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THE SCROLLAR INVARIANTS OF K -GONAL CURVES
HAVING A NODAL MODEL ON A SMOOTH QUADRIC AND
NODES ON FEW LINES

Abstract. We determine the scrollar invariants of the normalization C of a nodal curve Γ of type (k, a) on a smooth quadric $\mathbb{P}^1 \times \mathbb{P}^1$ associated to the g_k^1 defined by the pencil of lines of type $(0, 1)$ in case all nodes are contained in at most $k - 1$ lines of type $(1, 0)$. This result is very much related to the content of a paper by E. Ballico from 2002 [1], but our proof follows directly from an easy lemma not mentioned there. In particular the main theorem in [1] is a consequence of that lemma, and our approach shortens the proof considerably.

1. Introduction

Let C be a smooth irreducible complete curve of genus g defined over an algebraically closed field K and having a complete base point free linear system g_k^1 (we fix such g_k^1 on C from now on). The linear system g_k^1 has scrollar invariants $0 \leq e_1 \leq e_2 \leq \dots \leq e_{k-1}$: the integer $e_i + 2$ is the smallest integer n such that $h^0(C, ng_k^1) - h^0(C, (n-1)g_k^1) > i$. The Riemann-Roch Theorem implies that this inequality is equivalent to $h^0(C, \omega_C - (n-1)g_k^1) - h^0(C, \omega_C - ng_k^1) < k - i$. Those scrollar invariants satisfy the equality $e_1 + \dots + e_{k-1} = g - k + 1$.

Let $M = \mathbb{P}^1 \times \mathbb{P}^1$ and let Γ be an irreducible divisor of type (k, a) on M (here $k, a > 0$). Let C be the normalization of Γ . The pencil of lines of type $(0, 1)$ induces a base point free linear system g_k^1 on C . In our main result (Theorem 1) we determine the scrollar invariants of g_k^1 in case Γ is a nodal curve and all nodes are contained on at most $k - 1$ lines of the pencil of lines of type $(1, 0)$ on M .

This main result is strongly inspired by the results obtained in [1]. The main theorem in [1] is Theorem 2 in our paper. In part (i) of the proof of the main theorem in [1] our main result is obtained in the case both the choices of the lines of type $(1, 0)$ and the position of the nodes of Γ on those lines are general. Our main result implies this generality condition is not necessary. Moreover the proof of that part of the main theorem in [1] at the end consists of a reference to two remarks and part of the proof of a claim in Lemma 1 in [1]. The proof of that claim makes intensively use of the method d’Horace. It seems to me that in order to finish the arguments one needs the result of Lemma 3 in our paper. I do not see how this follows from the comments given in [1]. The proof of Lemma 3 is very elementary and easy and does not need the method d’Horace. This Lemma 3 is crucial for the proof of our main results without making use of the method d’Horace.

The main theorem in [1] is Theorem 2 in our paper. Determining which sequences $0 \leq e_1 \leq \dots \leq e_{k-1}$ do occur as scrollar invariants of smooth complete curves

of genus g is an important problem and the main theorem of [1] gives a very interesting result on this problem. Therefore it seems worthwhile to have arguments how this follows from Lemma 3 avoiding more complicated arguments as in [1]. Also in Theorem 2 the genus bound on $g(k, e)$ is a little bit better than the one obtained in [1]. It is important to mention that the idea for obtaining Theorem 2 using nodal curves on a smooth quadric is completely coming from [1].

2. Generalities

Let D be a divisor on a smooth complete variety X , then we denote by $\mathcal{O}(D)$ the associated invertible sheaf on X . It defines an element $c \in \text{Pic}(X)$. If the space of global sections $H^0(X, \mathcal{O}(D))$ is non-zero we write $|D|$ to denote the associated complete linear system. We also write $|c|$ or $|\mathcal{O}(D)|$ to denote this linear system. In case X is a curve, $\deg(D) = d$ and $\dim |c| = r$ then we say $|c|$ is a g_d^r on X . More general, a g_d^r on a curve X can be a linear subspace of dimension r of some complete linear system of degree d on X (so it need not be complete). If X is a curve and $|D|$ is a g_d^r we also write $|mg_d^r|$ to denote $|mD|$.

For any smooth complete variety X we write ω_X to denote the canonical line bundle on X . In case Ω_X is a canonical divisor (it need not be effective) and D is a divisor on X defining $c \in \text{Pic}(X)$ then we also write $|\omega_X - mD|$ (or $|\omega_X - mc|$) to denote $|\Omega_X - mD|$. In case X is a curve and $|D|$ is a g_d^r on X then we also write $|\omega_X - mg_d^r|$.

Now let $M = \mathbb{P}^1 \times \mathbb{P}^1$. There are two projections $pr_i : M \rightarrow \mathbb{P}^1$ ($i \in \{1, 2\}$). For $P \in \mathbb{P}^1$ the fiber $pr_i^{-1}(P)$ is an effective divisor on M . Referring to the classical imbedding $M \subset \mathbb{P}^3$ as a smooth quadric, we call the fiber $pr_i^{-1}(P)$ a line. The associated element of $\text{Pic}(M)$ is denoted by $(1, 0)$ in case $i = 1$ and $(0, 1)$ in case $i = 2$. It is well-known that $\text{Pic}(M) = \mathbb{Z} \cdot (1, 0) \oplus \mathbb{Z} \cdot (0, 1)$. For $a, b \in \mathbb{Z}$ we write (a, b) to denote $a \cdot (1, 0) + b \cdot (0, 1)$. We also write $\mathcal{O}(a, b)$ to denote an invertible sheaf associated to $(a, b) \in \text{Pic}(M)$. The intersection number $(a, b) \cdot (a', b')$ is given by $ab' + ba'$. One has $|(a, b)| \neq \emptyset$ if and only if $a \geq 0$ and $b \geq 0$ and in that case $\dim |(a, b)| = (a+1)(b+1) - 1 = ab + a + b$ and an element of $|(a, b)|$ is called a curve of type (a, b) . One has $\omega_M = \mathcal{O}(-2, -2)$. If γ is a curve on M and $P \in \gamma$ then $T_P(\gamma)$ is the tangent space of γ at P .

An important ingredient in the proofs of this paper is the following well-known lemma.

LEMMA 1. *If $a, b \geq 0$ then $H^1(\mathcal{O}(a, b)) = 0$.*

Proof. If one uses [4], Chapter III, Exercise 5.6 (a)(2) then it follows from Serre duality (especially [4], Chapter III, Corollary 7.7).

It can also be proved directly by means of induction starting with $H^1(\mathcal{O}_M) = 0$ (M is a rational surface) and using some exact cohomology sequences.

A short argument is as follows. By explicitly writing down above the two standard affine open subsets \mathbb{A}^1 of \mathbb{P}^1 (one omitting 0 and one omitting ∞) one finds

$$(pr_2)_*(\mathcal{O}(a, b)) = \mathcal{O}(b) \oplus \dots \oplus \mathcal{O}(b) \quad (a \text{ terms})$$

(here $\mathcal{O}(b)$ is an invertible sheaf of degree b on \mathbb{P}^1). From [4], Chapter V, Lemma 2.4 one knows $H^1(\mathcal{O}(a,b)) = H^1((pr_2)_*(\mathcal{O}(a,b)))$. On \mathbb{P}^1 one has $H^1(\mathcal{O}(b)) = H^0(\mathcal{O}(-2-b)) = 0$ for $b \geq 0$. \square

Let Z be a 0-dimensional subscheme of M . The set of divisors of $|(a,b)|$ containing Z is a linear subspace of $|(a,b)|$ denoted by $|(a,b) - Z|$. In case Z_1 and Z_2 are two disjoint 0-dimensional subschemes of M we write $Z_1 + Z_2$ to denote their union.

Now, let Γ be an irreducible nodal curve of type (a,b) on M having nodes at P_1, \dots, P_m and let $n : C \rightarrow \Gamma$ be the normalization of Γ . Write $n^{-1}(P_i) = \{P_{i1}, P_{i2}\}$.

LEMMA 2. *In the situation described above, with the same notations, we have*

$$|\omega_C| = \{n^{-1}(D \cdot \Gamma) - \sum_{i=1}^m (P_{i1} + P_{i2}) : D \in |(a-2, b-2) - P_1 - \dots - P_m|\}.$$

In particular

$$\dim(|\omega_C|) = \dim(|(a-2, b-2) - P_1 - \dots - P_m|).$$

Proof. See the first part of Remark 2 in [1]. It also follows from Lemma 2 in [3].

Let $\pi : Y \rightarrow M$ be the blowing-up of M at P_1, \dots, P_m . Let $E_i = \pi^{-1}(P_i)$ be the exceptional divisor above P_i for $1 \leq i \leq m$. The curve C is isomorphic to the proper transform of Γ on Y , hence it belongs to the linear system $|\pi^*(a,b) - 2E_1 - \dots - 2E_m|$. The canonical sheaf on Y is equal to $\omega_Y = \pi^*(\mathcal{O}_M(-2, -2)) \otimes \mathcal{O}_Y(E_1) \otimes \dots \otimes \mathcal{O}_Y(E_m)$. It is proved in Lemma 2 of [3] that restriction to C of the linear system $|\omega_Y(C)| = |\pi^*(a-2, b-2) - E_1 - \dots - E_m|$ is a bijection on $|\omega_C|$. The curves belonging to $|\pi^*(a-2, b-2) - E_1 - \dots - E_m|$ are exactly the curves $n^{-1}(D \cdot \Gamma) - E_1 - \dots - E_m$ with $D \in |(a-2, b-2) - P_1 - \dots - P_m|$. \square

3. Proofs

All results in this paper follow from the next elementary and easy to prove lemma; it also replaces the intensive use of the method d'Horace in the proof of [1]'s main theorem.

LEMMA 3. *Fix $a \in \mathbb{Z}_{\geq 0}$. On $M = \mathbb{P}^1 \times \mathbb{P}^1$ choose $k+1$ different lines L_1, \dots, L_{k+1} of type $(1,0)$ and for $1 \leq i \leq k+1$ choose an effective divisor D_i of degree y_i on L_i with $a \geq y_1 \geq y_2 \geq \dots \geq y_{k+1} \geq 0$ (so some divisors D_i are allowed to be 0). Then*

$$\dim(|(k,a) - D_1 - \dots - D_{k+1}|) = ka + k + a - y_1 - \dots - y_{k+1}.$$

Proof. In case $a = 0$ there is nothing to prove, so we assume $a \geq 1$.

First assume $k = 0$. One has $\dim(|(0,a)|) = a$ and each element of $|(0,a)|$ is the sum of a lines of type $(0,1)$. The fact that this sum needs to contain the divisor D_1 on L_1 imposes y_1 independent conditions on $|(0,a)|$. This proves the desired dimension claim in case $k = 0$.

Take $k \geq 1$ and assume the dimension claim holds for all smaller values of k . Let $k' = \max\{i : y_i \neq 0\}$ (hence $0 \leq k' \leq k+1$ with $k' = 0$ corresponding to $y_1 = 0$). In case $k' = 0$ there is nothing to prove, so we can assume $k' \geq 1$.

First assume $k' = 1$ and consider the exact sequence

$$0 \rightarrow \mathcal{O}_M(k-1, a) \rightarrow \mathcal{O}_M(k, a) \rightarrow \mathcal{O}_{L_1}(a) \rightarrow 0.$$

From $h^1(\mathcal{O}_M(k-1, a)) = 0$ (see Lemma 1) we find $|(k, a)|$ induces the complete linear system $|\mathcal{O}_{L_1}(a)|$ of degree a on $L_1 \cong \mathbb{P}^1$. We conclude again that the condition $D_1 \subset \Gamma$ for $\Gamma \in |(k, a)|$ induces y_1 independent linear conditions on $|(k, a)|$, again proving the desired dimension claim in this case.

We are also going to use induction on k' . So assume $k' \geq 2$ and the dimension claim holds for smaller values of k' (for the linear system $|(k, a)|$). Fix one more line L_{k+2} of type $(1, 0)$ and use the lines $L_2, L_3, \dots, L_{k+1}, L_{k+2}$ and the divisor $D_{k+2} = 0$ on L_{k+2} . So we use $y_2 \geq y_3 \geq \dots \geq y_{k+1} \geq y_{k+2} = 0$. For this situation the value of k' drops by one and from the inductive hypothesis we obtain

$$\dim(|(k, a)(D_2 - \dots - D_{k+1})|) = ak + a + k - y_2 - \dots - y_{k+1}.$$

Assume $|(k, a)(-D_2 - \dots - D_{k+1})|$ does not induce the complete linear system $|\mathcal{O}_{L_1}(a)|$ on L_1 . Then the condition $L_1 \subset \Gamma$ for $\Gamma \in |(k, a)(-D_2 - \dots - D_{k+1})|$ induces at most a independent linear conditions on $|(k, a)(-D_2 - \dots - D_{k+1})|$. Such elements are the sum of L_1 and a divisor in $|(k-1, a)(-D_2 - \dots - D_{k+1})|$, so it implies

$$\dim(|(k-1, a)(-D_2 - \dots - D_{k+1})|) = a(k-1) + a + (k-1) - y_2 - \dots - y_{k+1} + 1.$$

This contradicts the inductive hypothesis on k . Therefore $|(k, a)(-D_2 - \dots - D_{k+1})|$ induces the complete linear system $|\mathcal{O}_{L_1}(a)|$ on L and since $y_1 \leq a$ it implies

$$\dim(|(k, a)(-D_1 - \dots - D_{k+1})|) = ak + a + k - y_1 - \dots - y_{k+1}.$$

□

Now let L_1, \dots, L_{k-1} be different lines of type $(1, 0)$ on M and for $1 \leq i \leq k-1$ let D_i be a reduced divisor of degree y_i on L_i with $a \geq y_1 \geq \dots \geq y_{k-1} \geq 0$ (a being a fixed element of $\mathbb{Z}_{\geq 1}$). Let Γ be a nodal curve of type (k, a) on M having its set of nodes equal to $D_1 + \dots + D_{k-1}$. Let $n : C \rightarrow \Gamma$ be the normalization of Γ . The pencil of lines $|(0, 1)|$ induces a g_k^1 on C . In the next theorem we determine the scollar invariants of this g_k^1 .

As mentioned in the introduction this result is also mentioned (under some generality assumptions) in [1] (part (i) of the proof of the main theorem). However besides some indications with no details, there is no argument given in [1]. We give a complete proof only referring to our Lemma 3.

THEOREM 1. *Let C and g_k^1 be as described above. The scollar invariants e_i of g_k^1 are given by $e_i = a - y_i - 2$.*

Proof. Define $t \geq 1$ and $s_0 = 1 < s_1 < \dots < s_t = k$ such that $y_{s_0} > y_{s_1} > \dots > y_{s_{t-1}}$ with $\{y_1, \dots, y_{k-1}\} = \{y_{s_0}, y_{s_1}, \dots, y_{s_{t-1}}\}$ (so for $1 \leq i \leq t$ exactly $s_i - s_{i-1}$ integers y_i are equal to $y_{s_{i-1}}$). So we need to prove $e_1 = \dots = e_{s_1-1} = a - y_1 - 2$ (in case $t = 1$ this describes all scollar invariants); for $1 \leq l \leq t$ and $s_{l-1} \leq i < s_l$ one has $e_i = a - y_{s_{l-1}} - 2$. We are going to compute the function $f(n) = h^0(C, \omega_C - (n-1)g_k^1) - h^0(C, \omega_C - ng_k^1)$ for $n \in \mathbb{Z}_{\geq 1}$.

In case $m \leq a - 2 - y_1$, hence $a - 2 - m \geq y_1$, we have because of Lemma 3

$$\begin{aligned} \dim |\omega_C - mg_k^1| &= \dim |(k-2, a-2-m) - D_1 - \dots - D_{k-1}| = \\ &= (k-2)(a-2-m) + (k-2) + (a-2-m) - y_1 - \dots - y_{k-1}. \end{aligned}$$

This implies $f(m) = k - 1$.

Take $1 \leq l \leq t$ and $a - 2 - y_{s_{l-1}} < m \leq a - 2 - y_{s_l}$ (we take $y_{s_l} = 0$), hence $y_{s_l} \leq a - 2 - m < y_{s_{l-1}}$. For $1 \leq i \leq s_l - 1$ and $\Gamma \in |(k-2, a-2-m)(-D_1 - \dots - D_{k-1})|$ we need $\Gamma \cap L_i$ contains D_i . However $(k-2, a-2-m).L_i = a - 2 - m < y_{s_{l-1}} \leq y_i$ therefore Γ has to contain L_i . This implies

$$\begin{aligned} |(k-2, a-2-m) - D_1 - \dots - D_{k-1}| &= \\ &= |(k-1-s_l, a-2-m) - D_{s_{l-1}} - \dots - D_{k-1}| + L_1 + \dots + L_{s_{l-1}}. \end{aligned}$$

Since $\dim(|\omega_C - mg_k^1|) = \dim(|(k-2, a-2-m) - D_1 - \dots - D_{k-1}|)$ it follows

$$\dim(|\omega_C - mg_k^1|) = \dim(|(k-1-s_l, a-2-m) - D_{s_{l-1}} - \dots - D_{k-1}|).$$

Since $a - 2 - m \geq y_{s_l}$ we can use Lemma 3 and conclude

$$\dim |\omega_C - mg_k^1| = (k-1-s_l)(a-2-m) + (k-1-s_l) + (a-2-m) - y_{s_l} - \dots - y_{k-1}.$$

This implies

$$\begin{aligned} f(a-1-y_{s_{l-1}}) &= \dim(|\omega_C - (a-2-y_{s_{l-1}})g_k^1|) - \dim(|\omega_C - (a-1-y_{s_{l-1}})g_k^1|) = \\ &= (s_l - s_{l-1})y_{s_{l-1}} + (k - s_{l-1}) - (s_l - s_{l-1})y_{s_{l-1}} = k - s_{l-1}. \end{aligned}$$

For $a - y_{s_{l-1}} \leq m \leq a - 2 - y_{s_l}$ (this case does not occur if $y_{s_l} = y_{s_{l-1}} - 1$) one has $f(m) = k - s_l$. In particular we find $f(m) = 0$ for $m \geq a - y_{s_{l-1}}$.

As mentioned in the introduction, $e_i + 2$ is the minimal integer n such that $f(n) < k - i$. Since $f(n) = k - 1$ if and only if $n \leq a - 1 - y_1$ and, in case $t > 1$, $f(n) = k - s_1$ if and only if $a - y_1 \leq n \leq a - 1 - y_{s_1}$ we obtain $e_1 = \dots = e_{s_1-1} = a - y_1 - 2$. In case $t = 1$ we obtain $e_1 = \dots = e_{k-1} = a - y_1 - 1$. For $1 \leq l \leq t - 1$ we have $f(a - 1 - y_{s_{l-1}}) = k - s_{l-1}$ while $f(a - y_{s_{l-1}}) = k - s_l$. This implies $e_i = a - y_{s_{l-1}} - 2$ for $s_{l-1} \leq i < s_l$. We obtain $f(n) = 0$ if and only if $n \geq a - y_{s_{l-1}}$ while $f(a - y_{s_{l-1}} - 1) = k - s_{l-1}$. This implies $e_i = a - y_{s_{l-1}} - 2$ for $s_{l-1} \leq i \leq k - 1$. \square

In the next proposition we prove the existence of nodal curves on $M = \mathbb{P}^1 \times \mathbb{P}^1$ having a base point free pencil g_k^1 cut out by the pencil of lines $|(0, 1)|$ on M and such that all nodes are contained in the union of at most $k - 1$ lines of type $(1, 0)$ on M . In [1] this is proved in Lemma 1 and its proof is the main part of that paper. It is rather involved and it makes intensive use of the method d'Horace. As already mentioned in the introduction, we give a proof that only uses Lemma 3 and theorems of Bertini.

PROPOSITION 1. *On $M = \mathbb{P}^1 \times \mathbb{P}^1$ let L_1, \dots, L_{k-1} be different lines of type $(1, 0)$ and for each $1 \leq i \leq k - 1$ choose an effective reduced divisor D_i of degree y_i on L_i with $y_1 \geq \dots \geq y_{k-1} \geq 0$. Assume those divisors are taken such that there is no line of type $(0, 1)$ containing at least two points of $D_1 + \dots + D_{k-1}$. Let a be an integer with $a \geq (k - 1)y_1$ in case $\text{char}(K) = 0$ and $a \geq (k - 1)y_1 + 1$ in case $\text{char}(K) = p > 0$. There exists an irreducible nodal curve $\Gamma \in |(k, a)|$ having $D_1 + \dots + D_{k-1}$ as its set of nodes.*

Proof. Let \mathcal{L} be the linear system $|(k - 1, y_1) - D_1 - \dots - D_{k-1}|$ in case $\text{char}(K) = 0$ and $|(k - 1, y_1 + 1) - D_1 - \dots - D_{k-1}|$ in case $\text{char}(K) = p > 0$. As a first step we are going to prove that a general element $\gamma \in \mathcal{L}$ is smooth.

In case $\text{char}(K) = 0$, for $1 \leq i \leq k - 1$ the restriction of \mathcal{L} to L_i induces a linear subsystem of $D_i + |\mathcal{O}_{L_i}(y_1 - y_i)|$ (in case $y_1 = y_i$ this linear system has dimension 0). Applying Lemma 3 to the linear system $|(k - 2, y_1) - D_1 - \dots - \widehat{D}_i - \dots - D_{k-1}|$ (hat means omitted) we obtain equality. This shows a general element γ of \mathcal{L} intersects each line L_i ($1 \leq i \leq k - 1$) transversally (in particular γ is smooth at each point of D_i) and \mathcal{L} has no fixed point on L_i outside of D_i .

Take any line L of type $(1, 0)$ different from each line L_i ($1 \leq i \leq k - 1$). Applying Lemma 3 to the linear system $|(k - 2, y_1) - D_1 - \dots - D_{k-1}|$ we find \mathcal{L} induces a complete linear system by restriction to L . This implies \mathcal{L} has no base points outside of $L_1 \cup \dots \cup L_{k-1}$.

Because \mathcal{L} has no base points outside of $D_1 + \dots + D_{k-1}$ and a general element $\gamma \in \mathcal{L}$ is smooth at P belonging to $D_1 + \dots + D_{k-1}$, it follows from Bertini's Theorem (see e.g. [5], Theorem 4.1) that a general element $\gamma \in \mathcal{L}$ is smooth.

In case $\text{char}(K) = p > 0$, for $1 \leq i \leq k - 1$ the restriction \mathcal{L} to L_i induces a linear subsystem of $D_i + |\mathcal{O}_{L_i}(y_1 - y_i + 1)|$. Again from Lemma 3 we obtain equality and therefore \mathcal{L} has no base points on $L_i \setminus D_i$. In this case $y_1 - y_i + 1 > 0$, therefore for all $P \in L_i \setminus D_i$ there exists $\gamma \in \mathcal{L}$ with $P \in \gamma$ and $T_P(\gamma) \neq L_i$. Also $\gamma \in \mathcal{L}$ general is smooth at each point of D_i .

Moreover using any line L of type $(1, 0)$ different from each line L_i ($1 \leq i \leq k - 1$), again we find \mathcal{L} has no base points outside of $L_1 \cup \dots \cup L_{k-1}$. Also for each $P \in M \setminus (L_1 \cup \dots \cup L_{k-1})$, using the line L of type $(1, 0)$ containing P we find that there exists $\gamma \in \mathcal{L}$ smooth at P .

We are going to use Theorem 1 in [2] which is a weaker statement than Bertini's Theorem in the case $\text{char}(K) = 0$. First we need some notation. We use the quasi-projective variety $X = M \setminus \{D_1 \cup \dots \cup D_{k-1}\}$ and for $P \in X$ let $T_{P, \mathcal{L}}$ be the intersection

of all tangent spaces $T_p(\gamma)$ with $\gamma \in \mathcal{L}$ containing P (of course γ is restricted to X). Let $e_{\mathcal{L}}(P) = \dim(T_p(\mathcal{L}))$ and for $0 \leq f \leq 2$ let $X_f = \{P \in X : e_{\mathcal{L}}(P) \geq f\}$. From Theorem 1 in [2] we need to prove that for $0 \leq f \leq 2$ each subvariety Z of X_f satisfies $\dim(Z) \leq 2 - f$.

For $P \in X \cap L_i$ for some $1 \leq i \leq k-1$ we know there exists $\gamma \in \mathcal{L}$ smooth at P with $T_p(\gamma) \neq L_i$. But $\gamma' = L_1 \cup \dots \cup L_{k-1} \cup (y_1 + 1 \text{ general lines of type } (0, 1)) \in \mathcal{L}$ and $P \in \gamma'$ with $T_p(\gamma') = L_i$. This proves $T_{p,\mathcal{L}} = \{0\}$, hence $P \in X_0$. Since $e_{\mathcal{L}}(P) = 0$ is an open condition for $P \in X$ it implies $\dim(X_1) \leq 1$. We also proved that for $P \in X \setminus (L_1 \cup \dots \cup L_{k-1})$ a general $\gamma \in \mathcal{L}$ containing P is smooth at P , hence $X_2 = \emptyset$. So Theorem 1 in [2] implies a general element $\gamma \in \mathcal{L}$ is smooth on X , hence also on M . This finishes the first part of the proof.

Let \mathcal{L}_s be the linear subsystem of $|(k, a)|$ of curves singular at each point of $D_1 + \dots + D_{k-1}$. As a next step we prove a general element $\Gamma \in \mathcal{L}_s$ has ordinary nodes at $D_1 + \dots + D_{k-1}$. First we introduce some terminology used in the sequel of the proof.

A divisor in $|(k, a)|$ is called of type F_i ($1 \leq i \leq k-1$) if it is the sum of $\gamma \in \mathcal{L}$, the line L_i , all lines of type $(0, 1)$ through some point of $D_1 + \dots + \widehat{D}_i + \dots + D_{k-1}$ (called the fixed lines of type $(0, 1)$) and $a - (y_1 + \dots + \widehat{y}_i + \dots + y_{k-1} + y_1)$ more lines of type $(0, 1)$ (called the free lines of type $(0, 1)$). Each divisor of type F_i is singular at each point of $D_1 + \dots + D_{k-1}$. Choosing γ general and also choosing the free lines of type $(0, 1)$ general we conclude that a general divisor of type F_i has an ordinary node at each point of D_i . This proves that a general $\gamma \in \mathcal{L}_s$ has ordinary nodes at $D_1 + \dots + D_{k-1}$.

Next we are going to prove that a general element $\Gamma \in \mathcal{L}_s$ is smooth outside of $D_1 + \dots + D_{k-1}$. In case $\text{char}(K) = 0$, from Bertini's theorem, it is enough to prove that \mathcal{L}_s has no fixed points outside of $D_1 + \dots + D_{k-1}$.

Let $P \in L_i \setminus D_i$ for some $1 \leq i \leq k-1$. We use a divisor of type F_j for some $j \neq i$. In case P belongs to some line of type $(0, 1)$ through some point $Q \in D_1 + \dots + D_{k-1}$ then we take j such that $Q \in D_j$, otherwise we choose $j \neq i$ arbitrary. We use $\gamma \in \mathcal{L}$ such that $P \notin \gamma$ and we also take the free lines of type $(0, 1)$ not containing P . We obtain $\Gamma \in \mathcal{L}_s$ with $P \notin \Gamma$.

In case $P \notin L_1 \cup \dots \cup L_{k-1}$ we use a divisor of type F_i for some $1 \leq i \leq k-1$. In case the line of type $(0, 1)$ through P contains a point $Q \in D_1 + \dots + D_{k-1}$ we take i such that $Q \in D_i$, otherwise we take i arbitrarily. We choose $\gamma \in \mathcal{L}$ such that $P \notin \gamma$ and we take the free lines of type $(0, 1)$ not containing P . Again we obtain $\Gamma \in \mathcal{L}_s$ with $P \notin \Gamma$.

In case $\text{char}(K) > 0$, in order to use Theorem 1 in [2] as in the first step of the proof, we also need to show that in case $P \in L_i \setminus D_i$ for some $1 \leq i \leq k-1$ then there exists $\Gamma \in \mathcal{L}_s$ with Γ smooth at P and $T_p(\Gamma) \neq L_i$ and in case $P \notin L_1 \cup \dots \cup L_{k-1}$ there exists $\Gamma \in \mathcal{L}_s$ with P a smooth point of Γ . Such $\Gamma \in \mathcal{L}_s$ can be obtained by using divisors of type F_j as in the case of $\text{char}(K) = 0$ except that we take γ with $P \in \gamma$ as done in the first part of the proof.

Finally we need to show that a general element Γ of \mathcal{L}_s is irreducible. The divisors of type F_i already show \mathcal{L}_s does not have a fixed component. If a general element of \mathcal{L}_s would not be irreducible then in case $\text{char}(K) = 0$ another theorem of

Bertini implies \mathcal{L}_s is composed of a pencil (it need not be a linear pencil) (see e.g. [5] Theorem 5.3). The divisors of type F_i ($1 \leq i \leq k-1$) show this is not the case. In case $\text{char}(K) = p > 0$ one also has the possibility that the general element of \mathcal{L}_s is of the form $p^e U$ for some $e \geq 1$ and U an irreducible divisor (see [6], Section 1). In that case all divisors are of type $p^e U$ and the divisors of type F_i show this is not the case. \square

For the sake of completeness, we give the argument giving rise to the main theorem of [1] obtaining a slightly better bound. The arguments are those from [1], part (ii) of the proof of the Main Theorem.

THEOREM 2. *Let $e \in \mathbb{Z}_{\geq 1}$. In case $\text{char}(K) = 0$ let $A(e) = (k-1)((k-1)e-2)$ and in case $\text{char}(K) = p > 0$ let $A(e) = (k-1)((k-1)e-1)$. For each sequence $0 \leq e_1 \leq \dots \leq e_{k-1} = e_1 + e$ such that $g = k-1 + e_1 + \dots + e_{k-1} > A(e)$ there exists a smooth k -gonal curve C of genus g such that a linear system g_k^1 on C has scollar invariants (e_1, \dots, e_{k-1}) .*

Proof. Fix a sequence of integers $e = E_{k-1} \geq E_{k-2} \geq \dots \geq E_2 \geq E_1 = 0$. On $M = \mathbb{P}^1 \times \mathbb{P}^1$ choose $k-1$ different lines L_1, \dots, L_{k-1} of type $(1, 0)$ and choose an effective reduced divisor D_i on L_i of degree E_i such that no line of type $(0, 1)$ contains more than one point of $D_1 \cup \dots \cup D_{k-1}$. In case $\text{char}(K) = 0$ let $A = (k-1)e$ and in case $\text{char}(K) = p > 0$ let $A = (k-1)e + 1$. From Proposition 1 we know that for $a \geq A$ there exists an irreducible nodal curve in $|k, a|$ having its nodes exactly at $D_1 \cup \dots \cup D_{k-1}$. Let C be its normalization and let g_k^1 be the linear system on C induced by the pencil $|k, a|$ on M . From Theorem 1 we know the scollar invariants of g_k^1 are $a-2-e, a-2-E_{k-2}, \dots, a-2-E_2, a-2$.

In this way we obtain all scollar invariants e_1, e_2, \dots, e_{k-1} for a g_k^1 satisfying $e_{k-1} - e_{k-i} = E_i$ for $2 \leq i \leq k-1$ as soon as $e_{k-1} \geq A-2$. Varying E_{k-2}, \dots, E_2 we obtain all scollar invariants for a g_k^1 satisfying $e_{k-1} - e_1 = e$ as soon as $e_{k-1} \geq A-2$.

Now take $g > (k-1)(A-2)$ and $e_1 \leq \dots \leq e_{k-1}$ with $e_1 + \dots + e_{k-1} = g - k + 1$ and $e_{k-1} - e_1 = e$. In case $e_{k-1} \leq A-3$ we would obtain $g \leq (k-1)(A-2)$, a contradiction. Hence $e_{k-1} \geq A-2$ and we found the existence of a smooth curve C of genus g having a g_k^1 with scollar invariants e_1, \dots, e_{k-1} . \square

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