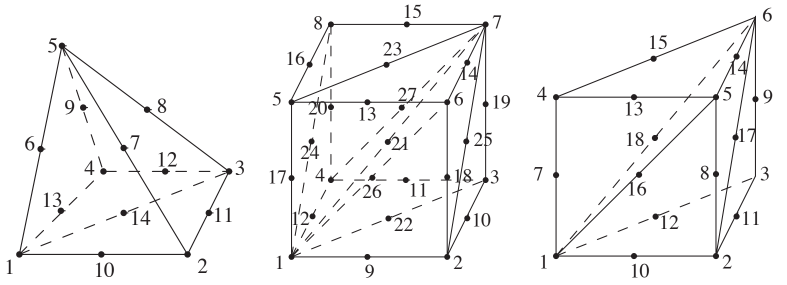
for all $f(x,y) \in \mathbb{P}_2$.

By Theorem 3, we obtain a 12-DOF element for plate bending problem, denoted by QS12. It can only exactly model quadratic displacement.

In addition, we have also constructed two spline elements for thick plate based on the Mindlin/Reissner theory [8].

3. The spline interpolation basis for 3D elements

In this section, we only briefly show that we have constructed three 3D spline elements for the pyramid, hexahedral and triangular prism. Similar to the case of 2D elements, dividing the solid elements into several sub-tetrahedrons, then by the B-net method and the tetrahedral volume coordinates, we have obtained the spline interpolation basis corresponding to boundary nodes as shown in Fig. 8. They are 13-node pyramid spline element [9], 21-node hexahedral spline element [10], and 15-node triangular prism spline element [11]. These three elements can exactly model the quadratic fields. The theoretical analysis and numerical examples were presented in the references.



(a) pyramid element (b) hexahedral element (c) Triangular prism element

Figure 8: The three 3D elements.

References

[1] Zienkiewicz O.C., Taylor R.L., *The finite element method. 5th ed*, Elsevier (Singapore) Pvt Itd., 2005.

[2] Lee N.S., Bathe K.J., *Effects of element distortion on the performance of isoparametric elements*, Int J Numer Methods Eng **36** (1993), 3553–76.

[3] Wang R.H., *Multivariate spline functions and their applications*, Science Press/ Kluwer Academic Publishers, New York 2001.

[4] Farin G., *Triangular Bernstein-Bézier patches*, Comput Aided Geom Des **3** (1986), 83–127.

[5] Li C.J., Wang R.H., *A new 8-node quadrilateral spline finite element*, J Comput Appl Math **195** (2006), 54–65.

[6] Chen J., Li C.J., Chen W.J., *A family of spline finite elements*, Comput Struct **88** (2010), 718–727.

[7] Chen J., Li C.J, *The cubic spline Hermite interpolation bases for thin plate bending quadrilateral elements (in Chinese)*, Sci Sin Math **45** (2015), 1523–1536.

[8] Chen J, *The quadrilateral Mindlin plate elements using the spline interpolation bases*, J Comput Appl Math **329** (2018), 68–83.

[9] Chen J., Li C.J., Chen W.J., *A 3D pyramid spline element*, Acta Mech Sin **27** (2011), 986–993.

[10] Li C.J., Chen J., Chen W.J., *A 3D hexahedral spline element*, Comput Struct **89** (2011), 2303–2308.

[11] Chen J., Li C.J, *A 3D triangular prism spline element using B-net method*, European Journal of Mechanics / A Solids **75** (2019), 485–496.

AMS Subject Classification: 65D07, 74S05

Juan CHEN,
School of Mathematics, Dongbei University of Finance and Economics
Dalian 116025, CHINA
e-mail: `chenjdufe@163.com`

Chong-Jun LI
School of Mathematical Sciences, Dalian University of Technology
Dalian 116024, CHINA
e-mail: `chongjun@dlut.edu.cn`

Lavoro pervenuto in redazione il 16-5-19.