Rendiconti Sem. Mat. Univ. Pol. Torino
Vol. 82, 1 (2023), 201 – 208

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VARIETIES WITH ULRICH TWISTED NORMAL, CONORMAL OR TANGENT BUNDLES

Dedicated to the memory of Gianfranco Casnati[2]

Abstract. This is an expanded version of the talk given at the conference *A workshop on Geometry and Commutative algebra-In occasion of the 60th birthday of Gianfranco Casnati* given at Politecnico di Milano on September 4, 2023.

1. Introduction.

To introduce Ulrich vector bundles, I often like to borrow a phrase of Beauville [3]: Ulrich bundles have had three lives, in the 80's in commutative algebra, entered algebraic geometry in 2003 with the seminal paper of Eisenbud-Schreyer [8] and then proliferated in recent years.

Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible complex variety of dimension $n \geq 1$ and codimension c. A vector bundle $\mathscr{E}$ on X is called Ulrich if it satisfies one of the following equivalent conditions [3, 8]:

1. $H^i(\mathscr{E}(-p)) = 0$ for all $i \geq 0$ and $1 \leq p \leq n$.
2. There is a linear resolution
   $0 \to \mathscr{O}_{\mathbb{P}^N}(-c)^{\oplus \beta_c} \to \dots \mathscr{O}_{\mathbb{P}^N}(-1)^{\oplus \beta_1} \to \mathscr{O}_{\mathbb{P}^N}^{\oplus \beta_0} \to \mathscr{E} \to 0.$
3. $\pi_* \mathscr{E}$ is trivial for any finite linear projection $\pi : X \to \mathbb{P}^n$.

The presence of such bundles often gives interesting consequences on the geometry of X and on the cohomology of sheaves on X. As a sample, whenever X carries an Ulrich bundle, its Chow form has a particularly simple presentation and the cone of cohomology tables of X is the same as the one of $\mathbb{P}^n$ (for these and more, see for example in [3, 7, 8] and references therein).

Perhaps the most challenging question in these matters is whether every $X \subseteq \mathbb{P}^N$ carries an Ulrich vector bundle (see for example [8, page 543]). This is only known for curves, in many cases for surfaces and more sporadically in higher dimension. This difference between curves and other varieties will also show up later.

It comes therefore very natural to ask if usual vector bundles associated to X, such as $T_X, \Omega_X, N_X, N_X^*$, can be Ulrich[3].

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[1] Research partially supported by PRIN "Advances in Moduli Theory and Birational Classification", GNSAGA-INdAM and the MIUR grant Dipartimenti di Eccellenza 2018-2022.

[2] A personal note is written in Section 7.

[3] This problem was considered, more generally for instanton bundles, by G. Casnati in [6].

Since Ulrich vector bundles are globally generated, it is better to consider twisted versions. We will therefore ask: *for which integers k one has that $T_X(k), \Omega_X(k), N_X(k), N_X^*(k)$ is an Ulrich vector bundle?*

A basic remark is in order: Chern classes of Ulrich bundles $\mathscr{E}$ satisfy strong numerical constraints given by Riemann-Roch and the vanishings $\chi(\mathscr{E}(-p)) = 0$ for $1 \le p \le n$. In particular, if $H \in |\mathscr{O}_X(1)|$, we have that

$$c_1(\mathscr{E})H^{n-1} = \frac{\mathrm{rk}(\mathscr{E})}{2}(K_X + (n+1)H)H^{n-1}. \tag{1}$$

This gives, if existing, only one possible value of k in the above question.

2. Varieties with $N_X(-k)$ Ulrich.

As above, we have $X \subset \mathbb{P}^N$, which we assume of codimension $c \ge 1$ and we denote its normal bundle by $N_X = N_{X/\mathbb{P}^N}$.

Since $N_X(-1)$ is globally generated, the natural question to be asked is: for which $k \ge 1$ we have that $N_X(-k)$ is an Ulrich vector bundle?

Certainly, there are some easy examples. If X is a linear space in $\mathbb{P}^N$, then $N_X(-1) \cong \mathscr{O}_{\mathbb{P}^n}^{\oplus c}$ is clearly an Ulrich vector bundle on X. Also, if $X \subset \mathbb{P}^3$ is a curve such that $H^0(N_X(-2)) = 0$, then also $H^1(N_X(-2)) = 0$, since $\chi(N_X(-2)) = 0$. Thus again $N_X(-1)$ is an Ulrich vector bundle. This is actually an occurrence of a bit less trivial family of examples, namely n-dimensional varieties $X \subset \mathbb{P}^{n+2}, 1 \le n \le 3$ such that $H^j(N_X(-2-j)) = 0$ for $0 \le j \le n-1$. Interesting examples of this type, but not all of them, are linear standard determinantal curves in $\mathbb{P}^3$, surfaces in $\mathbb{P}^4$ and threefolds in $\mathbb{P}^5$ (see [11] and also [10, Thm. 3.6]).

As it turns out, for varieties of degree at least 2 with $N_X(-k)$ Ulrich, (1) is equivalent to $k = 1$ and $c = 2$. And in fact, the above are the only examples, as the ensuing result shows (the very ampleness statement is an application of [13]):

THEOREM 1. *[11, Thm. 1]*

Let $X \subset \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \ge 1$ and let k be an integer. Then $N_X(-k)$ is an Ulrich vector bundle if and only if $k = 1$ and X is one of the following:

(i) *$X = \mathbb{P}^n$ embedded linearly in $\mathbb{P}^N$, or*

(ii) $1 \le n \le 3, N = n+2$ *and $X \subset \mathbb{P}^{n+2}$ is a variety such that $N_X(-1)$ is 0-regular, or, equivalently, with $H^j(N_X(-j-2)) = 0$ for $0 \le j \le n-1$.*

Moreover, in the latter case, if X does not contain a line, then $N_X(-1)$ is very ample.

We remark that the family of curves $X \subset \mathbb{P}^3$ satisfying $H^0(N_X(-2)) = 0$, thus giving examples of curves as in (ii) of Theorem 1, is very large (see for example [2,9]) and there is not much hope to classify them. On the other hand, for $n = 2, 3$, the only examples we know of varieties as in (ii) of Theorem 1 are linear standard determinantal schemes, that is smooth $X \subset \mathbb{P}^{n+2}$ with resolution

$$0 \to \mathcal{O}_{\mathbb{P}^{n+2}}(-s-1)^{\oplus s} \to \mathcal{O}_{\mathbb{P}^{n+2}}(-s)^{\oplus(s+1)} \to \mathcal{J}_{X/\mathbb{P}^{n+2}} \to 0.$$

3. Varieties with $N_X^*(k)$ Ulrich.

We now consider $X \subset \mathbb{P}^N$, which we assume of codimension $c \geq 1$ and we denote its conormal bundle by $N_X^* = N_{X/\mathbb{P}^N}^*$. We thus ask: for which k we have that $N_X^*(k)$ is an Ulrich vector bundle?

A first simple consequence can be drawn: if X is degenerate, then $N_X^*(k)$ is Ulrich if and only if $(X, H, k) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), 1)$.

On the other hand, suppose that X is nondegenerate. While in other cases (see Sections 2, 4) examples of surfaces and threefolds appeared, we find a very different result for the conormal bundle. In fact, we show that the answer to the above question is negative in dimension at least two.

THEOREM 2. *[1, Thm. 1]*

Let $X \subset \mathbb{P}^N$ be a smooth nondegenerate variety such that $N_X^*(k)$ is Ulrich. Then X is a curve.

Now, for curves the situation is wide. First of all, there are many examples, at least in $\mathbb{P}^3$, stemming from some classical works (see [1, Examples 8.1 and 8.2]).

We first prove that there is a bound (sharp in codimension 2) for the degree of a curve having Ulrich twisted conormal bundle.

THEOREM 3. *[1, Thm. 2]*

Let $X \subset \mathbb{P}^{c+1}$ be a smooth nondegenerate curve of degree d and codimension $c \geq 1$ such that $N_X^*(k)$ is Ulrich. Then $c \geq 2$ and

$$d \geq \frac{c+2}{2k+c}\binom{k+c}{c+1}.$$

Moreover this bound is sharp for $c = 2$ and $k \equiv 1, 3 \pmod 6$.

On the other hand, the examples mentioned above, are all subcanonical curves in $\mathbb{P}^3$. In fact, neither the fact of being subcanonical, nor of lying in $\mathbb{P}^3$, is a necessary condition, as we have examples, for unbounded genus, of non-subcanonical curves in $\mathbb{P}^3$ and in $\mathbb{P}^4$:

THEOREM 4. *[1, Thm. 3]*

(i) Let $X \subset \mathbb{P}^3$ be a general nonspecial curve of genus g and degree $d = 2g-2$. Then $N_X^*(4)$ is Ulrich and X is not subcanonical.

(ii) Let $X \subset \mathbb{P}^4$ be a general curve of genus $g \geq 3$ and degree $d = 5g-5$. Then $N_X^*(3)$ is Ulrich and X is not subcanonical.

4. Varieties with $T_X(k)$ Ulrich.

As above we have $X \subset \mathbb{P}^N$ of dimension $n \geq 1$ and we denote its tangent bundle by T_X. We thus ask: for which k we have that $T_X(k)$ is an Ulrich vector bundle?

By a well-known result [15], [16], $H^0(T_X(-1)) = 0$ unless $(X, \mathcal{O}_X(1)) = (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2)), (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. Since Ulrich bundles are globally generated, if $T_X(k)$ is Ulrich, then $k \geq 0$ unless $(X, \mathcal{O}_X(1), k) = (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1), -2)$ (and in the latter case $T_{\mathbb{P}^1}(-2)$ is Ulrich).

In the case $k = 0$, a recent result [5] gives a classification:

T_X is Ulrich if and only if $(X, \mathcal{O}_X(1)) = (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3)), (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$.

A new and simple proof of this is given also in [6] and in [12].

On the other hand, for $k \geq 1$, the question is more subtle and presents several intriguing questions and differences from the cases above.

In the case of curves, one sees that $k = 1$ is not possible, while the cases $k = 2, 3$ can be dealt with on any curve (see [12]). The novelty here is that, there is a sharp bound for k, showing also that for curves k can be as large as wanted.

THEOREM 5. *[12, Thm. 1]*

Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible curve of genus g. If $T_X(k)$ is an Ulrich line bundle, then

$$k \leq \frac{\sqrt{8g+1}-1}{2} \tag{1}$$

and equality holds if and only if k is even and either X is a curve of degree 6 and genus 3 lying on a smooth cubic or X is a curve of type $(\frac{k}{2}+1, k+2)$ on a smooth quadric. Also, in both cases, $T_X(k)$ is an Ulrich line bundle, hence the bound is sharp for every even $k \geq 0$. Moreover, if X has general moduli, then $k \leq 4$.

As far as we know, only curves show this kind of behavior, meaning that k is not bounded in terms of the dimension (a somewhat bad bound can also be given in terms of the degree, see [12]). As supporting evidence, we have:

THEOREM 6. *[12, Thm. 2]*

Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension n such that $2 \leq n \leq 12$. If $T_X(k)$ is an Ulrich vector bundle, then $k \leq n+1$.

We point out that, for $n \geq 2$, we know no examples with $k \geq 2$ and only one example with $k = 1$. As a matter of fact, the case $k = 1$ can be completely characterized, as follows:

THEOREM 7. *[12, Thm. 3]*

Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \geq 1$. Then $T_X(1)$ is an Ulrich vector bundle if and only if $(X, \mathcal{O}_X(1)) = (S_5, -2K_{S_5})$, where S_5 is a Del Pezzo surface of degree 5.

Moreover, for $k = 2$, we have

THEOREM 8. *[12, Thm. 4]*
Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \geq 4$. Then $T_X(2)$ is not an Ulrich vector bundle.

We do not know what happens for $k = 2, n = 3$, even though some evidence suggests that it might not be possible.

For surfaces, we can prove:

PROPOSITION 1. *[12, Prop. 6.2]*
Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible surface of degree d, $H \in |\mathcal{O}_X(1)|$. If $T_X(k)$ is an Ulrich vector bundle, the following hold:

(i) $0 \leq k \leq 3$.

Moreover, either

(ii) $k = 0$ *and* $(X, \mathcal{O}_X(1)) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$, *or*

(iii) $k = 1$ *and X is a Del Pezzo surface of degree* 5, *or*

(iv) $k = 2, q = 0$ *and X is a minimal surface of general type, or*

(v) $k = 3$, *X is a minimal surface of general type with* $2K_X \equiv 3H, K_X^2 = \frac{9d}{4}, \chi(\mathcal{O}_X) = \frac{d}{4}$. *Moreover X is a ball quotient.*

While the cases (ii) and (iii) actually occur, see Section 6 for cases (iv) and (v).

5. Varieties with $\Omega_X(k)$ Ulrich.

Consider a smooth irreducible variety $X \subseteq \mathbb{P}^N$. We ask if $\Omega_X(k), \Omega_{\mathbb{P}^N|X}(k)$ and $T_{\mathbb{P}^N|X}(k)$ can be Ulrich vector bundles for some integer k. In these cases it is not very difficult to prove the following:

PROPOSITION 2. *[11, Prop. 4.1]*
Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \geq 1$ and let k be an integer. Then

(i) $\Omega_X(k)$ *is an Ulrich vector bundle if and only if $k = 2$ and X is a line.*

(ii) $\Omega_{\mathbb{P}^N|X}(k)$ *is an Ulrich vector bundle if and only if $k = 2$ and X is a rational normal curve in $\mathbb{P}^N$.*

(iii) $T_{\mathbb{P}^N|X}(k)$ *is an Ulrich vector bundle if and only if either $N = 2, k = -1$ and X is a conic or $N = 1, k = -2$ and $X = \mathbb{P}^1$.*

6. Open questions.

All of the above results point out to the possible existence, that needs to be further investigated, of some special varieties $X \subset \mathbb{P}^N$.

Question 1. Is there a smooth $X \subset \mathbb{P}^{n+2}$ of dimension $n = 2, 3$ such that $N_X(-1)$ is 0-regular, and X is not a linear determinantal variety?

We have checked the classification of surfaces in $\mathbb{P}^4$ of degree less than 11 and the answer is no.

Question 2. Is there a minimal surface of general type X with $q = 0$ and $T_X(2)$ Ulrich?

A possible example would be a surface X with $2K_X$ very ample, $H^1(T_X) = 0$ and $K_X^2 = 5\chi(\mathcal{O}_X)$. We do not know if such surfaces exist.

Another interesting question, related to Ulrich twisted tangent bundles, arises for Fano varieties. As a matter of fact, if $X \subseteq \mathbb{P}^N$ is a smooth variety of dimension $n \geq 3$ with $T_X(k)$ Ulrich, then, by the above results, we see that $k \geq 2$. Thus it follows that $H^0(T_X) = 0$ and, as Ulrich bundles are aCM, also $H^1(T_X) = 0$.

Going through the classification of Fano threefolds one sees that the condition $H^0(T_X) = H^1(T_X) = 0$ is not possible [4, Appendix A]. On the other hand, there are examples of Fano varieties of even dimension such that $H^0(T_X) = H^1(T_X) = 0$ [14], but none of them has $T_X(k)$ Ulrich.

Question 3. Is there a Fano variety X of odd dimension $n \geq 5$ with $H^0(T_X) = H^1(T_X) = 0$?

7. A personal note about Gianfranco Casnati.

I first met Gianfranco while I was a graduate student at Brown University in the late ’80’s. We shared a room in a hotel for a conference. After that we met briefly several times, but we did not really familiarize. In 2017, I attended a very nice talk by him on Ulrich vector bundles. I got very interested and he was very welcoming and open to discuss and explain everything to someone that did not know much. I learned back then what kind of very good and elegant mathematician he was. And also a very pleasant person to be with. We started a collaboration over the following years, I visited Torino a few times and he made my stay very enjoyable, both for work and for the rest. I still find it very hard to believe that he is not with us anymore. But this is only physical, his friendship and legacy will stay forever within me and, I believe, within all the people he knew.

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AMS Subject Classification: 14J60, 14F06, 14J40.

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Lavoro pervenuto in redazione il 02.19.2024.