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ON THE TOTAL REDUCIBILITY OF THE HILBERT CURVE OF SOME SPECIAL VARIETIES

Dedicated to the memory of Gianfranco Casnati

Abstract. Hilbert curves of quadric fibrations over curves and of scrolls over surfaces are studied with reference to their total reducibility over $\mathbb{Q}$. The case of del Pezzo manifolds being discussed elsewhere, this completes the picture for polarized n-folds with unnormalized spectral value $n-1$.

1. Introduction

Let (X,L) be a complex polarized manifold of dimension n. The Hilbert curve of (X,L) is the affine plane curve of degree n defined by the Hilbert-like polynomial $\chi(xK_X + yL)$, where K_X is the canonical bundle of X, regarding x and y as complex variables. This notion was introduced in [2]: the natural expectation is that several properties of the polarized manifold (X,L) that one considers are encoded by its Hilbert curve Γ, as suggested by [2, Theorem 6.1]. In particular, if the nefvalue of (X,L) is $\tau := \frac{a}{b}$, with a,b relatively prime positive integers, then Γ is reducible and contains $a-1$ parallel lines of prescribed equations as components. Therefore it becomes important to understand the properties of the residual curve of the union of such lines in Γ, which is a plane curve, G, of degree $n-a+1$. This investigation has been carried out for several manifolds arising in adjunction theory [9], [10], [11], [12].

A question one could ask in particular is the following: when is the Hilbert curve of a special variety arising from adjunction theory totally reducible over $\mathbb{Q}$ (or over $\mathbb{R}$)? The question comes out naturally in view of [14], where the possibility that the Hilbert polynomial is totally reducible over $\mathbb{Q}$ is investigated for some classes of Fano manifolds. In general, if X is a Fano n-fold of index ι_X polarized by a positive multiple $L = rH$, of the fundamental line bundle $H(=\frac{1}{\iota_X}(-K_X))$, the Hilbert polynomial of (X,H) is

$$P(z) = \chi(zH) = A(z)\prod_{i=1}^{\iota_X -1}(z+i),$$

where $A(z)$ is a polynomial of degree $n+1-\iota_X$, whose coefficients, which can be computed by solving a suitable system of linear equations [13, Lemma 3.1], are rational numbers. For instance, for $\mathbb{P}^n$ we have $P(z) = \frac{1}{n!}\prod_{i=1}^{n}(z+i)$ and for the smooth hyperquadric $\mathbb{Q}^n \subset \mathbb{P}^{n+1}$, $P(z) = \frac{2}{n!}(z+\frac{n}{2})\prod_{i=1}^{n-1}(z+i)$. Here H denotes the ample generator of the Picard group of X when it is $\mathbb{Z}$, and $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)$ when

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$X = \mathbb{Q}^2$ ($\cong \mathbb{P}^1 \times \mathbb{P}^1$). Clearly, for any Fano manifold, $P(z)$ is totally reducible over $\mathbb{C}$; it is also totally reducible over $\mathbb{Q}$ for $\mathbb{P}^n$ and $\mathbb{Q}^n$, due to the above. Notice that this fact is true also for all Grassmannians [6], but not for all Fano manifolds [14]. Now, putting $z = ry - \iota_X x$ one can rewrite $P(z)$ in terms of x and y getting the equation of the Hilbert curve Γ of (X, L) in the complex affine plane (x, y). Furthermore, letting $x = \frac{1}{2} + u$ and $y = v$, we obtain the canonical equation of Γ in terms of coordinates (u, v), which makes evident the symmetry of this curve with respect to the origin of the (u, v)-plane; note that this origin corresponds to the half-canonical line bundle, whose class is the fixed point of the Serre involution acting on $\mathrm{Num}(X) \otimes \mathbb{C}$ [9]. Then, for $\left(\mathbb{Q}^n, \mathscr{O}_{\mathbb{Q}^n}(r)\right)$, r being a positive integer, the polynomial defining the canonical equation of Γ is

$$p\left(\frac{1}{2} + u, v\right) = (-1)^n \frac{2}{n!}(nu - rv) \prod_{i=1}^{n-1} \left(nu - rv + \frac{n}{2} - i\right). \tag{1}$$

Hence, in line with what already said, the Hilbert curve of $\left(\mathbb{Q}^n, \mathscr{O}_{\mathbb{Q}^n}(r)\right)$ is totally reducible over $\mathbb{Q}$. Looking outside the range of Fano manifolds, the same occurs when $X = \mathbb{P}(\mathscr{E}) \to B$ is a projective bundle over a smooth curve B and $L_f = \mathscr{O}_{\mathbb{P}^{n-1}}(r)$ for any fibre $f = \mathbb{P}^{n-1}$ of the bundle projection. Actually, in this case, letting q denote the genus of B and $d = L^n$, the canonical equation of Γ is defined by

$$p\left(\frac{1}{2} + u, v\right) = (-1)^{n-1} \frac{1}{n!} \left(n(2q-2)u + \frac{d}{r^{n-1}} v\right) \prod_{i=1}^{n-1} \left(nu - rv + \frac{n}{2} - i\right) \tag{2}$$

[9, Proposition 2.1]. Note that for $(X, L) = \left(\mathbb{Q}^2, \mathscr{O}_{\mathbb{Q}^2}(r)\right)$, i.e, $n = 2$ and $q = 0$, the right hand sides of both (1) and (2) simply reduce to $(2u - rv)^2$. From the adjunction theoretic point of view, for $r = 1$ all the above varieties correspond to the nefvalue of (X, L) being $\tau \geq n$ [1, Proposition 7.2.2]. Proceeding further for decreasing values of τ, the only polarized manifold whose nefvalue is between $n-1$ and n is $\left(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(2)\right)$ [1, Theorem 7.2.4] (for which $\tau = \frac{3}{2}$), and this case clearly fits into the discussion about the Hilbert polynomial of $\mathbb{P}^n$. So the obvious question is what happens when the nefvalue takes on the immediately lower value, namely $n-1$. By what we said we can neglect the case of $\left(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(2)\right)$ (in which case $\tau = 2$). Then, assuming that (X, L) does not contain (-1)-hyperplanes, so that it coincides with its adjunction theoretic reduction, condition $\tau = n-1$ corresponds to three possibilities: i) del Pezzo n-folds, ii) quadric fibrations over a smooth curve, and iii) scrolls over a smooth surface [1, Theorem 7.3.2] (see also [4, Theorem 11.8]). Case i) being completely settled by [14, Section 3], here we focus on cases ii) and iii), and investigating the total reducibility over $\mathbb{Q}$ of the corresponding Hilbert curves is exactly the aim of this paper.

In Section 2 some common features of the two cases are examined. The key point is that in both cases the residual curve G is simply a conic, which makes deciding on the reducibility elementary, at least in principle.

Then, the specific situation of quadric fibrations over curves is dealt with in Section 3 and the corresponding result, expressed by Theorem 1, solves the problem completely: for any adjunction theoretic quadric fibration, the Hilbert curve is totally reducible over $\mathbb{Q}$ if and only if there are no singular fibers.

The case of scrolls over surfaces appears more intricate and is addressed in Section 4. Here, for technical reasons we have to assume that the vector bundle giving rise to X is semistable in the sense of Bogomolov, but in spite of this extra assumption our results are only partial. First of all, the reducibility of G over $\mathbb{C}$ (in two, possibly coinciding, parallel lines) is characterized for adjunction theoretic scrolls: see Proposition 2. The reducibility over $\mathbb{Q}$, which is more delicate, requires a case-by-case analysis according to the Kodaira dimension of the base surface S and is summarized by Theorem 2. A similar analysis is needed to understand when G is reducible into two transverse lines. For scrolls over a surface S of Kodaira dimension ≤ 1 the result is given by Proposition 3. Unfortunately, when S is of general type we are only able to establish some restrictions on the numerical characters; however we provide a significant example in which S has ample cotangent bundle.

Finally, in Section 5, we consider the classical scrolls that are not adjunction theoretic, which are very few, due to a result of Fujita [5]. It turns out that G, hence Γ, is totally reducible over $\mathbb{Q}$ for all of them except for $X = \mathbb{P}\big(\mathscr{O}_{\mathbb{P}^2}(2) \oplus \mathscr{O}_{\mathbb{P}^2}(1)\big)$ polarized by the tautological line bundle.

2. Common aspects of the two cases

In both cases of quadric fibrations over a smooth curve B and of scrolls over a smooth surface S, the canonical equation in coordinates (u, v) of the Hilbert curve Γ of (X, L) has the following form

$$p\left(\frac{1}{2} + u, v\right) = (\alpha u^2 + 2\beta uv + \gamma v^2 + \varepsilon) \prod_{i=1}^{n-2} L_i = 0,$$

where $\alpha, \beta, \gamma, \varepsilon \in \mathbb{Q}$ and the L_i's are linear polynomials with rational coefficients, defining equally spaced parallel lines with the nef-value of (X, L) as slope and whose union is symmetric with respect to the origin; in fact $L_i = (n-1)u - v + \frac{1}{2}(n-1-2i)$. Moreover, in both cases we know the explicit expression of the residue polynomial of degree 2 [11] and [12]. First of all, Γ is totally reducible (over $\mathbb{C}$) if and only if so is the conic G, whose matrix is

$$A = \begin{bmatrix} \alpha & \beta & 0 \\ \beta & \gamma & 0 \\ 0 & 0 & \varepsilon \end{bmatrix},$$

where $\gamma > 0$ (see (5) and (8)). Set $\Delta := \alpha\gamma - \beta^2$. Due to the fact that $\det A = \Delta\,\varepsilon$, G is totally reducible over $\mathbb{C}$ if and only if either $\Delta = 0$ or $\varepsilon = 0$. So concerning the total reducibility over $\mathbb{Q}$ of Γ, i.e. of G, we get the following two possibilities:

a) $\varepsilon = 0$ and $\alpha u^2 + 2\beta uv + \gamma v^2 = \gamma(mu - v)(m'u - v)$ for some rationals m, m', in which case the conic G splits into two (possibly coinciding) lines through the origin, or

b) $\alpha u^2 + 2\beta uv + \gamma v^2 = \gamma(mu - v)^2$ and $\varepsilon = -\gamma h^2$ for some rationals m and h, in which case G splits into two parallel lines symmetric with respect to the origin.

Clearly, when both terms Δ and ε are zero, G consists of a single line with multiplicity 2.

In case a), $\Delta = -\frac{\gamma^2}{4}(m - m')^2$, i.e., $\Delta = -s^2$ for some $s \in \mathbb{Q}$. In case b), it is $-\frac{\varepsilon}{\gamma}$ which is the square of a rational number. In conclusion, we have

PROPOSITION 1. *With the notation as above,* Γ *is totally reducible over* $\mathbb{Q}$ *if and only if either*

a) $\varepsilon = 0$ *and* $\Delta = -s^2$ *for some* $s \in \mathbb{Q}$*, or*

b) $\Delta = 0$ *and* $\frac{\varepsilon}{\gamma} = -s^2$ *for some* $s \in \mathbb{Q}$.

In other words, the condition is that one of the two quantities Δ and $\frac{\varepsilon}{\gamma}$ vanishes, the other being the opposite of the square of a rational number. The above discussion also shows that Γ is totally reducible over $\mathbb{R}$ if and only if one of Δ and ε is zero and the other less than or equal to zero, γ being positive.

3. Quadric fibrations over curves

Let (X, L) be any quadric fibration over a smooth curve B of genus q and let $\pi : X \to B$ be the fibration morphism. Consider the rank-2 vector bundle $\mathscr{E} := \pi_* L$ and let $P := \mathbb{P}(\mathscr{E})$. Then X can be described as a smooth divisor of relative degree 2 inside P. In fact, $X \in |2\xi - \tilde{\pi}^*\mathscr{B}|$, where ξ is the tautological line bundle on P, $\tilde{\pi} : P \to B$ is the bundle morphism extending the fibration π and $\mathscr{B}$ is a line bundle on B. Moreover, $\xi_X = L$. From this description we see that $K_X + (n-1)L = \tilde{\pi}^*\mathscr{A}$, where $\mathscr{A} = K_B + \det\mathscr{E} - \mathscr{B}$. Let $a := \deg\mathscr{A}$. Then the following holds [12, §§ 0 and 1, in particular see Proposition 2].

i) If $a < 0$ then $(X, L) = \left(\mathbb{P}^1 \times \mathbb{P}^1, \mathscr{O}(2,1)\right)$;

ii) if $a = 0$, then (X, L) is also a del Pezzo manifold, hence, either

  ii-1) $n = 2$, X is any del Pezzo surface except $\mathbb{P}^2$ and $L = -K_X$, or

  ii-2) $n = 3$ and $(X, L) = \left(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \mathscr{O}(1,1,1)\right)$;

iii) if $a > 0$, then (X, L) is a quadric fibration in the adjunction theoretic sense [1, p. 81].

As to the canonical equation of Γ we have:

$$p\left(\frac{1}{2} + u, v\right) = 2(2u - v)(u - v)$$

in case i); in case ii-1), recalling that $L = -K_X$, we have

$$p\left(\frac{1}{2}+u, v\right) = \frac{1}{2}\left[d(u-v)^2 + \frac{8-d}{4}\right],$$

where $d = K_X^2$ is the degree of our del Pezzo surface. In particular, since $X \neq \mathbb{P}^2$, we conclude that Γ is totally reducible over $\mathbb{Q}$ if and only if $d = 8$. Due to the classification of del Pezzo surfaces, this corresponds to X being either $\mathbb{P}^1 \times \mathbb{P}^1$ or the Segre–Hirzebruch surface $\mathbb{F}_1$: in both cases $p(\frac{1}{2}+u, v) = 4(u-v)^2$, see also [14, Proposition 2.2]. On the other hand, in case ii-2),

$$p\left(\frac{1}{2}+u, v\right) = (2u-v)^3$$

[13, Theorem 3.3]. Finally, in case iii) we have [12, Proposition 3]

$$p\left(\frac{1}{2}+u, v\right) = \frac{(-1)^n}{n!} f(u,v) \prod_{i=1}^{n-2}\left((n-1)u - v + \frac{1}{2}\big(n-1-2i\big)\right),$$

the residual conic G of Γ having equation $f(u,v) = [u\ v\ 1]A\ {}^t[u\ v\ 1] = 0$, where A, up to a constant factor, is the matrix

$$A = \begin{bmatrix} A_\infty & 0 \\ 0 & (n-1)\frac{\mu}{4} \end{bmatrix},$$

with (see [12, Corollary 4])

$$A_\infty = \begin{bmatrix} (1-n)\big(2nc+2e-(n+1)b\big) & nc-(n-2)e-b \\ nc-(n-2)e-b & 2e-b \end{bmatrix}, \tag{3}$$

and

$$c = 2q-2, \quad e = \deg\mathscr{E}, \quad b = \deg\mathscr{B}. \tag{4}$$

In particular, $a = c+e-b$, and $\mu = 2e-(n+1)b$ is the number of singular fibers of π. Recalling the quantities related to the matrix A introduced in Section 2, we have that

$$\gamma = 2e-b = L^n = \deg(X,L) > 0 \tag{5}$$

[12, (3)]. Moreover, $\Delta = \det A_\infty$ and $\varepsilon = (n-1)\frac{\mu}{4}$. Now, a straightforward computation relying on (3) and (4) shows that

$$\det A_\infty = -n^2a^2,$$

so that $\Delta = -s^2$ with $s = na \in \mathbb{Z}$, and it can never be zero in case iii). Then, $\det A = (n-1)\frac{\mu}{4}\det A_\infty$, hence G is reducible if and only if $\mu = 0$. Moreover, in this case we have

$$f(u,v) = 2n\big((1-n)u+v\big)\left(cu + \frac{e}{n+1}\ v\right),$$

which shows that G is reducible over $\mathbb{Q}$. It thus follows that Γ is totally reducible over $\mathbb{Q}$ in case i), in case ii) when X is a del Pezzo surface of degree $d = 8$ or $(X, L) = \left(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \mathscr{O}(1,1,1)\right)$, and in case iii) when $\mu = 0$. In particular, concerning case iii) the above discussion proves the following result.

THEOREM 1. *Let (X, L) be an adjunction theoretic quadric fibration over a smooth curve B. Then its Hilbert curve is totally reducible over $\mathbb{Q}$ if and only if the fibration $\pi : X \to B$ has no singular fibers.*

In particular we can observe that, when reducible, G consists of two distinct lines cutting at the origin, one of which is parallel to the remaining lines constituting Γ.

4. Scrolls over surfaces

Now let us focus on the case of scrolls over a smooth surface S. So, let $X = \mathbb{P}(\mathscr{E})$ where $\mathscr{E}$ is an ample vector bundle of rank $n-1$ over S and let L be the tautological line bundle on X. Sometimes we refer to such a pair (X, L) as a classical scroll. Note that $\mathscr{E} = \pi_* L$, where, here, $\pi : X \to S$ denotes the scroll projection.

Recall that for a vector bundle $\mathscr{V}$ of rank r on a smooth surface S the Bogomolov number of $\mathscr{V}$ is

$$\delta(\mathscr{V}) := (r-1)c_1(\mathscr{V})^2 - 2rc_2(\mathscr{V}).$$

According to [3, Theorem p. 500] if $\mathscr{V}$ is H-stable for any ample line bundle H on S, then $\delta(\mathscr{V}) < 0$ (Bogomolov inequality). This provides a strong notion of instability: $\mathscr{V}$ is said B-unstable if $\delta(\mathscr{V}) > 0$. Consequently, in accordance with the usual terminology, we say that $\mathscr{V}$ is B-semistable if $\delta(\mathscr{V}) \leq 0$, properly B-semistable if equality occurs, and B-stable when the inequality is strict. In particular, if $\mathscr{V} = A \oplus B$, with A, B ample line bundles, we have $c_1(\mathscr{V}) = A + B$ and $c_2(\mathscr{V}) = A \cdot B$, hence $\delta(\mathscr{V}) = (A+B)^2 - 4A \cdot B = (A-B)^2$: we thus see that if S is a fibration over a smooth curve and $A - B$ is a linear combination of fibers, then $\mathscr{V}$ is properly B-semistable.

Let $\delta := \delta(\mathscr{E})$. Then, according to [11, Sec. 3], the canonical equation of the Hilbert curve Γ of (X, L) is defined by

$$p\left(\frac{1}{2} + u, v\right) = \frac{(-1)^n}{2(n-2)!}\, g(u,v) \prod_{i=1}^{n-2} \left((n-1)u - v + \frac{1}{2}\left(n-1-2i\right)\right),$$

the residual conic G of Γ having equation $g(u,v) = [u\ v\ 1]A\ {}^t[u\ v\ 1] = 0$, where A, up to a constant factor, is the matrix

$$A = \begin{bmatrix} A_\infty & 0 \\ 0 & \varepsilon \end{bmatrix},$$

with

$$A_\infty = \begin{bmatrix} K_S^2 + \frac{\delta}{n} & K_S \cdot \frac{c_1(\mathscr{E})}{n-1} - \frac{\delta}{n(n-1)} \\ K_S \cdot \frac{c_1(\mathscr{E})}{n-1} - \frac{\delta}{n(n-1)} & \frac{c_1(\mathscr{E})^2}{(n-1)^2} + \frac{\delta}{n(n-1)^2} \end{bmatrix}, \tag{6}$$

and

$$\varepsilon = \frac{1}{4}\left(8\chi(\mathscr{O}_S) - K_S^2 - \frac{\delta}{n}\right). \tag{7}$$

Concerning the other quantities related to the matrix A introduced in Section 2, we note that

$$\gamma = \frac{1}{(n-1)^2}\left(c_1(\mathscr{E})^2 + \frac{\delta}{n}\right) > 0. \tag{8}$$

Actually, since $\mathrm{rk}(\mathscr{E}) = n-1$, recalling the expression of δ, we have that $c_1(\mathscr{E})^2 + \frac{\delta}{n} = \frac{2(n-1)}{n}\big(c_1(\mathscr{E})^2 - c_2(\mathscr{E})\big)$ and, in turn, $c_1(\mathscr{E})^2 - c_2(\mathscr{E}) = L^n = \deg(X, L)$ [11, (6)]. Furthermore, $\Delta = \det A_\infty$ is given by

$$\Delta = \frac{1}{(n-1)^2}\left[\left(K_S^2 c_1(\mathscr{E})^2 - \big(K_S \cdot c_1(\mathscr{E})\big)^2\right) + \frac{\delta}{n}\big(K_S + c_1(\mathscr{E})\big)^2\right].$$

Set

$$\Phi := K_S^2 c_1(\mathscr{E})^2 - \big(K_S \cdot c_1(\mathscr{E})\big)^2 \qquad \text{and} \qquad \Psi := \frac{\delta}{n}\big(K_S + c_1(\mathscr{E})\big)^2,$$

so that

$$\Delta = \frac{1}{(n-1)^2}\,(\Phi + \Psi). \tag{9}$$

As to the term Φ, by the Hodge index theorem we have $\Phi \le 0$ for any surface S and any ample vector bundle $\mathscr{E}$, with equality if and only if K_S and $c_1(\mathscr{E})$ are linearly dependent over $\mathbb{Q}$.

Since our motivation is the study of pairs with nef-value $n-1$, we continue this section assuming that (X, L) is an adjunction–theoretic scroll, i.e. $K_X + (n-1)L = \pi^*\mathscr{A}$, where $\mathscr{A}$ is an ample line bundle on S. The canonical bundle formula immediately shows that

$$\mathscr{A} = K_S + c_1(\mathscr{E}),$$

hence this is equivalent to requiring that the adjoint bundle $K_S + c_1(\mathscr{E})$ is ample. This condition is satisfied except for a few classical scrolls, which we will cover in Section 5.

As to the summand Ψ on the right hand side of (9), we recall that for any classical scroll (X, L), $\mathscr{A}$ is nef by [17, Theorems 1 and 2] except when $(S, \mathscr{E}) = \big(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(1)^{\oplus 2}\big)$, in which case $\mathscr{A}^2 = 1$ and $\delta = 0$. Then we always have $\mathscr{A}^2 \ge 0$ and according to what we said before, the situations in which equality holds will be discussed in Section 5.

Now, having assumed that $\mathscr{A}$ is ample, it follows that the factor of Ψ consisting of $\mathscr{A}^2$ is strictly positive, hence $\Psi = 0$ if and only if $\delta = 0$.

From now on in this Section we will assume that

(10) $$\mathscr{E} \text{ is B-semistable, i.e.,} \delta \leq 0.$$

Under this assumption we have $\Psi \leq 0$, and equality holds if and only if $\mathscr{E}$ is properly B-semistable.

Summarizing the above discussion, if (X, L) is an adjunction theoretic scroll, and (10) holds, then we also have $\Psi \leq 0$, equality occurring only for $\delta = 0$. Thus recalling (9) we obtain

PROPOSITION 2. *Let (X, L) be an adjunction theoretic scroll over S, and assume that* (10) *holds; then $\Delta \leq 0$ with equality if and only if $\mathscr{E}$ is properly B-semistable and K_S and $c_1(\mathscr{E})$ are linearly dependent over $\mathbb{Q}$.*

According to [11, Theorem 4.1], $\mathscr{E}$ being properly B-semistable is exactly the condition ensuring that the conic G itself is the Hilbert curve of the $\mathbb{Q}$-polarized surface $\left(S, \frac{1}{\mathrm{rk}(\mathscr{E})} \det\mathscr{E}\right)$. Here is some further speculation on the vanishing of Δ.

REMARK 1. If the summand Φ of Δ and the term $\mathscr{A}^2$ (which is a factor of Ψ) are both zero, then $\Delta = 0$. In this case we have $K_S = \lambda c_1(\mathscr{E})$ for some $\lambda \in \mathbb{Q}$ and condition $\mathscr{A}^2 = 0$ reads as $(1+\lambda)^2 c_1(\mathscr{E})^2 = 0$, hence $\lambda = -1$. Therefore $\mathscr{A}$ is trivial, which prevents (X, L) from being an adjunction theoretic scroll. We will discuss this case in Section 5.

On the other hand if the summand Φ of Δ is zero and $\mathscr{E}$ is properly B-semistable, then $\Delta = 0$ as well, but from $K_S = \lambda c_1(\mathscr{E})$ with $\lambda \in \mathbb{Q}$ and the ampleness of $c_1(\mathscr{E})$ we can simply deduce that $K_S^2 \geq 0$, with equality if and only if $\lambda = 0$, i.e., when K_S is numerically trivial. We will study this case in detail shortly.

Next look at ε. Letting $\chi = \chi(\mathscr{O}_S)$, (7) becomes $\varepsilon = \frac{1}{4}(8\chi - K_S^2 - \frac{\delta}{n})$. First let us investigate the case in which G is reducible into two (possibly coinciding) parallel lines. According to the discussion that led to Proposition 1 it must be $\Delta = 0$, hence Proposition 2 implies that $\delta = 0$ and $K_S = \lambda c_1(\mathscr{E})$ for some $\lambda \in \mathbb{Q}$, up to numerical equivalence. Thus $K_S^2 = \lambda^2 c_1(\mathscr{E})^2$ and, due to the ampleness of $\mathscr{E}$, we see that $K_S^2 \geq 0$ with equality if and only if $\lambda = 0$, i.e. K_S is numerically trivial. According to the Enriques–Kodaira classification, condition $K_S^2 \geq 0$ implies that $\chi \geq 0$ (in other words, S cannot be a ruled surface over a curve of genus ≥ 2).

First suppose that $\lambda = 0$. In this case (7) shows that $\varepsilon = 2\chi$ since $K_S^2 = 0$. Moreover $\gamma = \frac{c_1(\mathscr{E})^2}{(n-1)^2}$ by (6). We thus get $\frac{\varepsilon}{\gamma} = (n-1)^2 \frac{2\chi}{c_1(\mathscr{E})^2} \geq 0$. Therefore for $\lambda = 0$ condition b) in Proposition 1 implies that $\chi \leq 0$, hence $\chi = 0$ and then it can be satisfied only when S is either an abelian or a bielliptic surface, K_S being numericall trivial. In this case, since $\varepsilon = 0$, G consists of a line over $\mathbb{Q}$ with multiplicity 2.

Next assume that $\lambda \neq 0$. Then $K_S^2 = \lambda^2 c_1(\mathscr{E})^2$ is strictly positive and this implies that

(11) S is either a rational surface or a surface of general type.

In both cases $\chi > 0$. From $\varepsilon = \frac{1}{4}(8\chi - K_S^2)$ and $\gamma = \frac{c_1(\mathscr{E})^2}{(n-1)^2}$ we get

$$\frac{\varepsilon}{\gamma} = \frac{(n-1)^2}{4}\,\frac{8\chi - \lambda^2 c_1(\mathscr{E})^2}{c_1(\mathscr{E})^2},$$

and so condition b) in Proposition 1 is satisfied if and only if $\frac{8\chi}{c_1(\mathscr{E})^2} - \lambda^2 = -s^2$ for some rational number s. Since $\lambda \neq 0$, we can also write $c_1(\mathscr{E}) = \lambda^{-1}K_S$, hence the above expression can be rewritten as

$$\frac{\varepsilon}{\gamma} = \frac{(n-1)^2\lambda^2}{4}\left(\frac{8\chi}{K_S^2} - 1\right). \tag{12}$$

Thus condition b) in Proposition 1 implies that $K_S^2 \geq 8\chi$. Moreover, $s = 0$ is equivalent to $\varepsilon = 0$, which exactly means that

$$K_S^2 = 8\chi. \tag{13}$$

Let's first suppose that $s = 0$ and recall (11). If S is rational, condition (13) simply says that S is a Segre–Hirzebruch surface $\mathbb{F}_e$ for some e. So, let $S = \mathbb{F}_e$; by using the notation as in [7, pp. 379–380], we can write $c_1(\mathscr{E}) = aC_0 + bf$ with $b > ae$ due to the ampleness and then $c_1(\mathscr{E})^2 = a(2b - ae)$. Moreover, since $K_S = -2C_0 - (2+e)f$ we have $K_S \cdot c_1(\mathscr{E}) = -2a - (2b - ae)$. We know that $\Psi = 0$ since $\delta = 0$, and then $\Delta = \frac{1}{(n-1)^2}\,\Phi$ where $\Phi = -\big(2a - (2b - ae)\big)^2$. Therefore $s = -\frac{1}{n-1}(2a + ae - 2b)$. So, condition $s = 0$ is satisfied only for $b = a(1 + \frac{e}{2})$. Since (X, L) is an adjunction theoretic scroll, by combining this with the ampleness of $\mathscr{A} = K_S + c_1(\mathscr{E})$ we get $a > 2$ and $(a-2)(1 + \frac{e}{2}) > (a-2)e$. This gives $e \leq 1$. Moreover, for $e = 0$ we get $b = a$ and then $c_1(\mathscr{E}) = \mathscr{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a, a) = \frac{a}{2}(-K_S)$; on the other hand, if $e = 1$ then a must be even since b is an integer, hence letting $a = 2\alpha$ we get $c_1(\mathscr{E}) = \alpha(2C_0 + 3f) = \alpha(-K_S)$. Furthermore, $a \geq 3$ and $\alpha \geq 2$ in the two cases respectively because $\mathscr{A}$ is ample.

On the other hand, if S is of general type then K_S is ample, being numerically equivalent to $\lambda c_1(\mathscr{E})$ (here $\lambda > 0$ necessarily). Let $\eta : S \to S_0$ be the birational morphism from S to its minimal model S_0 and suppose that η factors through t blowing-ups. Then $K_S^2 = K_{S_0}^2 - t$, hence, taking also into account the Miyaoka–Yau inequality, condition (13) implies that S_0 belongs to the region $8\chi \leq K^2 \leq 9\chi$ of the geographic plane (χ, K^2) and $t = K_{S_0}^2 - 8\chi$.

Finally suppose that $s \neq 0$. From (12) and the second condition in b) of Proposition 1 we get that $\frac{8\chi}{K_S^2} - 1$ is the opposite of the square of a non-zero rational number. This implies the strict inequality $K_S^2 > 8\chi$. Recall (11) again. If S is rational, this is impossible except when $S = \mathbb{P}^2$. On the other hand, if S is of general type, then K_S is ample, and keeping the notation as before, we have that its minimal model S_0 must satisfy the condition $8\chi < K^2_{S_0} \le 9\chi$ and the number of blowing-ups factoring $\eta : S \to S_0$ has to be small enough to preserve the bound $K^2_S > 8\chi$. Here is a summary of what we proved.

THEOREM 2. *Let (X, L) be an adjunction theoretic scroll over a smooth surface S with scroll projection $\pi : X \to S$, and suppose that the ample vector bundle $\mathscr{E} = \pi_* L$ is B-semistable. Then the Hilbert curve of (X, L) is totally reducible over $\mathbb{Q}$ with the residual conic G consisting of two parallel lines if and only if $\mathscr{E}$ is properly B-semistable and one of the following occurs:*

i) $\big(S, c_1(\mathscr{E})\big) = \big(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(a)\big)$ *for some integer $a \ge 4$;*

ii) $\big(S, c_1(\mathscr{E})\big) = \big(\mathbb{P}^1 \times \mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1\times\mathbb{P}^1}(\nu, \nu)\big)$ *for some integer $\nu \ge 3$, or* $\big(\mathbb{F}_1, \nu(-K_{\mathbb{F}_1})\big)$ *for some integer $\nu \ge 2$;*

iii) S is either an abelian or a bielliptic surface;

iv) S is a surface of general type with ample canonical bundle whose minimal model S_0 satisfies $8\chi \le K^2_{S_0} \le 9\chi$ and the birational morphism $S \to S_0$ factors through $K^2_{S_0} - 8\chi$ blowing-ups;

v) S is a surface of general type with ample canonical bundle whose minimal model S_0 satisfies $8\chi < K^2_{S_0} \le 9\chi$ and $\frac{8\chi}{K^2_S} - 1$ is the opposite of the square of a non-zero rational number.

In particular, note that the two lines constituting G coincide in cases ii) – iv) because $\varepsilon = 0$ too; on the other hand in case i), (12) gives $\frac{\varepsilon}{\gamma} = -\big(\frac{n-1}{2a}\big)^2$.

Now, still under the hypothesis (10), let us deal with the case where G is reducible into two transverse lines. In view of Theorem 2 we can suppose that $\Delta \ne 0$. According to Proposition 1 we need to analyze when $\varepsilon = 0$. By (7) this condition is equivalent to

$$K^2_S - 8\chi = -\frac{\delta}{n}, \tag{14}$$

which implies $K^2_S - 8\chi \ge 0$, in view of (10). Now, if $S = \mathbb{P}^2$, then $K^2_S = 9 = 8\chi + 1$, hence $\varepsilon = -\frac{1}{4}(1 + \frac{\delta}{n}) = 0$ if and only if $\delta = -n$, in particular this implies that $\mathscr{E}$ is B-stable. On the other hand, since we are on $\mathbb{P}^2$, we know that $\Phi = 0$. Moreover, $\Psi = \frac{\delta}{n}\mathscr{A}^2 = -\mathscr{A}^2$, hence letting $c := \deg c_1(\mathscr{E})$, we get $\mathscr{A}^2 = \big(\mathscr{O}_{\mathbb{P}^2}(c-3)\big)^2 = (c-3)^2$. Therefore the second condition in a) of Proposition 1 is satisfied by taking $s = \frac{c-3}{n-1}$.

Suppose $S \ne \mathbb{P}^2$ and let $\eta : S \to S_0$ be a birational morphism from S to either its minimal model or to a $\mathbb{P}^1$-bundle (in case S has negative Kodaira dimension), and let $t \ge 0$ be the number of blowing-ups η factors through. Then $K^2_S = K^2_{S_0} - t$, so if S_0 satisfies $K^2_{S_0} \le M\chi$ for some positive M, then, a fortiori, $K^2_S \le M\chi$ as well. Let us proceed in our analysis according to the Kodaira dimension $\kappa(S)$ of S.

If $\kappa(S) = -\infty$, then $K_S^2 = 8\chi - t \le 8\chi$, hence equality holds if and only if $S = S_0$ and in this case $\varepsilon = -\frac{\delta}{4n}$. So $\varepsilon = 0$ if and only if S is a $\mathbb{P}^1$-bundle and $\delta = 0$, i.e. $\mathscr{E}$ is properly B-semistable. Now look at Δ. Since $\delta = 0$ we have $\Psi = 0$, hence $\Delta = \frac{1}{(n-1)^2}\Phi$. Use notation as in [7, p. 373] again. Recalling that $K_S = -2C_0 + (2q-2-e)f$ up to numerical equivalence, where C_0 is a fundamental section of minimal self-intersection $-e$ and q is the genus of B, and writing $c_1(\mathscr{E}) = aC_0 + bf$, we get

$$\begin{aligned} \Phi &= K_S^2 c_1(\mathscr{E})^2 - \big(K_S \cdot c_1(\mathscr{E})\big)^2 \\ &= 8a(1-q)(2b-ae) - \big(2a(q-1) - (2b-ae)\big)^2 \\ &= -\big(2a(q-1) + (2b-ae)\big)^2. \end{aligned} \tag{15}$$

Thus the second condition in a) of Proposition 1 is satisfied, with $s = \frac{1}{n-1}\big(2a(q-1) + (2b-ae)\big)$, provided that it is not zero. It is immediate to check that $s = 0$ if and only if $\big(S, c_1(\mathscr{E})\big)$ is as in ii) of Theorem 2.

If $\kappa(S) = 0$, then $K_S^2 - 8\chi \le 0$, which contradicts (14), in view of (10), unless $S = S_0$ is either an abelian or a bielliptic surface; in these cases $K_S^2 = 8\chi = 0$, hence $\varepsilon = -\frac{\delta}{4n}$ again. In particular, $\varepsilon = 0$ if and only if $\mathscr{E}$ is properly B-semistable. Since Φ and Ψ are both zero, we get $\Delta = 0$, a contradiction.

If $\kappa(S) = 1$, then $K_S^2 - 8\chi \le 0$ with equality if and only if $S = S_0$ is an elliptic quasi-bundle in the sense of Serrano [16], and in this case $\varepsilon = -\frac{\delta}{4n}$ again. Once more, $\varepsilon = 0$ if and only if $\mathscr{E}$ is properly B-semistable. In this case $\Psi = 0$, but $\Phi = -\big(K_S \cdot c_1(\mathscr{E})\big)^2 < 0$, since $K_S^2 = 0$, $c_1(\mathscr{E})$ is ample and a multiple of K_S is effective. Then $\Delta = -\frac{1}{(n-1)^2}\big(K_S \cdot c_1(\mathscr{E})\big)^2$, hence the second condition in a) of Proposition 1 is satisfied with $s = \frac{1}{n-1} K_S \cdot c_1(\mathscr{E}) \neq 0$.

Before moving on to surfaces of general type, we collect the results of the previous discussion in a statement.

PROPOSITION 3. *Let (X, L) be an adjunction theoretic scroll over a smooth surface S of Kodaira dimension ≤ 1 with scroll projection $\pi : X \to S$, and suppose that the ample vector bundle $\mathscr{E} = \pi_* L$ is B-semistable. Then the Hilbert curve of (X, L) is totally reducible over $\mathbb{Q}$ with the residual conic G consisting of two transverse lines if and only if one of the following occurs:*

- *j) $S = \mathbb{P}^2$ and $\mathscr{E}$ is B-stable;*
- *jj) S is a $\mathbb{P}^1$-bundle over a smooth curve, $\mathscr{E}$ is properly B-semistable and $(S, \mathscr{E})$ is not as in* ii) *of Theorem 2;*
- *jjj) S is an elliptic quasi-bundle and $\mathscr{E}$ is properly B-semistable.*

Finally, let $\kappa(S) = 2$; recalling the meaning of t we get $K_{S_0}^2 = K_S^2 + t \ge 8\chi$ in view of (14) and (10). Recalling also the Miyaoka–Yau inequality, we thus conclude that S_0 must satisfy the conditions $8\chi \le K_{S_0}^2 \le 9\chi$. We have $K_{S_0}^2 = 8\chi + t - \frac{\delta}{n}$ by (14) again. As a consequence, $t = K_{S_0}^2 - 8\chi + \frac{\delta}{n}$ and the Miyaoka–Yau inequality shows that $t \le \chi + \frac{\delta}{n} \le \chi$. In particular, if $t = \chi$, then $K_{S_0}^2 = 9\chi$ and $\mathscr{E}$ is properly B-semistable: so, if $t = \chi$, then $\Psi = 0$, but $\Phi < 0$, since $\Delta \neq 0$. On the other hand it seems very difficult to squeeze out further information on $(S, \mathscr{E})$ from the second condition in a) of Proposition 1. This fact invalidates the attempt to summarize the situation in a meaningful statement for $\kappa(S) = 2$. However, here is a non-obvious example.

Example. Let S be a surface with ample cotangent bundle; there is a fairly extensive literature concerning these surfaces. Set $\mathscr{E} := \Omega_S^1$, the cotangent bundle of S. Then $K_S = c_1(\mathscr{E})$. In this case (X, L) is an adjunction theoretic scroll over a surface of general type with ample canonical bundle and $n = 3$. Moreover, $\Phi = 0$, and by using Noether's formula we see that $\delta = 5K_S^2 - 48\chi$. Then $\delta \le -3\chi$, due to the Miyaoka–Yau inequality, hence $\delta < 0$ (so $\mathscr{E}$ is B-stable). Thus $\Delta = \frac{1}{4}\Psi = \frac{1}{3}K_S^2\delta < 0$ and $\varepsilon = \frac{2}{3}(9\chi - K_S^2)$. Therefore $\varepsilon = 0$ if and only if $K_S^2 = 9\chi$. As is known, non-ruled surfaces satisfying this equality are quotients of the unit ball of $\mathbb{C}^2$. So, suppose that S is such a surface. Then Ω_S^1 is actually ample by a result of Miyaoka [15]. Note that in this case we get $\Delta = -(3\chi)^2$. Hence, according to a) in Proposition 1 we conclude that G is reducible into two transverse lines.

5. The case of classical non-adjunction theoretic scrolls

Here we enlarge the view including scrolls over S which fail to be adjunction theoretic in our discussion. According to [5, Main theorem] the pairs $(S, \mathscr{E})$ giving rise to such scrolls are those in the following list (which we will refer to as list $(*)$ in the sequel):

1) $(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(1)^{\oplus 3})$,
2) $(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(1)^{\oplus 2})$,
3) $(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(2) \oplus \mathscr{O}_{\mathbb{P}^2}(1))$,
4) $(\mathbb{P}^2, T_{\mathbb{P}^2})$,
5) $(\mathbb{P}^1 \times \mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)^{\oplus 2})$, and
6) $(S, \mathscr{E})$, where S is a $\mathbb{P}^1$-bundle over a smooth curve B and $\mathscr{E}_f = \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus 2}$ for every fiber $f \cong \mathbb{P}^1$.

Note that 5) also fits into case 6). Moreover, notice that pairs 3), 4), 5) fall within cases (2), (1), (3) in [14, p. 17] respectively, setting $m = 2$. So, the conclusion we will obtain concerning these three pairs could follow from the analysis of the Hilbert polynomials made in [14, §5]. However, due to the diversity of the approach, we prefer to discuss all six cases in the list $(*)$ from the unifying point of view adopted in this article.

First of all note that $\delta = 0$ in cases 1), 2), 5) of the list $(*)$, as well as in case 6), by [10, Proposition 3.5]. On the other hand, $\delta = 1$ in case 3) and $\delta = -3$ in case 4).

Moreover, in all cases of the list $(*)$ the adjoint bundle $\mathscr{A} := K_S + c_1(\mathscr{E})$, introduced in Section 4, is not ample. In particular, $\mathscr{A} = \mathscr{O}_S$ in cases 1) and 3) – 5), which means that $K_S = -c_1(\mathscr{E})$, hence $\Phi = 0$; in addition, since $\mathscr{A}^2 = 0$, we also have $\Psi = 0$ and then $\Delta = 0$ by (9). In case 2) $\mathscr{A}^2 = 1$; however $\delta = \Phi = 0$. Thus $\Delta = 0$ also in case 2). Now look at case 6). Since $\mathscr{E}_f = \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus 2}$ we know that $\mathscr{A} = K_S + c_1(\mathscr{E})$ is a linear combination of fibers (although $\mathscr{A} \neq \mathscr{O}_S$, in general). As a consequence, $\mathscr{A}^2 = 0$, so that $\Psi = 0$. When is it also $\Phi = 0$ or, equivalently, $\Delta = 0$? The answer is provided by the following more general result.

PROPOSITION 4. *Let $(S, \mathscr{E})$ be as in* 6) *of the list* $(*)$. *Then $\Delta = -s^2$ for some integer s. Moreover, $s = 0$ if and only if $(S, \mathscr{E}) = \left(\mathbb{P}^1 \times \mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)^{\oplus 2}\right)$ (namely in case* 5)*).*

Proof: Keeping the notation as in [7, p. 373], the fact that $\pi : S \to B$ is a $\mathbb{P}^1$-bundle implies that $K_S = -2C_0 + (2q - 2 - e)f$, up to numerical equivalence. On the other hand, $c_1(\mathscr{E}) = 2C_0 + \beta f$ for some integer β, up to numerical equivalence again, and the ampleness of $c_1(\mathscr{E})$ implies (see [7, p. 382])

$$\beta > \begin{cases} 2e & \text{if } e \geq 0 \\ e & \text{if } e < 0 \text{ (in which case } q > 0). \end{cases}$$

Specializing (15) by setting $a = 2$ and $b = \beta$ we see that $\Phi = -4(2q - 2 + \beta - e)^2$. Recall that $\delta = 0$, hence $\Delta = \frac{1}{4}\Phi$ since $\mathscr{E}$ has rank 2. Then, letting $s := 2q - 2 + \beta - e$ we obtain $\Delta = -s^2$. Now, the ampleness conditions above show that $s \geq 0$ (note that for $e = q = 0$ we have $\beta > 1$ because $\mathscr{E}$ itself is ample) and this is a strict inequality unless $q = 0$, in which case $\beta = e + 2$. So, if $s = 0$, then $S = \mathbb{F}_e$ and $c_1(\mathscr{E}) = -K_S$. Hence $c_1(\mathscr{E})^2 = 8$. Then the assertion follows from [8, Theorem 2.5].

Recalling what we said before about the other pairs in the list $(*)$ we get

COROLLARY 1. *Let $(S, \mathscr{E})$ be as in the list* $(*)$. *Then $\Delta = 0$ exactly for all pairs* 1) – 5).

As a consequence, for all (X, L) corresponding to these pairs, the conic G is reducible over $\mathbb{C}$. To decide about the reducibility over $\mathbb{Q}$ we have to look at $\frac{\varepsilon}{\gamma}$ according to b) in Proposition 1.

Recalling what we said about δ, and noting that in case 6) of the list $(*)$, $K_S^2 = 8(1 - q) = 8\chi(\mathscr{O}_S)$, a straightforward computation shows that the values of ε are $-\frac{1}{4}$ in cases 1) and 2), $-\frac{1}{3}$ in case 3), and 0 in cases 4) – 6). In particular, it turns out that G is reducible over $\mathbb{C}$ also in case 6). In passing let us note that G is also reducible over $\mathbb{R}$ in all cases, according to what we observed at the end of Section 2. On the other hand, $\gamma = 1$ in cases 1) and 2) and $\frac{7}{3}$ in case 3). Therefore the condition implying that G is reducible over $\mathbb{Q}$, namely the second condition in b) of Proposition 1, is satisfied in all cases except 3) of the list $(*)$. In particular, G consists of two parallel lines in cases 1) and 2), of a single line with multiplicity 2 in cases 4) and 5), and of two transverse lines in case 6), apart from 5).

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