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GENERICALLY NONREDUCED COMPONENTS OF HILBERT SCHEMES ON FOURFOLDS

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Abstract. We exhibit generically nonreduced components of the Hilbert scheme of at least 21 points on a smooth variety of dimension at least four. The result was announced in [19] and answers a question [1, Problem 3.8]. The method is similar to the one of [22, §6].

1. Introduction

Let X be a smooth quasi-projective variety over a field $\Bbbk$. The Hilbert scheme of d points $\mathrm{Hilb}_d(X)$ is a moduli space of central importance, with applications to combinatorics [11,12], algebra, enumerative geometry [29], and classical algebraic geometry [3]. Many of the applications are limited to the case when $\dim X \leq 2$ as in this case $\mathrm{Hilb}_d(X)$ is smooth [9]. For a good and gentle introduction to Hilbert schemes, see [4] or [27, Chapter 18]. See also [19] for a list of open problems.

The possible singularities of $\bigsqcup_d \mathrm{Hilb}_d(X)$ for $\dim X \geq 3$ are only partially understood. A point $[Z] \in \mathrm{Hilb}_d(X)$ is smooth for every $Z \subseteq X$ which can be embedded into a smooth surface. As a very particular case, this implies that $\mathrm{Hilb}_d(X)$ is smooth for $d \leq 3$. In contrast, the Hilbert scheme $\mathrm{Hilb}_d(X)$ is singular for every $d \geq 4$ and $\dim X \geq 3$, in fact for every $x \in X$, any degree d subscheme $Z \subseteq V(\mathfrak{m}_x^2)$ gives a singular point [27, Cor 18.30].

The singularities in the case $\dim X = 3$ are constrained as the Hilbert scheme is a critical locus [6]. Understanding the singularities is a very active research area, see for example [10,23,28,30].

The singularities in the case $\dim X \geq 16$ can be almost arbitrary: the Hilbert scheme satisfies Murphy's Law up to retraction, see [18]. For important singularity types, such as nonreduced ones, sharper bounds on $\dim X$ are known. Szachniewicz [34] proved that $\mathrm{Hilb}_d(\mathbb{A}^6)$ is nonreduced for every $d \geq 13$; it has an embedded component. See [7,31] for some results in similar direction on fixed loci.

One instance where *up to retraction* cannot be ignored is when we consider generic nonreducedness. In particular, the results above do not prove that the Hilbert scheme has any generically nonreduced components. Proving that such components do exist and already in codimension four is the main aim of the current article.

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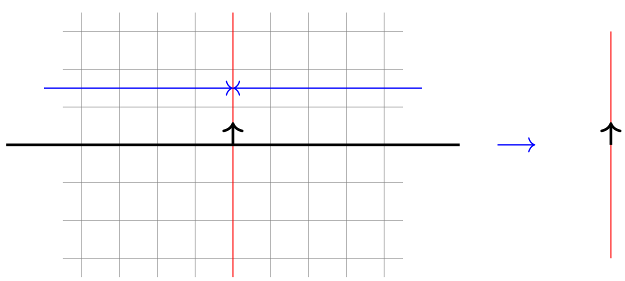
1.1. Generic nonreducedness

We work over a field of characteristic zero, in particular over a perfect field. An irreducible component of a finite type $\Bbbk$-scheme is either *generically smooth*, that is, its general point is smooth, or *generically nonreduced* which means that every point is nonreduced.

The problem is that generic nonreducedness does not propagate along retractions. For example, consider

$$\frac{\Bbbk[[y]]}{(y^2)} \hookrightarrow \frac{\Bbbk[[x,y]]}{(xy,y^2)}.$$

The source of this map is generically nonreduced, while the target is generically reduced. Geometrically speaking, the above map comes from a retraction of $V(xy,y^2) \subseteq \mathbb{A}^2$ onto $V(y^2) \subseteq \mathbb{A}^1$ by contracting the x axis:



Therefore, from [18] it does not follow that $\mathrm{Hilb}_d(\mathbb{A}^{16})$ admits generically nonreduced components. Neither it follows from subsequent paper of Szachniewicz [34]. In contrast, in the paper [22] the authors show that $\mathrm{Quot}_8(\mathbb{A}^4)$ has a generically nonreduced component.

The aim of the present note is to apply the method of Jelisiejew-Šivic to the case of the Hilbert scheme and show the following theorem, which resolves [1, Problem 3.8]. Let $\mathcal{L}_H \subseteq \mathrm{Hilb}_{\mathrm{pts}}(\mathbb{A}^n)$ denote the locus of $[Z]$ such that $Z = Spec(A)$ is an irreducible scheme corresponding to the local algebra A with Hilbert function $H_A = H$.

THEOREM 1. *Let* $\Bbbk$ *be a field of characteristic zero and let* $H = (1,4,10,s)$ *for* $s \in \{6,7,8,9\}$. *Then* $\mathcal{L}_H \subseteq \mathrm{Hilb}_{15+s}(\mathbb{A}^4)$ *is an irreducible component, and this component, with the scheme structure inherited from the Hilbert scheme, is generically nonreduced. Therefore, the Hilbert scheme* $\mathrm{Hilb}_d(\mathbb{A}^4)$ *admits generically nonreduced components for all* $d \geq 21$.

Prior to Theorem 1 it was not known whether $\mathrm{Hilb}_d(\mathbb{A}^4)$ or $\mathrm{Hilb}_d(\mathbb{A}^5)$ are *reduced* for all d. It remains an open question whether $\mathrm{Hilb}_d(\mathbb{A}^3)$ is nonreduced for d high enough and whether this scheme has generically nonreduced components, see [19, Problem XIV].

There are three main steps of the argument. First, the locus $\mathscr{L}_H$ is closed for the functions H as in theorem. Moreover, it is contained in a dominant Białynicki-Birula cell, which implies that on an open subset $U \subseteq \mathrm{Hilb}_{15+s}(\mathbb{A}^4)$ containing $\mathscr{L}_H$, the Hilbert scheme admits a retraction $\pi\colon U \to U^{\mathbb{G}_m}$ which maps any point $[Spec(S/I)] \in U$ to $Spec(S/\operatorname{in}(I))$, where $S = \Bbbk[x_1,\dots,x_4]$ and $\operatorname{in}(I)$ is the ideal of top degree forms.

Second, primary obstruction yields quadratic equations for the fibre

$$\pi^{-1}([Z]) \subseteq \left(T_{\mathrm{Hilb}_{15+s}(\mathbb{A}^4),[Z]}\right)_{<0}.$$

Third, for a chosen $[Z_0] \in \mathscr{L}_H$ we computer-check using *Macaulay2* that the quadrics alone cut out a 4-dimensional scheme in the affine space $\left(T_{\mathrm{Hilb}_{15+s}(\mathbb{A}^4),[Z_0]}\right)_{<0}$. It follows that $\dim \pi^{-1}([Z_0]) \le 4$. The fibre $\pi^{-1}([Z_0])$ is a cone and has a translation action by $\mathbb{A}^4$, so the fibre is equal to $\{Z_0 + v \mid v \in \mathbb{A}^4\}$ as a set and hence $\mathscr{L}_H$ contains an open neighbourhood of $[Z_0]$, so this locus is a component. A syzygetic argument shows that the containment $T_{\mathscr{L}_H,[Z]} \subseteq T_{\mathrm{Hilb}_{15+s}(\mathbb{A}^4),[Z]}$ is strict for every $[Z] \in \mathscr{L}_H$, hence $\mathscr{L}_H$ cannot be generically reduced.

1.2. Open questions and possible generalizations

Consider now $\mathrm{Hilb}_d(\mathbb{A}^n)$ and the unique very compressed Hilbert function $H = H_{n,d}$ given by the condition that there exists a δ such that

$$H_{n,d}(i) = \begin{cases} \binom{n+i-1}{i} = \dim \Bbbk[x_1,\dots,x_n]_i & \text{for } i < \delta \\ 0 & \text{for } i > \delta \\ d - \sum_{i=0}^{\delta-1}\binom{n+i-1}{i} & \text{for } i = \delta \end{cases}$$

The locus $\mathscr{L}_H \subseteq \mathrm{Hilb}_d(\mathbb{A}^n)$ is irreducible and closed also in this more general case. We then have three possibilities for a general $[Z] \in \mathscr{L}_H$:

SMOOTH the scheme $[Z]$ has only trivial negative tangents, so $\mathscr{L}_H$ is a component and $[Z]$ is a smooth point of $\mathrm{Hilb}_d(\mathbb{A}^n)$ on this component,

DEFORMS the scheme $[Z]$ has nontrivial negative tangents and some of them "integrate", that is, the fibre $\pi^{-1}([Z])$, which is a cone, contains more points than just $\mathbb{A}^n$. In this case $\mathscr{L}_H$ is not an irreducible component and without additional information we cannot say much about whether its points are reduced in $\mathrm{Hilb}_d(\mathbb{A}^n)$.

NONRED the scheme $[Z]$ has nontrivial negative tangents and $\pi^{-1}([Z])$ is, as a topological space, equal to $\mathbb{A}^n$. In this case $\mathscr{L}_H$ is a generically nonreduced component.

EXAMPLE 1. By [5] the case SMOOTH occurs for example for $H = (1,4,3)$. The case DEFORMS occurs for example for $H = (1,4,4)$. The case NONRED occurs for H as in Theorem 1.

We stress that above we look at a general point of $\mathscr{L}_H$. This makes a difference: for example for $H = (1,6,6)$ the case SMOOTH occurs, so $\mathscr{L}_H$ is a generically smooth component, however Szachniewicz [34] found an embedded component of $\mathrm{Hilb}_{13}(\mathbb{A}^6)$ inside $\mathscr{L}_H$. It is a completely open problem to understand whether having an embedded component is typical or exceptional for $\mathscr{L}_H$ which fall into the SMOOTH case.

One motivation to discuss the more general situation is the case $n = 3$, the Hilbert scheme of $\mathbb{A}^3$. Taking $d = 96$ and $H = H_{3,96}$ we get that $\mathscr{L}_H$ is too big to fit in the smoothable component of $\mathrm{Hilb}_{96}(\mathbb{A}^3)$, see [14]. A syzygetic argument, see Lemma 1 below, also shows that the case SMOOTH cannot hold. Moreover, it is known that the fibre $\pi^{-1}([Z])$ for a general $[Z]$ is cut out by quadrics only. Actually, this holds whenever $(T^2_{[Z]})_{<-2} = 0$, where $T^2_{[Z]} \subseteq \mathrm{Ext}^1(I_Z, \mathscr{O}_Z)$ is the Schessinger's functor, see [13, Chapter 3]. It is possible to obtain the quadrics explicitly using *Macaulay2*. However, the Gröbner basis computation necessary for determining $\dim \pi^{-1}([Z])$ is out of reach, at least using standard algorithms. We warn the reader that it is not clear, even intuitively, whether we should expect NONRED or DEFORMS in this case, since it may be that $\mathscr{L}_H$ lies in the closure of a compressed (not very compressed) component similar to the ones discussed in [15].

The question about $H = (1,4,10,s)$ in [1] is also formulated for $s = 10$. In this case one could try the approach above, however there are 50 negative tangents (see Lemma 1 below) and the approach is infeasible on our hardware. Of course, perhaps this is only a question of computational cost, however we prefer to leave the case $s = 10$ open, in the hope that it will stimulate further progress on understanding the Yoneda multiplication in $\mathrm{Ext}^\bullet(\mathscr{O}_Z, \mathscr{O}_Z)$ and in particular the primary obstruction.

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3. Preliminaries

We work over a field $\mathbb{k}$ of characteristic zero. The characteristic assumption will be used mostly for justifying the computations (we believe that the result holds for most characteristics). Let $S = \mathbb{k}[x_1, \ldots, x_n]$ be a polynomial ring and $\mathbb{A}^n = Spec(S)$. For a subscheme $Z \subseteq \mathbb{A}^n$ we denote by I_Z its ideal and by $\mathscr{O}_Z = S/I_Z$ its coordinate ring.

PROPOSITION 1. *The tangent space to* $[Z] \in \mathrm{Hilb}_d(\mathbb{A}^n)$ *is given by* $\mathrm{Hom}_S(I_Z, \mathscr{O}_Z)$. *This space is canonically isomorphic to* $\mathrm{Ext}^1_S(\mathscr{O}_Z, \mathscr{O}_Z)$.

Proof: A self-contained proof for $\mathrm{Hom}_S(I_Z, \mathscr{O}_Z)$ can be found in [32]; also the Ext functor naturally appears there. The isomorphism

$$\mathrm{Hom}_S(I_Z, \mathscr{O}_Z) \to \mathrm{Ext}^1_S(\mathscr{O}_Z, \mathscr{O}_Z)$$

follows from the long exact sequence obtained by applying $\mathrm{Hom}_S(-, \mathscr{O}_Z)$ to $0 \to I_Z \to S \to \mathscr{O}_Z \to 0$.

Further in the paper, when discussing $\mathrm{Hom}_S(-,-)$ and $\mathrm{Ext}^\bullet_S(-,-)$, we drop the subscript S from the notation.

If I_Z is presented as

$$S^{\oplus d_2} \to S^{\oplus d_1} \to I_Z \to 0,$$

then $\mathrm{Hom}(I_Z, \mathscr{O}_Z)$ is the kernel of the natural map $\mathrm{Hom}(S^{\oplus d_1}, \mathscr{O}_Z) \to \mathrm{Hom}(S^{\oplus d_2}, \mathscr{O}_Z)$. Computing this kernel is best performed with a computer.

We propose one example, which is straightforward, but it will be important in the following. Recall that when I_Z is graded, also the tangent space $\mathrm{Hom}_S(I_Z, \mathscr{O}_Z)$ is graded with

$$\mathrm{Hom}_S(I_Z, \mathscr{O}_Z)_i = \left\{\varphi\colon I_Z \to \mathscr{O}_Z \mid \varphi((I_Z)_j) \subseteq (\mathscr{O}_Z)_{i+j} \text{ for all } j\right\}.$$

LEMMA 1. *Suppose that* $[Z] \in \mathrm{Hilb}_{15+s}(\mathbb{A}^4)$ *is given by a* homogeneous *ideal* I_Z *and that* $\mathscr{O}_Z$ *is very compressed with Hilbert function* $(1,4,10,s)$. *Suppose further that* I_Z *is generated by cubics. Then*

$$\dim \mathrm{Hom}(I_Z, \mathscr{O}_Z)_0 = (20-s)s \quad \textit{and} \quad \dim \mathrm{Hom}(I_Z, \mathscr{O}_Z)_{-1} \geq 4s^2 - 55s + 200.$$

Proof: By assumption, the presentation of I_Z is

$$S(-5)^{\beta} \oplus S(-4)^{4\cdot(20-s)-35} \to S(-3)^{20-s} \to I_Z \to 0.$$

Let us first look at degree zero. If we consider the full linear space $\mathrm{Hom}_{\Bbbk}(I_Z, \mathscr{O}_Z)_0$, then any relation between generators of I_Z is mapped to $(\mathscr{O}_Z)_{\geq 4} = 0$, so $\mathrm{Hom}_{\Bbbk}(I_Z, \mathscr{O}_Z)_0 = \mathrm{Hom}(I_Z, \mathscr{O}_Z)_0$.

Let us now look at degree one. By similar considerations, for every linear map $\varphi \in \mathrm{Hom}_{\Bbbk}(I_Z, \mathscr{O}_Z)_{-1}$, the image of $S(-5)^{\beta}$ is zero and the image of $S(-4)^{4\cdot(20-s)-35}$ is contained in the s-dimensional space $(\mathscr{O}_Z)_3$. Thus, the relations in the presentation yield at most $s\cdot(4\cdot(20-s)-35)$ linear-algebraic conditions on the images of minimal homogeneous generators and so

$$\dim \mathrm{Hom}(I_Z, \mathscr{O}_Z)_{-1} \geq 10\cdot(20-s) - s\cdot(4\cdot(20-s)-35) = 4s^2 - 55s + 200,$$

as claimed.

PROPOSITION 2 (Very compressed loci). *Let H be any very compressed Hilbert function and δ be the largest index such that $H(\delta) \neq 0$. Then the very compressed locus* $\mathscr{L}_H$ *is closed in* $\mathrm{Hilb}_d(\mathbb{A}^n)$, *isomorphic to* $\mathbb{A}^n \times \mathrm{Gr}(H(\delta), \binom{n-1+\delta}{\delta})$ *and has dimension*

$$n + \binom{n-1+\delta}{\delta} - H(\delta).$$

Proof: A point $[Z] \in \mathrm{Hilb}_d(\mathbb{A}^n)$ lies in $\mathscr{L}_H$ if and only if, first, the support of $[Z]$ is a single point z and, second, the ideal I_Z is contained in $\mathfrak{m}_z^{\delta}$. The first condition is closed and the second is closed provided that the first one is satisfied. The description of $\mathscr{L}_H$ as a product is immediate, see [34, Proposition 2.27].

3.1. Białynicki-Birula decompositions

The general theory of Białynicki-Birula decompositions is beautiful but quite complicated, see [20,21,35]. We would like to apply it to the standard scalar torus action on the Hilbert scheme. We will see below that in this special case things simplify considerably. Therefore, we gather below only the necessary facts and restrict to the affine case and to the positive Białynicki-Birula decomposition, that is, when considering the limit at $t \to 0$.[1]

The following allows us to reduce to considering the affine case.

PROPOSITION 3 ([33], [20, Proposition 5.3(2)]). *Suppose that X is a quasi-projective scheme. Then there is a open cover $\{U_i\}$ by affine $\mathbb{G}_m$-stable schemes. Moreover, for every such cover the Białynicki-Birula decomposition X^+ of X is covered by the Białynicki-Birula decompositions U_i^+ of U_i.*

Next, a $\mathbb{G}_m$-action on an affine scheme $Spec(A)$ is the same as a $\mathbb{Z}$-grading on the algebra A. In this case, the Białynicki-Birula decomposition can be characterised explicitly as follows.

PROPOSITION 4. *Let $X = Spec(A)$ be an affine scheme with an action of $\mathbb{G}_m$. Then, the positive Białynicki-Birula decomposition X^+ of X is a closed subscheme X^+ given by the ideal generated by $A_{<0}$. The fixed locus of X is given by the ideal generated by $\{A_{<0}\} \cup \{A_{>0}\}$. The composition*

$$\frac{A}{A_{<0} \cdot A + A_{>0} \cdot A} \simeq \frac{A_0}{(A_{<0} \cdot A)_0} \hookrightarrow \frac{A_{\geq 0}}{(A_{<0} \cdot A)_{\geq 0}} \simeq \frac{A}{A_{<0} \cdot A}$$

gives a morphism $\pi\colon X^+ \to X^{\mathbb{G}_m}$. The canonical closed embedding $s\colon X^{\mathbb{G}_m} \to X^+$ is a section of π. We obtain the following diagram, where π and s are closed embeddings

$$\begin{tikzcd} X^+ \arrow[r, hook, "\theta"] \arrow[d, shift left, "\pi"] & X \\ X^{\mathbb{G}_m} \arrow[u, hook, shift left, "s"] & \end{tikzcd}$$

Thus, for every $x \in X^{\mathbb{G}_m}$, the fibre $\pi^{-1}(x)$ is given by spectrum of an $\mathbb{N}$-graded algebra $B \simeq \frac{A}{A_{<0} \cdot A + (\mathfrak{m}_x)_0 \cdot A}$, which satisfies $B_0 = \mathbb{k}$.

Proof: See for example [20, Proposition 4.5, Example 4.6].

[1] Be aware that in some articles by the author, notably [17], the sign X^+ denotes the *negative* Białynicki-Birula decomposition, that is, the one coming from considering $\lim_{t\to\infty}$.

For a homogeneous maximal ideal $\mathfrak{m}$ in a $\mathbb{Z}$-graded ring A, the cotangent space at $[\mathfrak{m}] \in Spec(A)$ is the subquotient $\mathfrak{m}/\mathfrak{m}^2$, so is also naturally graded. The tangent space at $[\mathfrak{m}]$ is also graded, the weights are opposite.

EXAMPLE 2. In the setup of Proposition 4, take $x \in X^{\mathbb{G}_m}(\Bbbk)$. Then the cotangent space at $x \in X^+$ is the non-negative part of the cotangent space of $x \in X$. Dualising, we obtain that

$$d\theta\colon T_{X^+,x} \to T_{X,x}$$

identifies $T_{X^+,x}$ with $(T_{X,x})_{\leq 0}$, the non-**positive** part of $T_{X,x}$.

The weights of the tangent space are crucial for comparing X^+ and X, as the following proposition says.

PROPOSITION 5 ([20, Proposition 1.6]). *Let X be a separated scheme locally of finite type (for example, this holds if X is quasi-projective). Assume that $x \in X^{\mathbb{G}_m}(\Bbbk)$ is such that $d\theta_x$ is surjective (that is, an isomorphism). Then up to restricting to a $\mathbb{G}_m$-stable affine neighbourhood of x we can assume that θ is an isomorphism.*

3.2. Białynicki-Birula decomposition of the Hilbert scheme of points

Let $\mathbb{G}_m = Spec(\Bbbk[t^{\pm 1}])$ be a one-dimensional torus and consider its action

$$\mathbb{G}_m \times \mathbb{A}^n \to \mathbb{A}^n$$

by rescaling: $\lambda \cdot (x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n)$ for every $\Bbbk$-point $(x_1, \ldots, x_n) \in \mathbb{A}^n(\Bbbk)$ and $\lambda \in \Bbbk^\times = \mathbb{G}_m(\Bbbk)$. For every closed subscheme $Z \subseteq \mathbb{A}^n$ and $\lambda \in \mathbb{G}_m(\Bbbk)$ we obtain a new closed subscheme $\lambda \cdot Z$ given by the closed embedding

$$Z \hookrightarrow \mathbb{A}^n \xrightarrow[\simeq]{\lambda\cdot} \mathbb{A}^n. \tag{3.1}$$

When we view a point as a closed subscheme, both definitions agree. A subscheme Z is a $\mathbb{G}_m$-fixed point if and only if its ideal I_Z is homogeneous.

Construction (3.1) generalizes readily to the case when Z is closed in $\mathbb{A}^n \times S$, for any scheme S. This yields an action $\mathbb{G}_m \times \mathrm{Hilb}_d(\mathbb{A}^n) \to \mathrm{Hilb}_d(\mathbb{A}^n)$, which on $\Bbbk$-points agrees with (3.1).

For any set A of d monomials in S, consider the locus $U_A \subseteq \mathrm{Hilb}_d(\mathbb{A}^n)$ which consists of $[Z] \in \mathrm{Hilb}_d(\mathbb{A}^n)$ such that A spans $\mathcal{O}_Z$. These loci are open and $\mathbb{G}_m$-stable, hence the corresponding Białynicki-Birula cells U_A^+ cover $(\mathrm{Hilb}_d(\mathbb{A}^n))^+$, see Proposition 3. The loci above are important for the computational aspects, see for example [26].

We would like now to understand when Proposition 5 can be applied in the case of Hilbert schemes, so we are interested in the weights on the tangent space.

LEMMA 2. *Let $[Z] \in \mathrm{Hilb}_d(\mathbb{A}^n)$ be a $\mathbb{G}_m$-fixed point. Then $(T_{\mathrm{Hilb}_d(\mathbb{A}^n),[Z]})_{>0}$ vanishes if and only if $\mathcal{O}_Z$ is very compressed.*

Proof: Take $S = \Bbbk[x_1,\dots,x_n]$. Suppose first that $\mathscr{O}_Z$ is very compressed. Then there exists an s such that $I_Z \subseteq S_{\geq s}$ and $(\mathscr{O}_Z)_{>s} = 0$. A tangent at Z of strictly positive degree i corresponds to a homomorphism $\varphi\colon I_Z \to \mathscr{O}_Z$ such that $\varphi((I_Z)_j) \subseteq (\mathscr{O}_Z)_{j+i}$. The source is nonzero only for $j \geq s$, but for such a j we have $i+j > s$, so the target is zero. It follows that $\varphi = 0$.

Suppose now that $\mathscr{O}_Z$ is not very compressed. This implies that there exists an s such that $I_s \neq 0$ and $(\mathscr{O}_Z)_{\geq s+1} \neq 0$. Pick a set of minimal generators of I_Z and let g be an element of lowest degree. Pick a socle element $h \in \mathscr{O}_Z$ of highest degree. Then $\deg(h) \geq s+1 > s \geq \deg(g)$. There exists a homomorphism $\varphi\colon I_Z \to \mathscr{O}_Z$ which satisfies $\varphi(g) = h$ and sends all other minimal generators to zero. It follows that φ is homogeneous of strictly positive degree, equal to $\deg(h) - \deg(g)$.

3.3. Primary obstructions

Primary obstructions govern the order two part of deformation theory and can be computed explicitly. We discuss them below.

Consider two tangent vectors at a point $[Z] \in \mathrm{Hilb}_d(\mathbb{A}^n)$. They yield maps

$$\varphi_i\colon Spec\left(\frac{\Bbbk[\varepsilon_i]}{(\varepsilon_i^2)}\right) \to \mathrm{Hilb}_d(\mathbb{A}^n),$$

for $i = 1,2$ and two elements

$$\varphi_1, \varphi_2 \in T_{\mathrm{Hilb}_d(\mathbb{A}^n),[Z]} \simeq \mathrm{Ext}^1(\mathscr{O}_Z, \mathscr{O}_Z).$$

The two tangent vectors span an at most 2-dimensional space and the corresponding morphism is

$$\varphi_{12}\colon Spec\left(\frac{\Bbbk[\varepsilon_1,\varepsilon_2]}{(\varepsilon_1,\varepsilon_2)^2}\right) \to \mathrm{Hilb}_d(\mathbb{A}^n),$$

which restricts to φ_1, φ_2 in the natural way. We may ask when φ_{12} does extend to a map $\widetilde{\varphi_{12}}$ from $Spec\big(\Bbbk[\varepsilon_1,\varepsilon_2]/(\varepsilon_1^2,\varepsilon_2^2)\big)$. Deformation theory [8, Chapter 5] implies that an extension exists if and only if an obstruction

$$ob_{\varphi_{12},\widetilde{\varphi}_{12}} \in \mathrm{Ext}^2(\mathscr{O}_Z, \mathscr{O}_Z)$$

vanishes. The key observation is that we can describe the obstruction explicitly.

THEOREM 2 ([22, Theorem 4.18]). *The obstruction $ob_{\varphi_{12},\widetilde{\varphi}_{12}}$ is equal to*

$$\varphi_1 \circ \varphi_2 + \varphi_2 \circ \varphi_1,$$

where $\circ$ denotes Yoneda's multiplication in $\mathrm{Ext}^\bullet(\mathscr{O}_Z, \mathscr{O}_Z)$ applied to $\varphi_1, \varphi_2 \in \mathrm{Ext}^1(\mathscr{O}_Z, \mathscr{O}_Z)$.

Let $\mu\colon \mathrm{Sym}^2 \mathrm{Ext}^1(\mathscr{O}_Z, \mathscr{O}_Z) \to \mathrm{Ext}^2(\mathscr{O}_Z, \mathscr{O}_Z)$ be given by

$$\mu(\varphi_1 \cdot \varphi_2) := \varphi_1 \circ \varphi_2 + \varphi_2 \circ \varphi_1.$$

Consider its transpose

$$\mu^\vee\colon \operatorname{Ext}^2(\mathscr{O}_Z,\mathscr{O}_Z)^\vee \to \left(\operatorname{Sym}^2\operatorname{Ext}^1(\mathscr{O}_Z,\mathscr{O}_Z)\right)^\vee \simeq \operatorname{Sym}^2\left(\operatorname{Ext}^1(\mathscr{O}_Z,\mathscr{O}_Z)^\vee\right).$$

As explained in [22, §4.2], Theorem 2 yields the following corollary. Recall that $\operatorname{Ext}^1(\mathscr{O}_Z,\mathscr{O}_Z)^\vee$ is the cotangent space at $[Z]\in \operatorname{Hilb}_d(\mathbb{A}^n)$.

COROLLARY 1. *Consider the complete local ring* $(\hat{\mathscr{O}}_{\operatorname{Hilb}_d(\mathbb{A}^n),[Z]},\mathfrak{m}_{[Z]})$. *Its truncation to second order satisfies*

$$\frac{\hat{\mathscr{O}}_{\operatorname{Hilb}_d(\mathbb{A}^n),[Z]}}{\mathfrak{m}^3_{[Z]}} \simeq \frac{\operatorname{Sym}^\bullet\operatorname{Ext}^1(\mathscr{O}_Z,\mathscr{O}_Z)^\vee}{\operatorname{Im}\left(\mu^\vee\colon \operatorname{Ext}^2(\mathscr{O}_Z,\mathscr{O}_Z)^\vee\to\operatorname{Sym}^2\operatorname{Ext}^1(\mathscr{O}_Z,\mathscr{O}_Z)^\vee\right)+\left(\operatorname{Ext}^1(\mathscr{O}_Z,\mathscr{O}_Z)^\vee\right)^3}.$$

3.4. Primary obstructions and Białynicki-Birula decompositions

As written, Corollary 1 does not directly involve the dimension of the local ring. Moreover, we would like to apply it for the fibre of the Białynicki-Birula decomposition. Both subtleties "cancel out": restriction to the fibre gives us an $\mathbb{N}$-grading which allows to pass from the third neighbourhood to the full complete local ring.

PROPOSITION 6. *Let* $[Z]\in\operatorname{Hilb}_d(\mathbb{A}^n)$ *be a* $\mathbb{G}_m$*-fixed* $\Bbbk$*-point and consider its Białynicki-Birula fibre* $Spec(A)$. *Assume that the subspace*

$$\left(T_{\operatorname{Hilb}_d(\mathbb{A}^n),[Z]}\right)_{\le -2}$$

is zero. Then there is a surjection of graded algebras

$$\frac{\operatorname{Sym}^\bullet\left(\operatorname{Ext}^1(\mathscr{O}_Z,\mathscr{O}_Z)_{<0}\right)^\vee}{\operatorname{Im}(\mu^\vee\colon \left(\operatorname{Ext}^2(\mathscr{O}_Z,\mathscr{O}_Z)_{<0}\right)^\vee\to\operatorname{Sym}^2\left(\operatorname{Ext}^1(\mathscr{O}_Z,\mathscr{O}_Z)_{<0}\right)^\vee)} \twoheadrightarrow A.$$

Proof: By Proposition 3 we already know that there is an open $\mathbb{G}_m$-stable neighbourhood $[Z]\in U\subseteq\operatorname{Hilb}_d(\mathbb{A}^n)$ such that the Białynicki-Birula fibre $\pi^{-1}([Z])$ is contained in U^+. Using Proposition 4 we conclude that $\pi^{-1}([Z])$ is a spectrum of an $\mathbb{N}$-graded algebra B with $B_0=\Bbbk$.

We now employ Corollary 1. To make the notation lighter, we put $E:=\operatorname{Ext}^1(\mathscr{O}_Z,\mathscr{O}_Z)$. The complete local ring $\hat{B}$ of $[Z]\in Spec(B)$ is a quotient of the complete local ring of $[Z]\in\operatorname{Hilb}_d(\mathbb{A}^n)$. Hence, also the truncation $\hat{B}/\mathfrak{m}^3$ is a quotient of the truncation of

$$\frac{\hat{\mathscr{O}}_{\operatorname{Hilb}_d(\mathbb{A}^n),[Z]}}{\mathfrak{m}^3}\simeq\frac{\operatorname{Sym}^\bullet E^\vee}{\operatorname{Im}\left(\mu^\vee\colon \operatorname{Ext}^2(\mathscr{O}_Z,\mathscr{O}_Z)^\vee\to\operatorname{Sym}^2 E^\vee\right)+(E^\vee)^3}.$$

The cotangent space of $[Z]\in Spec(B)$ has no nonpositive weights, so $\hat{B}/\mathfrak{m}^3$ is in fact a quotient of

$$\frac{\operatorname{Sym}^\bullet(E_{<0})^\vee}{\operatorname{Im}\left(\mu^\vee\colon \left(\operatorname{Ext}^2(\mathscr{O}_Z,\mathscr{O}_Z)_{<0}\right)^\vee\to\operatorname{Sym}^2(E_{<0})^\vee\right)+\left((E_{<0})^\vee\right)^3}$$

We now lift this from infinitesimal second order to a more global situation using the $\mathbb{N}$-grading. Consider the map of graded rings

$$p\colon \operatorname{Sym}^\bullet E_{<0} \to B.$$

By Example 2, this map is an isomorphism on cotangent spaces. Since B is $\mathbb{N}$-graded with $B_0 = \Bbbk$, the map p is a surjection, by induction on the degree. Moreover, again thanks to the grading and to the fact that all $E_{<0} = E_{-1}$, we obtain an isomorphism $\hat{B}/\mathfrak{m}^3 \simeq B/B_{\geq 3}$.

We summarize the situation on a commutative diagram

$$\begin{tikzcd}
\operatorname{Sym}^\bullet E_{<0} \arrow[rr, two heads] \arrow[d, two heads] & & B \arrow[d] \arrow[dr] & \\
\dfrac{\operatorname{Sym}^\bullet (E_{<0})^\vee}{\operatorname{Im}\left(\mu^\vee\colon \left(\operatorname{Ext}^2(\mathscr{O}_Z,\mathscr{O}_Z)_{<0}\right)^\vee \to \operatorname{Sym}^2 (E_{<0})^\vee\right) + \left((E_{<0})^\vee\right)^3} \arrow[r, "\simeq"] & & \dfrac{\hat{B}}{\mathfrak{m}^3} \arrow[r, "\simeq"] & \dfrac{B}{B_{\geq 3}}
\end{tikzcd}$$

This shows that the image $p(\operatorname{Im}\mu^\vee)$ in B_2 is zero. The claim follows.

3.5. Computational input

As mentioned in the introduction, currently there is not enough knowledge about the Ext algebra to perform a conceptual analysis of the primary obstruction. In this section we include a somewhat brute-force check of specific examples.

The package *MatricesAndQuot* is available as an auxiliary file for the arXiv version of [22]. Needless to say, many alternatives exist, in particular the computation can be performed using Ilten's *VersalDeformations* package [16] or Lella's *HilbertAndQuotSchemesOfPoints.m2* package [25] or the framework [2].

PROPOSITION 7 (Key computational output). *Let $H = (1,4,10,s)$ For every $s = 6,7,8,9$ there is an example of $[Z] \in \mathscr{L}_H \subseteq \operatorname{Hilb}_{15+s}(\mathbb{A}^4)$ such that the algebra*

$$\frac{\operatorname{Sym}^\bullet \left(\operatorname{Ext}^1(\mathscr{O}_Z,\mathscr{O}_Z)_{<0}\right)^\vee}{\operatorname{Im}(\mu^\vee\colon \left(\operatorname{Ext}^2(\mathscr{O}_Z,\mathscr{O}_Z)_{<0}\right)^\vee \to \operatorname{Sym}^2 \left(\operatorname{Ext}^1(\mathscr{O}_Z,\mathscr{O}_Z)_{<0}\right)^\vee)}$$

is 4-dimensional.

Proof: This is an explicit *Macaulay2* computation. See code below. We work over characteristic 17 for efficiency. By semicontinuity the same result (for the ideals given by obvious lifts of generators) is true in characteristic zero. See [24, (10.12)-(10.13)] for a detailed discussion of this method.

```
S = (ZZ/17)[x_1 .. x_4];
loadPackage("MatricesAndQuot", Reload=>true);
I9 = ideal(x_2*x_3^2, x_2^2*x_3+x_1*x_4^2+x_4^3,
```

```
x_1^2*x_2+x_1*x_3^2+x_2*x_4^2,
x_1^3+x_2^2*x_4+x_2*x_4^2, x_3*x_4^2, x_1^2*x_4,
x_1*x_2*x_4+x_3^2*x_4+x_1*x_4^2, x_1*x_2*x_3+x_3^3+x_1^2*x_4,
x_2^3+x_2*x_3*x_4+x_4^3, x_1^3+x_2^3+x_3^2*x_4,
x_1^2*x_3+x_1*x_2*x_4+x_2*x_4^2);
assert(degree I9 == 24); -- case (1,4,10,9)
assert(dim primaryObstruction(S^1/I9) == 4);
I8 = I9 + ideal(x_1*x_2^2);
assert(degree I8 == 23); -- case (1,4,10,8)
assert(dim primaryObstruction(S^1/I8) == 4);
I7 = I8 + ideal(x_1*x_3^2);
assert(degree I7 == 22); -- case (1,4,10,7)
assert(dim primaryObstruction(S^1/I7) == 4);
I6 = I7 + ideal(x_1^2*x_3);
assert(degree I6 == 21); -- case (1,4,10,6)
assert(dim primaryObstruction(S^1/I6) == 4);
```

The total computation time is a few minutes, the case `I9` takes most of it.

3.6. Proof of Theorem 1

We proceed to the proof of our main theorem. Recall that we consider Hilbert functions $H = (1,4,10,s)$ for $s \in \{6,7,8,9\}$.

Proof: [Proof of Theorem 1] We follow the strategy outlined in the introduction. Fix an s, let $d = 1+4+10+s$, and pick a point $[Z] \in \mathcal{L}_H$ as in Proposition 7. The fibre $\pi^{-1}([Z])$ is connected as it is a cone. The group scheme $(\mathbb{A}^4,+)$ acts on the fibre $\pi^{-1}([Z])$ by translations. From this and from $\dim \pi^{-1}([Z]) = 4$ it follows that the fibre is, as a set, equal to the $(\mathbb{A}^4,+)$-orbit of the cone point $[Z]$. By semicontinuity of fibre dimensions, the same holds for fibres near $[Z]$. It follows that, as a set, $\mathcal{L}_H$ contains an open neighbourhood of $[Z]$.

From Proposition 2 and Lemma 1 it follows that for every point $[Z'] \in \mathcal{L}_H$, the tangent spaces to $\mathrm{Hilb}_d(\mathbb{A}^4)$ and $\mathcal{L}_H$ differ already in degree -1. This can happen only if the component of the Hilbert scheme that topologically is equal to $\mathcal{L}_H$ has no smooth points, so it is generically nonreduced.

To obtain generically nonreduced components of $\mathrm{Hilb}_d(\mathbb{A}^4)$ for $d \geq 21$ consider a scheme Z as above for $d = 21$ and enlarge it to a scheme

$$Z \sqcup \bigsqcup_{d-21} Spec(\Bbbk)$$

embedded (in any way) into $\mathbb{A}^4$.

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