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INTEGERS IN A RATIONAL SEQUENCE

Abstract. We discuss a peculiar rational sequence whose terms, in very specific cases, are natural numbers. We also discuss the possibility of having prime numbers appearing in the sequence.

1. Introduction

In this paper we analyze the rational sequence

$$a_d(n) = \frac{d^n + 1}{2^n} \qquad d \in \mathbb{N}, d > 1 \tag{1}$$

and we determine the conditions that ensure that $a_d(n)$ is an integer, or even a prime number.

The sequence (1) can contain integers only if d is odd and, in this case, we prove that the number of integers in the sequence is finite. Then, we proceed to study whether, for a fixed d, the sequence $\big(a_d(n)\big)$ contains prime numbers, and we identify a few conditions to ensure it.

2. Integers in the rational sequence

To prove the main theorems we need the lifting-the-exponent (LTE) lemma, which provides a formula for computing the p-adic valuation ν_p of special forms of integers.

LEMMA. (Lifting-the-exponent lemma). For any integers x, y and positive integers n, p, where n is odd and p is a prime such that $p \nmid x$ and $p \nmid y$, $(n, p) = 1$ and $p \mid x + y$, then $\nu_p(x^n + y^n) = \nu_p(x + y)$.

The lemma, often used in olympic problem solving contests [3], belongs to the folklore and it is typically attributed to Lucas [4] and Carmichael [1]. It is related to Hensel's lemma [2, 5], and it is very useful in several applications, see Sanna [6].

It is straightforward to prove that

THEOREM 1. *If $d \in \mathbb{N}$ is even, then $\big(a_d(n)\big)$ does not contain natural numbers.*

Proof. If $d \in \mathbb{N}$ is an even number, then the numerator in (1) is odd, for all n. □

The situation is more interesting and more intricate if we consider an odd d.

THEOREM 2. *If $d \in \mathbb{N}$ is odd, then $a_d(n) \notin \mathbb{N}$, $\forall n$ even.*

Proof. $d \in \mathbb{N}$ odd implies either $d \equiv 1 \pmod 4$ or $d \equiv 3 \pmod 4$. Then we have

$$d^2 \equiv 1 \pmod 4$$

and thus

$$d^{2k} \equiv 1 \pmod 4,\ \forall k \in \mathbb{N},$$

which is equivalent to

$$d^n \equiv 1 \pmod 4,\ \forall n \in \mathbb{N} \text{ even}$$

and therefore

$$d^n + 1 \equiv 2 \pmod 4,\ \forall n \in \mathbb{N} \text{ even}.$$

The number 4 does not divide $d^n + 1$, which implies

$$\frac{d^n + 1}{2^n} \notin \mathbb{N},\ \forall n \text{ even}$$

□

THEOREM 3. *For every odd $d \in \mathbb{N}$, the sequence $\left(a_d(n)\right)$ has at most a finite number of terms that belong to $\mathbb{N}$. Moreover, the sequences $\left(a_d(n)\right)$ that contain the maximum number of natural terms are those with $d = 2^k - 1$.*

Proof. Using the LTE lemma with $p = 2$, $x = d$ and $y = 1$, we can prove that

$$\frac{d^n + 1}{2^n} \in \mathbb{N} \quad \Longleftrightarrow \quad \frac{d + 1}{2^n} \in \mathbb{N}$$

which implies

$$d + 1 = 2^n m \tag{2}$$

for some $m \in \mathbb{N}$. Fixing the value of d, equation (2) has a finite number of solutions and this number is maximum if the integer d has the form $d = 2^k - 1$. □

THEOREM 4. *For every odd $d \in \mathbb{N}$, the integer terms of $\left(a_d(n)\right)$ are only among those with $n \leq \log_2(d+1)$.*

Proof. Equation (2) states a condition:

$$d + 1 = 2^n m\,, \qquad m \in \mathbb{N},\ d \text{ odd}.$$

Thus we have

$$d + 1 = 2^n m \geq 2^n$$

and so

$$\log_2(d+1) \geq n.$$

□

COROLLARY 1. *Let $d = 2^k - 1$.*
If k is odd then the only integer terms in the sequence are $a_1,\ a_3,\ a_5,\ \ldots,\ a_k$.
If k is even then the only integer terms in the sequence are $a_1,\ a_3,\ a_5,\ \ldots,\ a_{k-1}$.

In general, the problem regarding how many integer terms are present in the rational sequence (1) depends on the p-adic valuation of $d+1$.

THEOREM 5. *Given d such that $\nu_2(d+1) = 1$, that is there exists an odd integer H such that $d+1 = 2H$, then $a_1 \in \mathbb{N}$ and $\left(a_d(n)\right)_{n\geq 2}$ does not contain integers.*

Proof. If $d+1 = 2H$, then equation (2) becomes

$$\begin{aligned} d+1 &= 2^n m \\ 2H &= 2^n m \end{aligned} \tag{3}$$

The only solution to equation (3) is $n = 1$ and $m = H$. □

$d = 2^k - 1$ and $\nu_2(d+1) = 1$ are the extreme cases. In the remaining ones, the number of integers in the sequence $\left(a_d(n)\right)$ assumes values in-between.

3. Prime numbers in the rational sequence

We now wonder whether, for a fixed d, the sequence $\left(a_d(n)\right)$ not only contains integers, but some of them are prime numbers. The answer to this question is not easy, but we can make some observations.

By inspection of a few terms, it is easy to identify that $d = 7$ brings to $a_3 = 43$, which is a prime number. In the case $d = 31$, there are only two integers in the sequence, $a_1 = 16$ and $a_3 = 3724$, none of which is prime. This is sufficient to state that the sequence $\left(a_d(n)\right)$ may contains prime numbers, but it doesn't always occur.

More precisely, we can state the following theorems.

THEOREM 6. *The term a_1 is prime if and only if $d = 2p-1$, where p is a prime number.*

Proof. The proof follows immediately from the definition of the sequence in equation (1). □

THEOREM 7. *$\left(a_d(n)\right)$ may contain primes for $n \geq 2$ only if $d = 2^k - 1$ and k is an odd integer. In this case only the term a_k may be a prime.*

Proof. From Theorem 1 we know that only terms with odd n of the sequence $\left(a_d(n)\right)$ may be integer. From the LTE lemma it follows that

$$\frac{d+1}{2^n} \text{ divides } \frac{d^n+1}{2^n} = a_d(n) \tag{4}$$

for every odd n. If $a_d(n)$ is prime and $d > 1$, this implies that

$$\frac{d+1}{2^n} = 1$$

and then

$$d = 2^n - 1.$$

This prove the first part of the Theorem.

Furthermore from Equation 4 it follows that if

$$\frac{d+1}{2^n} > 1 \quad \Longleftrightarrow \quad n < \log_2(d+1)$$

then $a_d(n)$ is not prime.

It is known from Theorem 4 that $a_d(n)$ is integer if $n \leq \log_2(d+1)$ and then none of the integer terms in the sequence can be a prime number, except for a_k; where k is such that $d = 2^k - 1$. □

Theorem 7 states that a_k may be prime only for $d = 2^k - 1$, with odd k but, unfortunately, this does not always occur.

For example for $d = 7$ ($k = 3$, $d = 2^k - 1$), we have a_1 and a_3 integers, and $a_3 = 43$ is prime. On the other hand, for $d = 31$ ($k = 5$, $d = 2^k - 1$) we have a_1, a_3 and a_5 integers, but $a_5 = 894661$ is not prime (it is divisible by 41).

To analyze the primality of the values of all the sequences, as d varies in the odd naturals, it is convenient to set $d = 2^{2k+1} - 1$, $k \geq 1$, and consider the sequence

$$c_k = \frac{\left(2^{2k+1} - 1\right)^{2k+1} + 1}{2^{2k+1}}. \tag{5}$$

Thanks to the above theorems, it is possible to prove that this new sequence is an integer sequence and contains all the possible prime numbers belonging to all the sequences under study.

By calculations, we can verify that the prime numbers that we are looking for are very few. More precisely, we have verified that there exist only two primes in all the sequences with $d \leq 10^{100}$: they belong to the sequences with $d = 2^3 - 1 = 7$ and with $d = 2^{43} - 1 = 8796093022207$, respectively.

Our conjecture is the following.

CONJECTURE. There are only finitely many values of d such that the corresponding sequence $\left(a_d(n)\right)$ contains prime numbers.

We currently have only computational evidence and heuristic arguments for this conjecture. We look forward to achieving further results in a future work.

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