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ALGEBRAIC MANIPULATIONS OF SYMBOLS

Abstract. This short note reviews how algebraic properties of subdivision symbols are linked to the analytic/geometric properties of the corresponding schemes with particular emphasis on the reproduction capability. Also, with the aim at underlining the advantage of symbols manipulation, it shows, with an example in the Hermite case, how algebraic conditions for polynomial reproduction translate into a very simple algorithm.

1. Introduction

From the two pioneering papers [1] and [12], subdivision schemes attracted many scientists for both the simplicity of their basic ideas and the mathematical elegance emerging in their analysis. They are, essentially, iterative methods for different types of data (scalars, vectors, matrices as well as other entities) generation based on refinement rules that can be easily and efficiently implemented on a computer. Their domain of application is vast since they emerge in different contexts ranging from surface generation [2], [16] to image analysis [9], [17]. A nice aspect of linear subdivision schemes is that many of their properties can be translated into algebraic properties of Laurent polynomials which makes any practical verification of them particularly simple. Moreover, since they can be essentially described by repeated multiplication of matrices, many analysis tools are based on numerical linear algebra so that they also acquire a strategic significance for the integration and cooperation between different areas of research opening up new perspectives and providing original application ideas.

The aim of this short note is to show how algebraic properties of subdivision symbols are linked to analytic/geometric properties of corresponding schemes with particular emphasis on their reproduction capability. Also, aiming at underlining the advantage of symbols manipulation, we will show with an example how in the Hermite case they translate into a very *simple* algorithm.

To fix the notation, for $d \geqslant 1$, we recall that a univariate (vector) subdivision scheme based on the *matrix masks* $\{\mathcal{A}_n,\ n \geqslant 0\}$ with $\mathcal{A}_n = \{A_n(l) \in \mathbb{R}^{d\times d}, l \in \mathbb{Z}\}$, acts on a sequence of vector data $f_n = \{f_n(l) \in \mathbb{R}^d,\ l \in \mathbb{Z}\}$ applying the *refinement rules*

$$f_{n+1}(i) = \sum_{j\in\mathbb{Z}} A_n(i-2j) f_n(j) \quad \forall\, i \in \mathbb{Z}, \quad n \geqslant 0. \tag{1.1}$$

The subdivision scheme -based on the repetition of the previous rules- is *scalar* when $d = 1$, *stationary* if $\mathcal{A}_n = \mathcal{A}$ for all $n \geqslant 0$, *Hermite* if $\mathcal{A}_n = D^{-n-1}\mathcal{A}D^n,\ n \geqslant 0$ where $D = diag(1,\ \frac{1}{2},\ \ldots, \frac{1}{2^d})$ and the sequence of initial vectors consists of function values and subsequent derivatives.

Loosely speaking, a subdivision scheme is said to be *convergent* if the denser and denser sequence of generated vectors is approaching a continuos function vector and, under convergence, the limit associated with the δ-sequence is the so called *basic limit function* whose smoothness is the smoothness of the scheme. For a formal definition of convergence and of basic limit functions we refer the reader to [6] or to [12].

Regardless of the specific type of subdivision scheme we are dealing with, to any sequence of masks $\{\mathcal{A}_n,\ n \geqslant 0\}$ we associate the matrix Laurent polynomials

$$\mathbf{A}_n(z) = \sum_{l\in\mathbb{Z}} A_n(l) z^l, \quad z \in \mathbb{C}\backslash\{0\}, \quad n \geqslant 0, \tag{1.2}$$

also called *subdivision symbols* and the *sub-symbols*

$$\mathbf{A}_{n,e}(z) = \sum_{l\in\mathbb{Z}} A_n(2l) z^{2l}, \quad \mathbf{A}_{n,o}(z) = \sum_{l\in\mathbb{Z}} A_n(2l+1) z^{2l+1}, \tag{1.3}$$

which are related by the equation $\mathbf{A}_n(z) = \mathbf{A}_{n,e}(z) + \mathbf{A}_{n,o}(z)$, by means of which the refinement rules can be riformulated as

$$\mathbf{f}_{n+1}(z) = \mathbf{A}_n(z)\mathbf{f}_n(z^2), \ \ n \geqslant 0 \quad \text{where} \quad \mathbf{f}_n(z) = \sum_{l\in\mathbb{Z}} f_n(l) z^l, \quad z \in \mathbb{C}\backslash\{0\}.$$

The mentioned Laurent polynomials play a very crucial role and, in fact, their algebraic properties are connected to analytic and geometric properties of the subdivision scheme. For example, in the full rank case (see [5, 10] for the notion or *rank*), reproduction of constant sequences is guaranteed by

$$\mathbf{A}_n(1) = 2I, \ \ \mathbf{A}_n(-1) = 0, \ \ n \geqslant 0,$$

while *interpolation* of the initial data as well as of all data generated during the process is equivalent to

$$\mathbf{A}_n(z) + \mathbf{A}_n(-z) = 2I, \ \ n \geqslant 0.$$

Also, symmetry of the basic limit functions is strongly connected to symbols parity and anti-parity, i.e.,

$$\mathbf{A}_n(z) = \mathbf{A}_n(z^{-1}) \quad \text{or} \quad z\mathbf{A}_n(z) = \mathbf{A}_n(z^{-1}), \ \ n \geqslant 0.$$

Even more, sufficient conditions on the subdivision symbols for convergence and regularity can also be given involving associated symbols of Taylor operators and difference operators (see for example, [4, 15]).

From a geometrical point of view, subdivision schemes can be both *primal* (at each iteration they retain or modify the given vectors and create a 'new' vector in between two 'old' ones) or *dual* (they discard all given vectors after creating two new ones in between any pair of them). Algebraically, this is connected with the choice of the *parameter values* t_i^n, with $t_i^n < t_{i+1}^n$ for $i \in \mathbb{Z}$, to which we attach the vectors generated by the scheme. More precisely, the primal parametrization is such that $t_i^n = \frac{i}{2^n}$ while the dual one is given by $t_i^n = \frac{i-\frac{1}{2}}{2^n}$. In this paper, we consider the parametrization

$t_i^n = \frac{i+\tau}{2^n}$ which includes primal and dual cases. We simply say that τ is the *parametric shift* or the *parameterization of the scheme* (see [13], for example).

We continue with the notion of *reproduction* for $\mathcal{V}$, a given space of function vectors. For details we refer the reader to [3, 13] concerning the vector case and the scalar case, respectively.

Definition 1. A subdivision scheme with masks $\{\mathcal{A}_n,\ n \geqslant 0\}$ and parametrization τ *reproduces* $\mathcal{V}$ if for any initial vector $f_0 = \{g(j+\tau) \in \mathbb{R}^d,\ j \in \mathbb{Z}\}$ with $g \in \mathcal{V}$, the sequence f_n, still in $\mathcal{V}$, is

$$\{f_n(j) = g((j+\tau)/2^n),\ j \in \mathbb{Z}\}, \quad \text{for all} \quad n > 1. \tag{1.4}$$

The nice fact is that, in particular, the reproduction of *polynomial* and *exponential polynomial* function spaces also translates into algebraic conditions on the symbols.Though they look complicated conditions, their algebraic nature makes them simple to verify and to translate into a Matlab code.

Being here impossible to deal with all instances of reproduction, we continue by analysing the set of conditions for *reproduction of polynomials* which is strongly connected with the approximation order of the basic limit function of the scheme (see [8] and reference thereirin). After recalling the well known conditions for scalar stationary schemes we present the more recent set of conditions for polynomial reproduction of Hermite schemes given in [7] to put in evidence that they are the 'natural'generalization of the scalar ones. Though we here consider the case $d = 2$ only, similar conclusions can be drawn for $d > 2$ and for other type of schemes/reproductions as well (see again [7] but also [4]).

2. Polynomial reproduction of stationary scalar and Hermite schemes

We start with the simplest situation, i.e. with the *polynomial* reproduction of a stationary scalar subdivision scheme with mask $a = \{a(l) \in \mathbb{R},\ l \in \mathbb{Z}\}$, parametrization τ, symbol $\mathbf{a}(z) = \sum_{l\in\mathbb{Z}} a(l)z^l$ and k-th derivatives

$$\mathbf{a}^{(k)}(z) := \sum_{l\in\mathbb{Z}} q_{k,0}(l/2)\, a(l) z^{l-k}, \quad k \geqslant 1.$$

For $i \in \mathbb{Z}$, the polynomials $q_{k,i} \in \prod_k$ involved above and later on, are defined by

$$q_{0,i}(x) := 1, \quad q_{k,i}(x) := \prod_{r=0}^{k-1}(2x+i-r), \quad k > 0. \tag{2.5}$$

Polynomial reproduction means that for any $p \in \Pi_m$, starting with the initial sequence

$$f_0 = \{p(j+\tau),\ j \in \mathbb{Z}\},$$

the refined sequence is

$$f_n = \{p((j+\tau)/2^n),\ j \in \mathbb{Z}\} \quad \text{for all} \quad n \in \mathbb{N}.$$

The algebraic result characterising polynomial reproduction reads as follows (see [3] for details even in the multivariate setting).

THEOREM 1. *Let $a = \{a(l) \in \mathbb{R},\ l \in \mathbb{Z}\}$ be the scalar mask of a convergent stationary subdivision scheme with parametrization* τ *and symbol* $\boldsymbol{a}(z) = \sum_{l\in\mathbb{Z}} a(l)z^l$. *It* reproduces polynomials up to degree $m \geqslant 1$ *if and only if*

$$\boldsymbol{a}^{(k)}(1) = 2q_{k,2\tau}(-\frac{\tau}{2}) \quad \textit{and} \quad \boldsymbol{a}^{(k)}(-1) = 0, \quad \textit{for all} \quad k = 0,\dots,m. \tag{2.6}$$

Remark 2. Theorem 1 implies that the correct parametric shift is $\tau = \mathbf{a}^{(1)}(1)/2$. Also, it provides a 'practical' tool to construct new scalar schemes with prescribed polynomial reproduction.

We continue by considering Hermite subdivision schemes i.e. assuming that the subdivision masks are $\{\mathcal{A}_n = D^{-n-1}\mathcal{A}D^n,\ n \geqslant 0\}$, $\mathcal{A} = \{A(l) \in \mathbb{R}^{d\times d},\ l \in \mathbb{Z}\}$, where $D = diag(1,\ \frac{1}{2},\ \dots, \frac{1}{2^d})$. In the Hermite situation we consider reproduction of *polynomials* up to a certain degree, say m, in the sense that for any $p \in \Pi_m$ starting with the initial sequence

$$f_0 = \left\{ \begin{bmatrix} p(j+\tau) \\ \vdots \\ p^{(d-1)}(j+\tau) \end{bmatrix},\ j \in \mathbb{Z}, \right\}$$

the refined sequence is

$$f_n = \left\{ \begin{bmatrix} p((j+\tau)/2^n) \\ \vdots \\ p^{(d-1)}((j+\tau)/2^n) \end{bmatrix},\ j \in \mathbb{Z} \right\} \quad \text{for all } n \in \mathbb{N}.$$

The k-th derivatives of symbol associated with $\mathcal{A}$ are

$$\mathbf{A}^{(k)}(z) := \sum_{l\in\mathbb{Z}} q_{k,0}(l/2)A(l)z^{l-k}, \quad k \geqslant 1,$$

with $q_{k,i} \in \Pi_k$ the polynomials defined in (2.5). Also for $\ell, k \in \mathbb{N}$, we define a set of coefficients

$$\alpha_{k,\ell} = 2(\ell-1)!\binom{k}{\ell},\ \ell = 1,\dots,k, \tag{2.7}$$

and the polynomials $\tilde{q}_{k,i} \in \prod_{k-1}$ based on them

$$\tilde{q}_{0,i} := 0, \quad \tilde{q}_{k,i}(x) := \sum_{\ell=1}^{k} (-1)^{\ell}\alpha_{k,\ell}q_{k-\ell,i}(x), \quad k > 0, \quad i \in \mathbb{Z}. \tag{2.8}$$

In the recent paper [7], the following Theorem, direct generalizatin of Theorem 1, is proven.

THEOREM 2. *Let* $\{\mathcal{A}_n = \boldsymbol{D}^{-n-1}\mathcal{A}\boldsymbol{D}^n,\ n \geqslant 0\}$ *be the matrix masks of a convergent Hermite subdivision scheme with parametrization* τ. *It* reproduces constants *if and only if*

$$\boldsymbol{A}(1)\boldsymbol{e}_1 = 2\boldsymbol{e}_1, \quad \boldsymbol{A}(-1)\boldsymbol{e}_1 = \boldsymbol{0}, \tag{2.9}$$

where $\boldsymbol{e}_1 = \binom{1}{0}$, $\boldsymbol{e}_2 = \binom{0}{1}$, *and* $\boldsymbol{0} = \binom{0}{0}$. *Moreover, it* reproduces polynomials up to degree $m \geqslant 1$ *if and only if it reproduces constants and for all* $k = 1,\dots,m$

$$\boldsymbol{A}^{(k)}(1)\boldsymbol{e}_1 + \sum_{\ell=1}^{k}(-1)^{\ell}\alpha_{k,\ell}\cdot\boldsymbol{A}^{(k-\ell)}(1)\boldsymbol{e}_2 = \begin{bmatrix} 2q_{k,2\tau}(-\frac{\tau}{2}) \\ \tilde{q}_{k,2\tau}(-\frac{\tau}{2}) \end{bmatrix} \tag{2.10}$$

$$\boldsymbol{A}^{(k)}(-1)\boldsymbol{e}_1 + \sum_{\ell=1}^{k}\alpha_{k,\ell}\cdot\boldsymbol{A}^{(k-\ell)}(-1)\boldsymbol{e}_2 = \boldsymbol{0}, \tag{2.11}$$

with $\alpha_{k,\ell}$, $\ell = 1,\dots,k$ *given in* (2.7).

Remark 3. When $m = 1$ the previous result allows us to identify the correct parametrization corresponding to the choice $\tau = \frac{1}{2}((\mathbf{A}^{(1)}(1))_{11} - 2(\mathbf{A}^{(1)}(0))_{12})$ and, again, it is a useful 'tool ' even for the construction of new Hermite subdivision schemes.

3. An algorithm for polynomial reproduction

To stress the advantage of symbols manipulation, we will show with an example how the conditions in Theorem 2 translate into a simple algorithm. To this purpose we consider the Hermite subdivision scheme proposed in [14]. Its mask is supported in $[0,4] \cap \mathbb{Z}$ with non-zero elements $\mathcal{A} = \{A_0,\ \dots\ A_4\}$ given by

$$\begin{bmatrix} \theta & -\frac{\theta}{2} \\ -\frac{3\omega}{2} & \frac{\omega}{2} \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{2} & -\frac{1}{8} \\ \frac{3}{4} & -\frac{1}{8} \end{bmatrix}, \quad \begin{bmatrix} 1-2\theta & 0 \\ 0 & \frac{1+4\omega}{2} \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{2} & \frac{1}{8} \\ -\frac{3}{4} & -\frac{1}{8} \end{bmatrix}, \quad \begin{bmatrix} \theta & -\frac{\theta}{2} \\ -\frac{3\omega}{2} & \frac{\omega}{2} \end{bmatrix}$$

where $\theta, \omega \in \mathbb{R}$ are free parameters. The algebraic conditions (2.9), (2.10) and (2.11) allow us to verify that this scheme reproduces polynomials up to degree 3 while polynomials of degree 4 are *not* reproduced. Obviously, in consideration of the support size of the scheme, 4 is the highest possible degree of polynomial reproduction.

The algorithm for checking the polynomial reproduction, translation of the mentioned conditions, follows.

Algorithm for checking polynomial reproduction up to degree $0 \leqslant m \leqslant 4$

1. Input the mask $\mathcal{A} = \{A_0,\ \dots\ A_4\}$
2. Define the matrices (symbol and its derivatives at $z = 1$)

$S := A_0 + A_1 + A_2 + A_3 + A_4;$

$S_1 := A_1 + 2A_2 + 3A_3 + 4A_4;$

$S_2 := 2A_2 + 6A_3 + 12A_4;$

$S_3 := 6A_3 + 24A_4;$

$S_4 := 24A_4;$

3 Define the matrices (symbol and its derivatives at $z = -1$)

$M := A_0 - A_1 + A_2 - A_3 + A_4;$

$M_1 := A_1 - 2A_2 + 3A_3 - 4A_4;$

$M_2 := 2A_2 - 6A_3 + 12A_4;$

$M_3 := 6A_3 - 24A_4;$

$M_4 = 24A_4;$

4. Compute the correct parametrization $\tau = \frac{(S)_{1,1} - 2(S)_{1,2}}{2}$;
5. Set the coefficients

$$\begin{array}{llll} \alpha_{1,1} = 2; & & & \\ \alpha_{2,1} = 4; & \alpha_{2,2} = 2; & & \\ \alpha_{3,1} = 6; & \alpha_{3,2} = 6; & \alpha_{3,3} = 4; & \\ \alpha_{4,1} = 8; & \alpha_{4,2} = 12; & \alpha_{4,3} = 16; & \alpha_{4,4} = 12; \end{array}$$

6. Check the reproduction of degree $0 \leqslant m \leqslant 4$

If $S\binom{1}{0} = 2\binom{1}{0} = \binom{0}{0}$ & $M\binom{1}{0} = \binom{0}{0}$ set $m := 0$

If $S_1\binom{1}{0} - \alpha_{1,1}S_1\binom{0}{1} = \binom{2\tau}{-2}$ & $M_1\binom{1}{0} + \alpha_{1,1}M_1\binom{1}{0} = \binom{0}{0}$ set $m := 1$

If $S_2\binom{1}{0} + (-\alpha_{2,1}S_1 + \alpha_{2,2}S)\binom{0}{1} = \binom{2\tau(\tau-1)}{-\alpha_{2,1}\tau+\alpha_{2,2}}$
&
$M_2\binom{1}{0} + (\alpha_{2,1}M_1 + \alpha_{2,2}M)\binom{0}{1} = \binom{0}{0}$ set $m := 2$

If $S_3\binom{1}{0} + (-\alpha_{3,1}S_2 + \alpha_{3,2}S_1 - \alpha_{3,3}S)\binom{0}{1} = \binom{2\tau(\tau-1)(\tau-2)}{-\alpha_{3,1}\tau(\tau-1)+\alpha_{3,2}\tau-\alpha_{3,3}}$
&
$M_3\binom{1}{0} + (\alpha_{3,1}M_2 + \alpha_{3,2}M_1 + \alpha_{3,3}M)\binom{0}{1} = \binom{0}{0}$ set $m := 3$

If $S_4\binom{1}{0} + (-\alpha_{4,1}S_3 + \alpha_{4,2}S_2 - \alpha_{4,3}S_1 + \alpha_{4,4}S)\binom{0}{1} = \binom{2\tau(\tau-1)(\tau-2)(\tau-3)}{-\alpha_{4,1}\tau(\tau-1)(\tau-2)+\alpha_{4,2}\tau(\tau-1)-\alpha_{4,3}\tau+\alpha_{4,4}}$
&
$M_4\binom{1}{0} + (\alpha_{4,1}M_3 + \alpha_{4,2}M_2 + \alpha_{4,3}M_1 + \alpha_{4,4}M)\binom{0}{1} = \binom{0}{0}$ set $m := 4$

6. Output the reproduction degree m.

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