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FAMILIES OF CRITICAL LOCI IN MULTIVIEW GEOMETRY

Dedicated to the memory of Gianfranco Casnati

Abstract. Linear projections from $\mathbb{P}^k$ to $\mathbb{P}^h$ model pinhole cameras in the context of Computer Vision or Multiview Geometry. It is well known that, given two sets of n projections $P_1,\dots,P_n$ and $Q_1,\dots,Q_n$, there exist sets of points that have the same images when projected from the two different sets of projection. Such points fill the so–called critical locus for the reconstruction problem for the two sets of projections. In the present paper, we address the problem of describing the critical loci that arise when we keep fixed $Q_1,\dots,Q_n$ and we allow $P_1,\dots,P_n$ to vary. In particular, we construct a suitable space that parameterizes the projections $P_1,\dots,P_n$, provide an embedding of such space into a suitable Grassmann variety, and construct a map from that space to the Hilbert scheme of closed subschemes in $\mathbb{P}^k$. The subscheme of the Grassmannian corresponding to projections for which the critical locus is the whole $\mathbb{P}^k$ is completely characterized, while the fibers of the map above are studied in the case of two projections.

1. Introduction

The process of modeling photos capturing static three-dimensional scenes from pinhole cameras typically involves linear projections from $\mathbb{P}^3$ to $\mathbb{P}^2$. Similarly, in computer vision, linear projections from $\mathbb{P}^k$ to $\mathbb{P}^h$, are utilized to describe videos or images of dynamic and segmented scenes. Consequently, a "camera" in this context can be identified with a linear projection $P:\mathbb{P}^k \dashrightarrow \mathbb{P}^h$.

In this setting, the *reconstruction* problem can be stated as follows: given a set of points in $\mathbb{P}^k$ with unknown coordinates, referred to as the "scene," and n images of it in n target spaces $\mathbb{P}^{h_i}$, $i = 1,\dots,n$, captured by unknown cameras, the objective is to recover the positions of the cameras and scene points in the ambient space $\mathbb{P}^k$.

Corresponding points in the target images represent images of the same point in the scene. In principle, having enough images along with corresponding points in those images should allow a successful projective reconstruction. However, there are sets of points in the ambient space $\mathbb{P}^k$, for which the projective reconstruction fails. These point configurations are called "critical," indicating that there exist alternative sets of points and cameras, not projectively equivalent, that produce identical images in the target spaces.

Critical loci are algebraic varieties, specifically in the class of determinantal varieties, and have been extensively studied by numerous authors. Consequently, there exists a broad literature dedicated to this subject. The classical case of projections from $\mathbb{P}^3$ to $\mathbb{P}^2$ is analysed in [12, 14, 15, 18–21]. The case of projections onto $\mathbb{P}^2$

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from higher dimensional spaces is investigated first in [2,3,9]. Later, in [1,4–6] the study of the ideal of critical loci has been formalized making use of the so-called Grassmann tensor introduced in [16]. For the definition of critical locus, see section 2.

In this paper, we approach the analysis of critical loci from a different perspective. Specifically, previous investigations have focused on studying the ideal of critical loci with fixed pairs of projection matrices $P_1,\ldots,P_n$ and $Q_1,\ldots,Q_n$. Indeed, to fully describe the equations of the critical locus, knowledge of both the Q_i's and the P_i's is necessary.

From both a theoretical and a practical point of view, it can be useful to fix only the projections $Q_1,\ldots,Q_n$, and to let the projections $P_1,\ldots,P_n$ vary. In practice, one might only have knowledge of the positions where the cameras are placed, denoted as $\overline{Q}_1,\ldots,\overline{Q}_n$. The objective then becomes determining the sets of points in the scene that should be avoided to prevent encountering critical configurations for any choice of $P_1,\ldots,P_n$. In essence, the goal is to identify all possible critical loci associated with pairs of n projections $P_1,\ldots,P_n$ and $\overline{Q}_1,\ldots,\overline{Q}_n$. For instance, in the simplest scenario of two projections from $\mathbb{P}^3$ to $\mathbb{P}^2$, it is enough to consider a set of points that is not contained in any of the quadrics passing through the two projection centers of $\overline{Q}_1$ and $\overline{Q}_2$. This ensures that the set is non-critical for any choice of P_1 and P_2.

Hence the analysis translates into the problem of describing all possible critical loci $\mathscr{X}$, as $P_1,\ldots,P_n$ vary. Of course, we get families of critical loci $\mathscr{X}$ depending of the projections $P_1,\ldots,P_n$, and a map from the space of projections to a space of critical loci.

By increasing the generality of the previous problem, one can describe all possible critical loci, as both $P_1,\ldots,P_n$ and $Q_1,\ldots,Q_n$ vary. The guess for both this and the previous item, is that the possible critical loci belong to some given family of projective varieties, arising from the numerical constrains of the data.

When dealing about families of algebraic varieties, the natural space where to consider them is the Hilbert scheme. However, by describing the state of art on critical loci, it becomes evident that numerical data are not enough to completely list the possible structure of schemes that happen to be critical loci. So, we are forced to consider the full Hilbert scheme that parameterizes all closed subschemes in $\mathbb{P}^k$. On the other hand, the projections $P_1,\ldots,P_n$ form a space and we show that it is possible to embed this space into a suitable Grassmann variety, providing then a map from this space / Grassmannian to the Hilbert scheme. The problem of describing the critical loci when $P_1,\ldots,P_n$ vary splits then in two different problems: the first is to characterize the subschemes of the Grassmann variety for which the critical locus does not change, i.e. the fibers of the map, while the second is to characterize the image of the map itself. We completely characterize the subscheme corresponding to projections for which the critical locus is the whole $\mathbb{P}^k$ in terms of the geometry of the Grassmann variety. In details, we prove that, in such a case, the points of the Grassmannian associated to the projections $(P_1,\ldots,P_n)$ fill either suitable Schubert subvarieties, or suitable (projections of) Veronese varieties.

Moreover, we give a full description of the fiber of the map over a general point in the case of two projections with critical locus a hyperquadric in $\mathbb{P}^k$. In this setting, we show that the fibre is a conic in the general case, and we describe also the exceptional situations in which the fibre consists either of two conics, or of a surface. This last two cases can occur only if the points in the Grassmannian associated to the two couples of projections fulfil some extra relations that we explicitly determine.

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The paper is structured as follows: in section 2, we introduce the setting of multiple view geometry and we recall the definition of the critical locus, the construction of its defining ideal, and some of its properties. In section 3, we provide an overview on some critical loci already known in literature, but we adopt a point of view that emphasizes their family structures. Section 4 is devoted to the construction of the space of projections $P_1,\ldots,P_n$, and of the map from this space to the full Hilbert scheme of $\mathbb{P}^k$. We show that the map above factors as an embedding of the parameter space into a suitable Grassmann variety, and a map from the Grassmannian to the Hilbert scheme. The rest of the section is devoted to the study of the case when the critical locus associated to the two n–tuples of projections is $\mathbb{P}^k$. To show difficulties and potentialities of the study of families of critical loci, in section 5 we present a careful analysis of the simplest case, namely two projections from $\mathbb{P}^k$ to $\mathbb{P}^{h_1},\mathbb{P}^{h_2}$ with critical locus either a hyperquadric or the whole $\mathbb{P}^k$. We prove that, given a critical locus $\mathscr{X}$ associated to the projections $\overline{P}_1,\overline{P}_2$ and $\overline{Q}_1,\overline{Q}_2$, there can be either a conic, or two conics, or a surface in the Grassmann variety parameterizing couples P_1,P_2 for which the critical locus is the same hyperquadric $\mathscr{X}$.

2. Multiview Geometry

In this section, we recall some standard facts and definitions from Computer Vision, or better, Multiple View Geometry.

The standard mathematical model of *camera*, is a linear projection P from $\mathbb{P}^k$ onto $\mathbb{P}^h$, from a linear subspace C of dimension $k-h-1$, called *center of projection*. The target space $\mathbb{P}^h$ is called *view*. A *scene* is a set of points $\mathbf{X}_i \in \mathbb{P}^k$, $i=1,\ldots,N$.

Of course, once homogeneous coordinates in $\mathbb{P}^k$ and $\mathbb{P}^h$ are fixed, we can identify P with a $(h+1)\times(k+1)$ matrix of maximal rank, defined up to a multiplicative constant. In such a setting, C is the right annihilator of P.

Given a set of n cameras $P_j : \mathbb{P}^k \setminus C_j \to \mathbb{P}^{h_j}$, $j=1,\ldots,n$, that project the same scene in $\mathbb{P}^k$, the images $P_j(\mathbf{X}) \in \mathbb{P}^{h_j}$ of the same scene point $\mathbf{X} \in \mathbb{P}^k$ in the different views are said to be *corresponding points*. We always assume that the number of points in the scene is much larger than the one needed to set up reference frames in the views. This assumption is needed when one has to consider different N–tuples of points, to avoid that the different N–tuples are always projectively equivalent.

Analogously, the subspaces $L_1,\dots,L_n$, one in each view $\mathbb{P}^{h_1},\dots,\mathbb{P}^{h_n}$, respectively, are said to be *corresponding* if there is at least a scene point $\mathbf{X}$ such that $P_j(\mathbf{X}) \in L_j$ for $j = 1,\dots,n$.

In this context, one of the main problems considered in literature, called *projective reconstruction of a scene*, consists in determining, up to projective transformations, the set of n cameras $P_j : \mathbb{P}^k \setminus C_j \to \mathbb{P}^{h_j}$, $j = 1,\dots,n$, and the position of the points of the scene in $\mathbb{P}^k$ starting from many enough corresponding points in the views.

In this paper, we study the cases in which the projective reconstruction of a scene is ambiguous, according to the following definition.

DEFINITION 1. *Given two sets of n projections $Q_j, P_j : \mathbb{P}^k \dashrightarrow \mathbb{P}^{h_j}, j = 1,\dots,n$, two sets of points $\{\mathbf{X}_1,\dots,\mathbf{X}_N\}$ and $\{\mathbf{Y}_1,\dots,\mathbf{Y}_N\}$, $N \gg 0$, in $\mathbb{P}^k$ are said to be conjugated critical configurations, associated to the projections $\{Q_1,\dots,Q_n\}$ and $\{P_1,\dots,P_n\}$ if, for all $i = 1,\dots,N$ and $j = 1,\dots,n$, we have $Q_j(\mathbf{X}_i) = P_j(\mathbf{Y}_i)$.*

Once conjugate critical configurations are given, we define the main object we are going to study.

DEFINITION 2. *Given two sets of n projections $Q_j, P_j : \mathbb{P}^k \dashrightarrow \mathbb{P}^{h_j}$, $j = 1,\dots,n$, as above, the locus $\mathscr{X} \subseteq \mathbb{P}^k$ containing all possible critical configurations $\{\mathbf{X}_1,\dots,\mathbf{X}_N\}$ is called critical locus for the associated projections.*

While in Definition 1 there is a perfect symmetry between the two sets of points $\{\mathbf{X}_1,\dots,\mathbf{X}_N\}$ and $\{\mathbf{Y}_1,\dots,\mathbf{Y}_N\}$, in Definition 2 we focus on one of the two sets. It is possible to define the analogous critical locus $\mathscr{Y}$ containing all points $\{\mathbf{Y}_1,\dots,\mathbf{Y}_N\}$, and finally, to define a unified critical locus $\mathscr{U}$ containing the couples $(\mathbf{X},\mathbf{Y})$ such that $Q_j(\mathbf{X}) = P_j(\mathbf{Y})$ for all $j = 1,\dots,n$. In [8], we analyse the relationship between the three critical loci $\mathscr{X},\mathscr{Y},\mathscr{U}$, and the projections $\mathscr{U} \to \mathscr{X}$ and $\mathscr{U} \to \mathscr{Y}$.

Now, we describe the defining ideal of $\mathscr{X}$. The main idea is the use of Grassmann tensors introduced by Hartley and Schaffalitzky in [16], to describe the constraints between corresponding subspaces. In case the subspaces contain the images of $\mathbf{X}$, the constraints reduce to the vanishing of the maximal minors of the matrix

$$M_{\mathscr{X}} = \begin{pmatrix} P_1 & Q_1(\mathbf{X}) & 0 & 0 & \dots & 0 & 0 \\ P_2 & 0 & Q_2(\mathbf{X}) & 0 & \dots & 0 & 0 \\ \vdots & & & & & & \vdots \\ P_n & 0 & 0 & 0 & \dots & 0 & Q_n(\mathbf{X}) \end{pmatrix}. \tag{1}$$

$M_{\mathscr{X}}$ is a $(n+\sum_{i=1}^{n} h_i)\times(n+k+1)$ matrix, the last n columns of which are of linear forms, while the first $k+1$ columns are of constants. The discussion above is summarized in the following statement (see [7] for a thorough discussion on critical loci and their defining ideals).

PROPOSITION 1. *The ideal $I(\mathscr{X})$ of the critical locus $\mathscr{X}$ is generated by the maximal minors of $M_{\mathscr{X}}$, and so $\mathscr{X}$ is a determinantal variety. Moreover, $\mathscr{X}$ contains the center of Q_j, $j = 1,\ldots,n$.*

Finally, the expected dimension of the critical locus $\mathscr{X}$ is

$$k - \left(1 + (n - k - 1 + \sum_{i=1}^{n} h_i) - n\right) = 2k - \sum_{i=1}^{n} h_i, \tag{2}$$

since $\mathscr{X}$ is determinantal. When the critical locus has the expected dimension, its degree is

$$\binom{n - k - 1 + \sum_{i=1}^{n} h_i}{n-1}. \tag{3}$$

3. State of the art on critical loci

In this section, we resume the known results on critical loci, in order to describe the families of algebraic varieties that arise in this context.

We only consider the case in which not all the views are $\mathbb{P}^1$. Indeed, in such a case, the reconstruction is never unique, as proven in [16] first and in [17] with techniques from algebraic geometry.

In the first subsection, we consider the classical case of projections from $\mathbb{P}^3$ to $\mathbb{P}^2$. The main reference on this case is [15], and we quote the results therein. In it, one can also find an ample list of references on critical loci. Another, more recent, paper on the subject is [11]. In the second and last subsection, we consider the case of projections from $\mathbb{P}^k$, where $k \geq 4$.

3.1. State of the art in the classical case of $\mathbb{P}^3$

Let us consider the two sets of n linear projections $P_i, Q_i : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$, $i = 1,\ldots,n$, and the corresponding critical locus $\mathscr{X}$. At first, we restrict our analysis to the case $\mathscr{X}$ has the expected dimension.

When $n = 2$, one of the first cases for which a complete classification was given, the critical loci are quadrics through the centers of projection of Q_1, Q_2. A thorough analysis of critical quadrics over the reals is in [15]. When one considers critical quadrics over the complex ground field, their result can be summarized as follows.

PROPOSITION 2. *Every quadric surface $Q \subset \mathbb{P}^3$ can be critical for suitable couples of two projections. As far as the positions of the centres are concerned, the only two possibilities that do not occur are the following:*

1. *Q is a union of two planes, and the centers lie in different planes, none of them on the common line;*

2. *Q is a cone and the centers lie on the same line in it, none of them being the vertex.*

The two exceptions above have been proven in [4] (the first case only, because in the quoted paper only reducible cases were of interest), and in [10] (both cases).

For two projections, it is enough to prove that one quadric for each possible rank is a critical locus for suitable projections to get that every quadric with the same rank is so, since all quadrics with the same rank are equivalent up to projective transformations.

For $n = 3$, in the same paper [15], the authors prove that, in the general case, the critical locus is a reduced 0-dimensional scheme with degree 10 or 7, according to the fact the centers of projections are considered to be critical or not. For the critical locus of 7 points, it is proven in [11] that these points have to satisfy some geometrical conditions, and so they are not general in the component of $\mathcal{H}ilb_7(\mathbb{P}^3)$ containing reduced schemes. In [15], the authors classify not only the 0–dimensional critical loci, but all admissible critical loci for a couple of three projections from $\mathbb{P}^3$ to $\mathbb{P}^2$, with a slightly different definition of criticality. Their classification follows from the careful analysis of the relative position of the quadric surfaces (irreducible, or reducible, does not matter) which are critical for a couple of two projections over the three given ones. The paper [15] has been recently reconsidered and improved in [11]. The cases when $\dim \mathcal{X} \geq 1$ are: (i) $\mathcal{X}$ is the union of a plane and a plane conic not in the plane above; (ii) $\mathcal{X}$ is an elliptic quartic curve, or a rational quartic curve with a singular double point; (iii) $\mathcal{X}$ is a rational normal curve; (iv) $\mathcal{X}$ is the union of at most two, possibly reducible, conics; (v) $\mathcal{X}$ is the union of a conic and a line, which do not meet; (vi) $\mathcal{X}$ is the union of two skew lines.

3.2. State of the art in the higher dimensional case

Possible generalizations concern the number n of views, the dimension k of the scene space, and the dimensions $h_1, \dots, h_n$ of the views.

In [7], smooth critical loci have been classified for any n, k and $h_1, \dots, h_n$.

For $n = 2$, in the general case, the smooth, codimension c, critical locus is a minimal degree variety. Conversely, with the only exception of Veronese surfaces in $\mathbb{P}^5$, every codimension c minimal degree variety embedded in $\mathbb{P}^k$ with $c+2 \leq k \leq 2c+1$, is the critical locus for suitable pairs of projections ([7], Propositions 5.1, 5.2). In this setting, general means that the ideal of the maximal minors of $M_{\mathcal{X}}$ drops rank in the expected codimension c. Furthermore, from the well-known classification of minimal degree varieties, $\mathcal{X}$ is singular as soon as $k \geq 2c+2$. Hence, a smooth minimal degree variety is embedded in $\mathbb{P}^k$ only for $c+2 \leq k \leq 2c+1$, and so no one of them escapes from being critical locus for the reconstruction problem, except Veronese surfaces in $\mathbb{P}^5$.

For $n = 3$, the following classification is proven in ([7], Theorem 6.1). A smooth codimension c variety $\mathscr{X} \subset \mathbb{P}^k$ is the critical locus for a suitable pair of three projections from $\mathbb{P}^k$ if and only if

1. $\mathscr{X}$ is a cubic curve in $\mathbb{P}^2$, or
2. $\mathscr{X}$ is a cubic surface in $\mathbb{P}^3$, or
3. $\mathscr{X}$ is a Bordiga surface in $\mathbb{P}^4$.

For sake of completeness, we recall that a Bordiga surface is the blow–up of $\mathbb{P}^2$ at 10 general points, embedded in $\mathbb{P}^4$ via the complete linear system of plane quartics through the 10 points above.

In the proof of Theorem 6.1 in [7], the pairs of three projections that exhibit a smooth variety in the previous list as a critical locus is explicitly given.

For $n = 4$ views, the smoothness of the critical locus $\mathscr{X}$ implies $k = 3$, and $h_1 = \cdots = h_4 = 1$. Even if this case is not interesting from a Multiview Geometry point of view, it has been considered for completeness from a geometrical perspective. In such a case, under some generality assumptions, $\mathscr{X}$ is a smooth, determinantal, quartic surface containing 4 skew lines. In [7], only a partial converse is proven. In particular, given 4 pairwise skew lines, there is a smooth quartic surface containing them that is critical for a pair of four projections from $\mathbb{P}^3$ to $\mathbb{P}^1$. The given lines are centers of the projections, of course. A parameter count suggests that not every smooth, determinantal, quartic surface containing four skew lines is critical for a pair of four projections from $\mathbb{P}^3$ to $\mathbb{P}^1$ ([7], Remark 7.2).

The cases above are the only ones for which a critical locus $\mathscr{X}$ can be smooth.

The case when the critical locus $\mathscr{X}$ has dimension greater than the expected one has been investigated in this setting, too.

In [4], the authors classify all the critical loci admissible for a couple of three projections $\mathbb{P}^4 \dashrightarrow \mathbb{P}^2$, by using the same definition of criticality we use in this paper. The result on critical loci follows from the classification of 4×3 matrices of linear forms dropping rank in codimension 1. The possible cases are the following.

1. $\mathscr{X}$ is the union of a hyperplane and a degree 3 surface.
2. $\mathscr{X}$ contains a hyperplane H. The residual scheme is a 2-dimensional linear space and a twisted cubic curve in H.
3. $\mathscr{X}$ contains a hyperplane H. The residual scheme is a quadric surface and a line in H.
4. $\mathscr{X}$ contains a cone over a smooth quadric surface in $\mathbb{P}^3$. The residual scheme is the vertex of the cone.
5. $\mathscr{X}$ contains a smooth quadric hypersurface. The residual scheme is a line contained in the quadric.
6. $\mathscr{X}$ is the union of a smooth quadric hypersurface and a 2–dimensional linear space.

4. Families of critical loci: the general setting

In this section, we face the following problem: we fix the projections $Q_1,\dots,Q_n$, and we want to describe all possible critical loci $\mathscr{X}$, as $P_1,\dots,P_n$ vary.

Of course, we get critical loci $\mathscr{X}$ depending of the projections $P_1,\dots,P_n$, and a map from the space of projections to a space of critical loci. This section is devoted to the construction of the spaces above, to the study of the quoted map, and to the thorough analysis of the case when the critical locus coincides with $\mathbb{P}^k$.

To construct the space of projections, we fix reference frames in $\mathbb{P}^k$ and in all target spaces $\mathbb{P}^{h_1},\dots,\mathbb{P}^{h_n}$. So, the i–th projection becomes a full rank $(h_i+1)\times(k+1)$ matrix, defined up to a multiplicative constant. Hence, it belongs to the open subset W_i of $\mathbb{P}(\mathrm{Mat}(h_i+1,k+1))$ parameterizing the full rank matrices. Since we have n projections, the natural space to consider is $W_1\times\cdots\times W_n$ that is a subspace of $\mathbb{P}(\mathrm{Mat}(h_1+1,k+1))\times\cdots\times\mathbb{P}(\mathrm{Mat}(h_n+1,k+1))$. We recall that our standard assumption is that no point in $\mathbb{P}^k$ belongs to all projection centers, which translates into the open condition $\mathrm{rank}(\mathscr{P})=k+1$, where

$$\mathscr{P}=\begin{pmatrix} P_1\\ \vdots\\ P_n\end{pmatrix}.$$

Finally, we call W the subspace of $W_1\times\cdots\times W_n$ containing the full rank matrices as above.

The target set contains all possible critical loci we get by choosing an element in W and the projections $Q_1,\dots,Q_n$ we fixed at the beginning of the section. Such critical loci are subschemes in $\mathbb{P}^k$, whose ideal is generated by the maximal minors of $M_{\mathscr{X}}$, as explained in Section 2. Hence, they are points of the Hilbert scheme $\mathscr{H}=\mathscr{H}ilb(\mathbb{P}^k)$ that parameterizes closed subschemes in $\mathbb{P}^k$.

The map $\varphi: W\to\mathscr{H}$ is defined as $\varphi(\mathscr{P})=[I(\mathscr{X})]$, where, as said before, $I(\mathscr{X})$ is the ideal of the maximal minors of $M_{\mathscr{X}}$, and $[I(\mathscr{X})]$ is the point in $\mathscr{H}$ that defines $\mathscr{X}\subseteq\mathbb{P}^k$.

LEMMA 1. *φ is well–defined.*

Proof. Because of the construction of W,

$$\begin{pmatrix} c_1P_1\\ \vdots\\ c_nP_n\end{pmatrix}\qquad\text{and}\qquad\begin{pmatrix} P_1\\ \vdots\\ P_n\end{pmatrix}$$

define the same point in W for every choice of non–zero constants $c_1,\dots,c_n$. When we take a maximal minor in $M_{\mathscr{X}}$, we fix certain rows in each view. To avoid trivial cases, we assume that we select at least a row from every view, so to get a $k+1+n$ square matrix with non–zero columns. We can compute the determinant of such a matrix by using a generalized Laplace rule, by taking minors in the first $k+1$ columns and minors in the n last ones. Of course, a minor in the last n columns is non–zero when we select a row from each view, and so the $k+1$ complementary minor contains the same number of rows from each projection P_i, independently from the minor in the last n columns. Then, we can collect $c_1^{\alpha_i} \dots c_n^{\alpha_n}$ in the determinant, where α_i is the number of rows we selected from the i–th view. Hence, the ideal $I(\mathscr{X})$ does not depend on the constants $c_1, \dots, c_n$. □

From Definition 1 of critical configurations, the sets of points $\{\mathbf{X}_1, \dots, \mathbf{X}_N\}$ and $\{\mathbf{Y}_1, \dots, \mathbf{Y}_N\}$ can be thought in different $\mathbb{P}^k$'s. It follows that we can change the reference frame in the $\mathbb{P}^k$ containing $\{\mathbf{Y}_1, \dots, \mathbf{Y}_N\}$ without changing the critical locus $\mathscr{X}$. This implies that $\varphi(\mathscr{P}) = \varphi(\mathscr{P}M)$ for every $M \in PGL(k)$. Hence, φ factorizes through the projection of W in $\mathbb{G}rass(k+1, V)$, Grassmann variety of the $(k+1)-$dimensional subspaces in a vector space V of dimension $n + \sum_{i=1}^n h_i$. So, we have $\varphi = \psi \circ \pi$, where $\pi : W \to \mathbb{G}rass(k+1, V)$ is the projection, and $\psi : \mathbb{G}rass(k+1, V) \to \mathscr{H}$ closes the commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{\varphi} & \mathscr{H} \\ {\scriptstyle\pi}\downarrow & \nearrow_{\psi} & \\ \mathbb{G}rass(k+1, V) & & \end{array} \quad . \tag{4}$$

The map ψ is defined on the whole $\mathbb{G}rass(k+1, V)$ by setting $\psi([\mathscr{P}])$ to be the closed scheme whose defining ideal is the one of the maximal minors of $M_{\mathscr{X}}$, where the first $k+1$ columns of $M_{\mathscr{X}}$ are a basis of $[\mathscr{P}]$ as a subspace of V. If we identify $\mathbb{G}rass(k+1, V)$ with its Plücker embedding in $\mathbb{P}^v$, then the generators of $I(\mathscr{X})$ will be bihomogeneous polynomials of degree 1 in the Plücker coordinates of $\mathbb{P}^v$, and degree n in the homogeneous coordinates $(x_0 : \dots : x_k)$ of $\mathbb{P}^k$. In the following, we study ψ.

We spend a few words on the target space $\mathscr{H}$. For general choices of the projections $Q_1, \dots, Q_n$ and $P_1, \dots, P_n$, the critical locus $\mathscr{X}$ is expected to have dimension and degree as in (2) and (3). For particular choices, the dimension can increase and the degree can vary, as in Section 3. However, the schemes that appear as critical loci belong to a finite list of families with different Hilbert polynomials. Hence $\mathscr{H}$ is a very large target space for our purposes. However, since we do not have a precise classification of the Hilbert polynomials that can be obtained in all possible degenerations of n views, we are forced to take $\mathscr{H}$ as target set.

The first results on ψ concern the case $\mathscr{X} = \mathbb{P}^k$.

PROPOSITION 3. *Let $J = \{j_1, \dots, j_r\}$ be a subset of $\{1, \dots, n\}$, and let I be the complementary set. Let $[\mathscr{P}] \in \mathbb{G}rass(k+1, V)$ be a point such that the rank of the matrix obtained by stacking $P_{j_1}, \dots, P_{j_r}$ is less than $k+1-\sum_{i \in I} h_i$. Then, $\psi([\mathscr{P}]) = 0$, and so the critical locus associated to the associated projections $P_1, \dots, P_n$ and $Q_1, \dots, Q_n$ is $\mathscr{X} = \mathbb{P}^k$.*

Proof. We recall the construction of the generators of $I(\mathscr{X})$, maximal minors of $M_{\mathscr{X}}$. To avoid trivial cases, one has to select at least a row from every view. If we select all rows from $P_i, i \in I$, i.e. $\sum_{i\in I} h_i + n - r$, the remaining rows from $P_j, j \in J$ are $k+1+n-(\sum_{i\in I} h_i + n - r) = k+1+r-\sum_{i\in I} h_i$. Since we develop the minor by generalized Laplace rule, we take a $r \times r$ minor from the last n columns of $M_{\mathscr{X}}$ and so the minor from $P_j, j \in J$, has order $k+1-\sum_{i\in I} h_i$. Our assumption on the rank of the matrix obtained by stacking $P_{j_1}, \dots, P_{j_r}$ guarantees that this last minor vanishes. Of course, this holds also if we select less rows from $P_i, i \in I$, and so $I(\mathscr{X}) = 0$, as claimed. □

REMARK 1. Since the matrices that appear in Proposition 3 are defined in terms of rank, they vary in some Schubert varieties in $\mathbb{G}rass(k+1, V)$.

Proposition 3 does not cover all cases when the critical locus is the whole $\mathbb{P}^k$. In particular, this can happen also when all $P_1, \dots, P_n$ have maximal rank.

THEOREM 1. *Assume $h_1 \geq h_2 \geq \cdots \geq h_n$ be the dimensions of the views, and $h_1 > 1$. In the same notation as before, and if $[\mathscr{P}] \in \pi(W)$, then $\psi([\mathscr{P}]) = [0]$ if and only if $P_i = \lambda_i Q_i$, for $(\lambda_1 : \dots : \lambda_n) \in \mathbb{P}^{n-1}$.*

Proof. Assume $P_i = \lambda_i Q_i$ for $(\lambda_1 : \dots : \lambda_n) \in \mathbb{P}^{n-1}$. When we choose a maximal submatrix of $M_{\mathscr{X}}$, and compute its determinant by generalized Laplace rule, every $k+1$ minor in the first $k+1$ columns is obtained by selecting a constant number of rows from every $P_i, i = 1, \dots, n$. Hence, the generator of $I(\mathscr{X})$ does not depend on $\lambda_1, \dots, \lambda_n$. Since every point in $\mathbb{P}^k$ is critical when $P_i = Q_i, i = 1, \dots, n$, we get the if part of the statement.

For the only if part, we need ([16], Theorem 5.1), main result of the quoted paper. Indeed, in [16], the authors prove that it is possible to reconstruct the projection matrices if a Grassmann tensor is known. Such a tensor is known when a suitably large number of corresponding subspaces in the views is given. The codimensions $(\alpha_1, \dots, \alpha_n)$ of the subspaces in the views is referred to as profile, and it holds $\alpha_1 + \cdots + \alpha_n = k+1$. Moreover, the reconstruction is unique, up to a multiplicative scalar for each matrix, if at least one view is not $\mathbb{P}^1$. Assume first that $k+1 = \sum_i h_i$, so that the critical locus is expected to be a hypersurface. In this setting, the only possible profile is $(\alpha_1, \dots, \alpha_n) = (h_1, \dots, h_n)$, so that corresponding subspaces in the views are a collection of points, one in each view. Because of the definition of corresponding subspaces, this means that there exists a point $\mathbf{X} \in \mathbb{P}^k$ such that $(Q_1(\mathbf{X}), \dots, Q_n(\mathbf{X}))$ is the only choice for a collection of corresponding subspaces. Since $k+1 = \sum_i h_i$, the Grassmann tensor for the given profile is actually the multilinear map defined by the vanishing of the determinant of $M_{\mathscr{X}}$. To saturate the tensor, one has to evaluate it at a suitably large number of collections of corresponding points. The assumption $\psi([\mathscr{P}]) = 0$ means that there is no restriction on $\mathbf{X}$, since the critical locus is $\mathscr{X} = \mathbb{P}^k$. Hence, $\mathscr{X} = \mathbb{P}^k$ determines the Grassmann tensor, and this implies that the projection matrices are uniquely reconstructed, up to multiplicative constants. Since $P_i = Q_i$ for every $i = 1, \dots, n$, is a condition that guarantees the vanishing of $\det(M_{\mathscr{X}})$, we have the claim.

Assume now that $\sum_i h_1 > k+1$. We only stress the differences with the discussed case, but for the construction and properties of the Grassmann tensor we refer to [16]. Since $\sum_i \alpha_i = k+1$, there are different possible profiles. Let us choose one of them. For such a choice, the corresponding spaces are generated by $Q_i(\mathbf{X})$ and other general points in the view. Once we vary $\mathbf{X} \in \mathbb{P}^k$ and the other points in the views sufficiently many times, we reconstruct the Grassmann tensor associated to the chosen profile. Thanks to ([16], Theorem 5.1), this determines the projection matrices $P_1, \ldots, P_n$, up to a multiplicative scalar, as before. As in the previous case, since $P_i = Q_i$ for every $i = 1, \ldots, n$ guarantees that the Grassmann tensor vanishes for every profile, then we have that the solution above does not depend on the chosen profile. □

REMARK 2. The description of the locus of matrices in Theorem 1 is the following. When $(\lambda_1 : \ldots : \lambda_n)$ varies in $\mathbb{P}^{n-1}$, the $k+1$ minors of $[\mathscr{P}]$ define a point in $\mathbb{G}rass(k+1, V)$ whose Plücker coordinates are monomials in $\lambda_1, \ldots, \lambda_n$ of degree $k+1$. The variable λ_j has degree at most $h_j + 1$ in whatever coordinate. Hence, the variety described by such points is the image of $\mathbb{P}^{n-1}$ via the linear system

$$|(k+1)H - (k-h_1)A_1 - \cdots - (k-h_n)A_n|$$

where $A_1 = (1, 0, \ldots, 0), \ldots, A_n = (0, \ldots, 0, 1)$ are the fundamental points and H is the hyperplane divisor in $\mathbb{P}^{n-1}$. In other words, this locus is a Veronese variety, or a projection of it. The linear system above can have fixed components. E.g., this happens when

$$\sum_{j \neq i} (h_j + 1) < k+1 \leq \sum_j h_j$$

since in this case λ_i appears in every monomial.

REMARK 3. As said in Section 3, in case every view is $\mathbb{P}^1$, it is known that there are two different pairs of associated projections, the first $P_1, \ldots, P_n$ and $Q_1, \ldots, Q_n$, the second $P'_1, \ldots, P'_n$ and $Q_1, \ldots, Q_n$ that give the same critical locus, but $(P'_1, \ldots, P'_n) \neq (\lambda_1 P_1, \ldots, \lambda_n P_n)$. In such case, there are two Veronese varieties for which $I(\mathscr{X}) = 0$.

5. Families of critical loci: the case of 2 views

In this section, we carefully study the case $n = 2$, and $h_1 + h_2 = k+1$.

In such a case, the critical locus is either a quadric hypersurface in $\mathbb{P}^k$, or is the whole $\mathbb{P}^k$.

The Grassmann variety to be considered is $\mathbb{G}rass(k+1, V)$ where $\dim(V) = h_1 + h_2 + 2$, that is dual to $\mathbb{G}rass(2, V)$. From now on, we'll work in this last Grassmann variety. We recall that the Plücker embedding identifies $\mathbb{G}rass(2, V)$ with a subvariety of $\mathbb{P}^\nu$, where $\nu = \binom{k+3}{2} - 1$, whose ideal is generated by the Plücker relations

$$QR(i,j,h,l) : p_{ij}p_{hl} - p_{ih}p_{jl} + p_{il}p_{jh} = 0,$$

for every $1 \le i < j < h < l \le h_1 + h_2 + 2$. With abuse of notation, we call $\mathbb{G}rass(2, V)$ the subvariety in $\mathbb{P}^v$ we get by the Plücker embedding.

Let us fix the projections $\overline{Q}_1, \overline{Q}_2$, and $\overline{P}_1, \overline{P}_2$, and let $\mathscr{X}$ be the critical quadric hypersurface associated to them. Our aim is to describe all possible P_1, P_2 such that the critical quadric associated to $\overline{Q}_1, \overline{Q}_2$, and P_1, P_2 is $\mathscr{X}$. The defining equation of $\mathscr{X}$ is the determinant of $M_{\mathscr{X}}$ (see equation (1)). By the generalized Laplace rule, such determinant is the sum of products of maximal minors of $\mathscr{P}$ by products of linear forms in the last two columns. In order to get a non–zero summand, we have to select a linear form from $\overline{Q}_1(X)$ and another one from $\overline{Q}_2(X)$. The corresponding maximal minor of $\mathscr{P}$ is obtained by erasing a row from P_1 and a row from P_2. By ([13], Corollary 2.4), the generators of $\overline{Q}_1(X)\overline{Q}_2(X)$ are linearly dependent. More precisely, for a general choice of $\overline{Q}_1$ and $\overline{Q}_2$, a generator is a linear combination of the remaining ones. Then, to get the same quadric hypersurface as critical locus, we have to assign the maximal minors (that is to say, Plücker coordinates) we get by erasing a row from P_1 and a row from P_2, in such a way that they are proportional to the corresponding minors in $\overline{P}_1, \overline{P}_2$. Of course, because of the previous remark, we can add to such minors also multiples of the maximal minors of $\overline{Q}_1, \overline{Q}_2$, since, for this choice, the critical locus is the whole $\mathbb{P}^k$.

Let $\hat{p}, \hat{q} \in \mathbb{G}rass(2, V)$ be the dual coordinates of $[\overline{P}_1, \overline{P}_2]^T$ and $[\overline{Q}_1, \overline{Q}_2]^T$, respectively. The previous argument shows that we have to study the intersection of $\mathbb{G}rass(2, V)$ with the linear space $L(\hat{p}, \hat{q})$ defined by $p_{ab} = \alpha\hat{p}_{ab} + \beta\hat{q}_{ab}$, where $1 \le a \le h_1 + 1, h_1 + 2 \le b \le h_1 + h_2 + 2$.

Since in the rest of the section computations are heavy, we summarize the results in the following statement.

THEOREM 2. *The intersection between* $\mathbb{G}rass(2, V)$ *and* $L(\hat{p}, \hat{q})$ *is equal to two conics, for general choices of the points* $\hat{p}, \hat{q}$*, and either three conics or a surface when* $\hat{p}, \hat{q}$ *satisfy certain closed conditions.*

The general case is proved in Propositions 4, 5. The three conics case is considered in Proposition 7. This case happens when the points $\hat{p}, \hat{q}$ belong to the subvariety $\mathscr{E}$ (see (11)). The surface case is treated in Proposition 8. This last case happens if $\hat{p}, \hat{q}$ belong to the subvariety $\mathscr{D}$ (see (12)). For completeness, we have that $\mathscr{D} \subseteq \mathscr{E}$ and that none of them is empty (see Proposition 6).

According to the indices, we divide the $QR(i, j, h, l)$ in the following 5 cases:

Case (1) $l \le h_1 + 1$;

Case (2) $h \le h_1 + 1, l \ge h_1 + 2$;

Case (3) $j \le h_1 + 1, h \ge h_1 + 2$;

Case (4) $i \le h_1 + 1, j \ge h_1 + 2$;

Case (5) $i \ge h_1 + 2$.

Let us consider first the quadratic relations restricted to $L(\hat{p},\hat{q})$ in cases (2) and (4).

In case (2), we get

$$p_{ij}(\alpha\hat{p}_{hl}+\beta\hat{q}_{hl})-p_{ih}(\alpha\hat{p}_{jl}+\beta\hat{q}_{jl})+p_{jh}(\alpha\hat{p}_{il}+\beta\hat{q}_{il})=0, \tag{5}$$

while in case (4), we get

$$p_{hl}(\alpha\hat{p}_{ij}+\beta\hat{q}_{ij})-p_{jl}(\alpha\hat{p}_{ih}+\beta\hat{q}_{ih})+p_{jh}(\alpha\hat{p}_{il}+\beta\hat{q}_{il})=0, \tag{6}$$

Hence, we get two homogeneous linear systems in the indeterminates $p_{cd}, 1\le c<d\le h_1+1$ the first, and $p_{cd}, h_1+2\le c<d\le h_1+h_2+2$ the latter. Let us analyze the first linear system, and let us consider the three indeterminates $p_{i_1i_2}, p_{i_1i_3}, p_{i_2i_3}$, with $i_1<i_2<i_3\le h_1+1$ and the corresponding subsystem. A row of the coefficient matrix of this subsystem is

$$\begin{pmatrix}\alpha\hat{p}_{i_3l}+\beta\hat{q}_{i_3l} & -(\alpha\hat{p}_{i_2l}+\beta\hat{q}_{i_2l}) & \alpha\hat{p}_{i_1l}+\beta\hat{q}_{i_1l}\end{pmatrix}$$

for $l=h_1+2,\dots,h_1+h_2+2$. Since $h_2\ge 2$, there are at least three rows.

The determinant of the maximal minor obtained by choosing l_1,l_2,l_3 with $h_1+2\le l_1<l_2<l_3\le h_1+h_2+2$ is equal to

$$\alpha^2\,\beta\,D_{i_1,i_2,i_3,l_1,l_2,l_3}(\hat{p},\hat{p},\hat{q})+\alpha\,\beta^2\,D_{i_1,i_2,i_3,l_1,l_2,l_3}(\hat{q},\hat{q},\hat{p}) \tag{7}$$

where

$$\begin{aligned} D_{i_1,i_2,i_3,l_1,l_2,l_3}(\hat{p},\hat{p},\hat{q}) &= \begin{vmatrix}\hat{p}_{i_3l_1} & -\hat{p}_{i_2l_1} & \hat{q}_{i_1l_1}\\ \hat{p}_{i_3l_2} & -\hat{p}_{i_2l_2} & \hat{q}_{i_1l_2}\\ \hat{p}_{i_3l_3} & -\hat{p}_{i_2l_3} & \hat{q}_{i_1l_3}\end{vmatrix} + \begin{vmatrix}\hat{p}_{i_3l_1} & -\hat{q}_{i_2l_1} & \hat{p}_{i_1l_1}\\ \hat{p}_{i_3l_2} & -\hat{q}_{i_2l_2} & \hat{p}_{i_1l_2}\\ \hat{p}_{i_3l_3} & -\hat{q}_{i_2l_3} & \hat{p}_{i_1l_3}\end{vmatrix} + \\ &+ \begin{vmatrix}\hat{q}_{i_3l_1} & -\hat{p}_{i_2l_1} & \hat{p}_{i_1l_1}\\ \hat{q}_{i_3l_2} & -\hat{p}_{i_2l_2} & \hat{p}_{i_1l_2}\\ \hat{q}_{i_3l_3} & -\hat{p}_{i_2l_3} & \hat{p}_{i_1l_3}\end{vmatrix}, \end{aligned} \tag{8}$$

and $D_{i_1,i_2,i_3,l_1,l_2,l_3}(\hat{q},\hat{q},\hat{p})$ is analogously defined by exchanging the roles of $\hat{p}$ and $\hat{q}$ in $D_{i_1,i_2,i_3,l_1,l_2,l_3}(\hat{p},\hat{p},\hat{q})$, since the coefficients of α^3 and of β^3 identically vanish by using the relations QR for suitable indices.

In the second linear system, we consider the subsystem associated to the indeterminates $p_{l_2l_3}, p_{l_1l_3}, p_{l_1l_2}$, with $h_1+2\le l_1<l_2<l_3\le h_1+h_2+2$. The order 3 minor associated to the rows $i_3>i_2>i_1$ is the same as before, since the matrices are the transposes of the matrices in the case above, up to change the signs of the second column and row.

To have non trivial solutions, all such determinants must vanish. This happens, in the general case, only if either $\alpha=0$, or $\beta=0$. There are two more cases to consider that happen when $\hat{p},\hat{q}$ satisfy some constraints. If

$$\det\begin{pmatrix} D_{i_1,i_2,i_3,l_1,l_2,l_3}(\hat{p},\hat{p},\hat{q}) & D_{i_1,i_2,i_3,l_1,l_2,l_3}(\hat{q},\hat{q},\hat{p})\\ D_{i_4,i_5,i_6,l_4,l_5,l_6}(\hat{p},\hat{p},\hat{q}) & D_{i_4,i_5,i_6,l_4,l_5,l_6}(\hat{q},\hat{q},\hat{p})\end{pmatrix}=0 \tag{9}$$

for every $(i_1,i_2,i_3) = (i_4,i_5,i_6)$ and $h_1+2 \le l_1 < l_2 < l_3 \le h_1+h_2+2, h_1+2 \le l_4 < l_5 < l_6 \le h_1+h_2+2$, or $(l_1,l_2,l_3) = (l_4,l_5,l_6)$ and $1 \le i_1 < i_2 < i_3 \le h_1+1, 1 \le i_4 < i_5 < i_5 \le h_1+1$, then there is a third case in which all determinants vanish, namely

$$(\hat{\alpha},\hat{\beta}) = (-D_{i_1,i_2,i_3,l_1,l_2,l_3}(\hat{q},\hat{q},\hat{p}), D_{i_1,i_2,i_3,l_1,l_2,l_3}(\hat{p},\hat{p},\hat{q})),$$

up to a multiplicative non zero scalar. In such a case, it is possible to show that the whole linear system has non trivial solutions, and, in the general case, there is one solution, up to a multiplicative scalar (see the proof of Proposition 7).

Finally, if

$$D_{i_1,i_2,i_3,l_1,l_2,l_3}(\hat{p},\hat{p},\hat{q}) = D_{i_1,i_2,i_3,l_1,l_2,l_3}(\hat{q},\hat{q},\hat{p}) = 0 \tag{10}$$

for every $1 \le i_1 < i_2 < i_3 \le h_1+1$, and $h_1+2 \le l_1 < l_2 < l_3 \le h_1+h_2+2$, then all the determinants above vanish for every (α,β), and again, the whole linear system has non trivial solutions, one in the general case, up to a multiplicative scalar (see the proof of Proposition 8).

To summarize, the cases to consider are the following:

(A) $\alpha = 0$;

(B) $\beta = 0$;

(C) all determinants in (9) vanish;

(D) all polynomials in (10) vanish.

Now, we consider the four cases, one at a time.

Case (A), namely $\alpha = 0$. The linear space $L(\hat{p},\hat{q})$ is $p_{ab} = \beta\hat{q}_{ab}$, with $1 \le a \le h_1+1, h_1+2 \le b \le h_1+h_2+2$, and we have to intersect it with $\mathbb{G}rass(2,V)$.

It is easy to check that the only solution of the linear system defined by the quadratic equations in case (2) is $p_{ij} = \nu\hat{q}_{ij}$, for $1 \le i < j \le h_1+1$, while the only solution of the linear system defined by the quadratic equations in case (4) is $p_{ij} = \mu\hat{q}_{ij}$, for $h_1+2 \le i < j \le h_1+h_2+2$.

If we substitute the computed Plücker coordinates in $QR(i,j,h,k)$ with $k \le h_1+1$ or with $h_1+2 \le i$, we get the identity $0 = 0$ because the Plücker coordinates of $\hat{q}$ satisfy all $QR(i,j,h,k)$'s.

The last quadratic relations to consider are $QR(i,j,h,k)$ with $j \le h_1+1, h_1+2 \le h$, i.e. the ones in case (3). If we substitute the computed Plücker coordinates, we get

$$\nu\ \mu\ \hat{q}_{ij}\hat{q}_{hk} - \beta^2\hat{q}_{ih}\hat{q}_{jk} + \beta^2\hat{q}_{ik}\hat{q}_{jh} = 0.$$

With the use of the quadratic relations, we can rewrite the equation as

$$\nu\ \mu\ \hat{q}_{ij}\hat{q}_{hk} - \beta^2\ \hat{q}_{ij}\hat{q}_{hk} = 0$$

equivalent to $\beta^2 = \nu\,\mu$. So, the points in the intersection of $\mathbb{G}rass(2,V)$ and $L(\hat{p},\hat{q})$ are parameterized by $(c:d) \in \mathbb{P}^1$, and they are

$$p_{ij} = \begin{cases} c^2\hat{q}_{ij} & \text{if } 1 \le i < j \le h_1+1 \\ cd\,\hat{q}_{ij} & \text{if } i \le h_1+1, h_1+2 \le j \\ d^2\hat{q}_{ij} & \text{if } h_1+2 \le i < j \le h_1+h_2+2 \end{cases}$$

with the condition $\nu = c^2, \beta = cd, \mu = d^2$. We have then proved

PROPOSITION 4. *The intersection between the Grassmannian* $\mathbb{G}rass(2,V)$ *and the linear space* $p_{ab} = \beta\hat{q}_{ab}$*, with* $1 \le a \le h_1+1, h_1+2 \le b \le h_1+h_2+2$*, for a general* $\hat{q} \in \mathbb{G}rass(2,V)$ *is the degree* 2 *Veronese embedding of* $\mathbb{P}^1$.

The assumption "the point $\hat{q} \in \mathbb{G}rass(2,V)$ is general" means that the linear systems we have solved have one solution, up to a scalar. It is worth noting that the critical locus for (P_1,P_2) with dual Plücker coordinates as in the previous Proposition is the whole $\mathbb{P}^k$, since $\det(M_{\mathscr{X}})$ identically vanish for such choices.

Case (B), namely $\beta = 0$, is analogous and we do not report the computations. We have the following result.

PROPOSITION 5. *The intersection between the Grassmannian* $\mathbb{G}rass(2,V)$ *and the linear space* $p_{ab} = \alpha\hat{p}_{ab}$*, with* $1 \le a \le h_1+1, h_1+2 \le b \le h_1+h_2+2$*, for a general* $\hat{p} \in \mathbb{G}rass(2,V)$ *is the degree* 2 *Veronese embedding of* $\mathbb{P}^1$

$$p_{ij} = \begin{cases} c^2\hat{p}_{ij} & \text{if } 1 \le i < j \le h_1+1 \\ cd\,\hat{p}_{ij} & \text{if } i \le h_1+1, h_1+2 \le j \\ d^2\hat{p}_{ij} & \text{if } h_1+2 \le i < j \le h_1+h_2+2 \end{cases}.$$

REMARK 4. The critical locus, in case (B), is a hyperquadric $\mathscr{X}$ in $\mathbb{P}^k$. For every (P_1,P_2) whose dual Plücker coordinates are as in the previous Proposition, the critical locus is the same as for $(\overline{P}_1,\overline{P}_2)$. Also for this result, $\hat{p}$ general means that the two linear systems have one non trivial solution each, up to a multiplicative scalar.

Before analysing the cases $(C),(D)$, we investigate the existence of points $\hat{p},\hat{q}$ that satisfy the conditions (9) and (10). As said at the beginning of this section, the multiview geometry problem we are addressing is to describe the projections P_1,P_2 such that the critical locus associated to (P_1,P_2) and $(\overline{Q}_1,\overline{Q}_2)$ is the same quadric as the one associated to $(\overline{P}_1,\overline{P}_2)$ and $(\overline{Q}_1,\overline{Q}_2)$. Hence we fix the point $\hat{q}$, and let $\hat{p}$ vary. We set

$$\mathscr{E} = \{\hat{p} \in \mathbb{G}rass(2,V) \mid \hat{p} \text{ satisfies equations (9)}\} \tag{11}$$

and

$$\mathscr{D} = \{\hat{p} \in \mathbb{G}rass(2,V) \mid \hat{p} \text{ satisfies equations (10)}\} \tag{12}$$

for a given point $\hat{q}$. Since $\mathscr{D}$ is contained in $\mathscr{E}$, too, we focus on $\mathscr{D}$.

PROPOSITION 6. *The set $\mathscr{D}$ is not empty, for every choice of $\hat{q}$.*

Proof. Because of the transitivity of $\mathbb{G}rass(2,V)$, it suffices to choose a couple of projections $(\overline{Q}_1, \overline{Q}_2)$ general enough, and to find points in $\mathscr{D}$. Assume the point $[\mathscr{Q}] \in \mathbb{G}rass(k+1,V)$ obtained by stacking $\overline{Q}_1, \overline{Q}_2$ is the identity matrix of order $k+1$ with two more rows equal to

$$\begin{pmatrix} -1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 0 & \dots & 0 \end{pmatrix}.$$

Then, the dual Plücker coordinates of $[\mathscr{Q}]$ are $\hat{q}$ where

$$\hat{q}_{ij} = \begin{cases} 1 & \text{if } (i,j) = (1,2), (1, h_1+h_2+2), (h_1+h_2+1, h_1+h_2+2), \\ -1 & \text{if } (i,j) = (2, h_1+h_2+1), \\ 0 & \text{otherwise} \end{cases} .$$

Equations $D_{\dots}(\hat{q},\hat{q},\hat{p}) = 0$ become $\hat{p}_{i_3,l_1} = 0$ with $3 \le i_3 \le h_1+1, h_1+2 \le l_1 \le h_1+h_2$. On the other hand, equations $D_{\dots}(\hat{p},\hat{p},\hat{q}) = 0$ become

$$\hat{p}_{i_1 i_2}\hat{p}_{l_1 l_2} = 0$$

where $1 \le i_1 < i_2 \le h_1+1, (i_1,i_2) \ne (1,2)$, and $h_1+1 \le l_1 < l_2 \le h_1+h_2$, or $3 \le i_1 < i_2 \le h_1+1, h_1+2 \le l_1 < l_2 \le h_1+h_2+2$, and $(l_1,l_2) \ne (h_1+h_2+1, h_1+h_2+2)$, and moreover

$$\hat{p}_{2,i_3}\hat{p}_{l_1,h_1+h_2+1} - \hat{p}_{1,i_3}\hat{p}_{l_1,h_1+h_2+2} = 0$$

with $3 \le i_3 \le h_1+1, h_1+2 \le l_1 \le h_1+h_2$.

Every $2 \times (h_1+h_2+2)$ matrix of rank 2 with column $C_j = (0,0)^T$ for $j = 3,\dots,h_1+1$, or for $j = h_1+2,\dots,h_1+h_2$, has Plücker coordinates that satisfy all the previous equations, and so we get the claim. □

REMARK 5. Because of the points in $\mathscr{D}$ we have found, we expect $\mathscr{D}$ to contain the Grassmannian varieties $\mathbb{G}rass(2,h_1+3)$ and $\mathbb{G}rass(2,h_2+3)$. Numerical experiments confirm that $\mathscr{D}$ has a top dimensional component equal to $\mathbb{G}rass(2,h_1+3)$, in the case $h_1 \ge h_2$, but is not irreducible. The locus $\mathscr{E}$ has dimension higher than $\mathscr{D}$, as numerical experiments confirm.

Case (C). We have the following result.

PROPOSITION 7. *Let us assume that $\hat{p} \in \mathscr{E}$ for a given $\hat{q}$. Then, the intersection between the Grassmannian $\mathbb{G}rass(2,V)$ and the linear space $L(\hat{p},\hat{q})$ defined by $p_{ab} = \eta\left(\hat{\alpha}\hat{p}_{ab} + \hat{\beta}\hat{q}_{ab}\right)$, with $1 \le a \le h_1+1, h_1+2 \le b \le h_1+h_2+2$, for general $\hat{p},\hat{q} \in \mathscr{E}$ is a degree 2 Veronese embedding of $\mathbb{P}^1$.*

Proof. Let us consider two subsystems from the quadratic equations in range (2), associated to (i_1,i_2,i_3,l) and to (i_1,i_2,i_4,l) with $i_3 \ne i_4$, and $i_3,i_4 \le h_1+1, h_1+2 \le l$. Then, p_{i_1,i_2} is a variable in both systems. The value of p_{i_1,i_2} is an order 2 minor of the coefficient matrix, obtained by selecting the second and third column. Since the two subsystems have coefficient matrices that differ for the first column only, the value of p_{i_1,i_2} does not depend on the subsystem, and this proves that the global linear system from the quadratic equations in range (2) has non–zero solutions. Because of the generality assumptions, there is a solution, up to a multiplicative scalar, equal to

$$p_{ab} = \nu\left[(\hat{\alpha}\hat{p}_{al_1} + \hat{\beta}\hat{q}_{al_1})(\hat{\alpha}\hat{p}_{bl_2} + \hat{\beta}\hat{q}_{bl_2}) - (\hat{\alpha}\hat{p}_{al_2} + \hat{\beta}\hat{q}_{al_2})(\hat{\alpha}\hat{p}_{bl_1} + \hat{\beta}\hat{q}_{bl_1})\right]$$

for $1 \le a < b \le h_1 + 1$.

The same argument applies to the linear system from the quadratic equations in range (4), and its solution is

$$p_{ab} = \mu\left[(\hat{\alpha}\hat{p}_{i_1a} + \hat{\beta}\hat{q}_{i_1a})(\hat{\alpha}\hat{p}_{i_2b} + \hat{\beta}\hat{q}_{i_2b}) - (\hat{\alpha}\hat{p}_{i_2a} + \hat{\beta}\hat{q}_{i_2a})(\hat{\alpha}\hat{p}_{i_1b} + \hat{\beta}\hat{q}_{i_1b})\right]$$

for $h_1 + 2 \le a < b \le h_1 + h_2 + 2$.

When substituting these coordinates in the quadratic equations QR in range (1) or range (5), they are identically satisfied.

Finally, when substituting in the quadratic relations QR in range (3), we get

$$\eta^2 = \mu\,\nu\left[(\hat{\alpha}\hat{p}_{i_1k_1} + \hat{\beta}\hat{q}_{i_1k_1})(\hat{\alpha}\hat{p}_{i_2k_2} + \hat{\beta}\hat{q}_{i_2k_2}) - (\hat{\alpha}\hat{p}_{i_1k_2} + \hat{\beta}\hat{q}_{i_1k_2})(\hat{\alpha}\hat{p}_{i_2k_1} + \hat{\beta}\hat{q}_{i_2k_1})\right]$$

and so we have proved that, if the assumptions are fulfilled, the intersection is a conic, also in this case. □

Case (D). With computations analogous to case (C), the solutions of the linear systems associated to the quadratic relations in ranges (2) and (4) are

$$p_{ab} = \nu\left[(\alpha\hat{p}_{al_1} + \beta\hat{q}_{al_1})(\alpha\hat{p}_{bl_2} + \beta\hat{q}_{bl_2}) - (\alpha\hat{p}_{al_2} + \beta\hat{q}_{al_2})(\alpha\hat{p}_{bl_1} + \beta\hat{q}_{bl_1})\right]$$

for $1 \le a < b \le h_1 + 1$, and

$$p_{ab} = \mu\left[(\alpha\hat{p}_{i_1a} + \beta\hat{q}_{i_1a})(\alpha\hat{p}_{i_2b} + \beta\hat{q}_{i_2b}) - (\alpha\hat{p}_{i_2a} + \beta\hat{q}_{i_2a})(\alpha\hat{p}_{i_1b} + \beta\hat{q}_{i_1b})\right]$$

for $h_1 + 2 \le a < b \le h_1 + h_2 + 2$, respectively.

When we substitute the solutions in the quadratic relations in ranges (1) and (5), as in the previous case, they are identically satisfied.

Now, we consider the quadratic relations in range (3), where we use the previous computations. Let us consider $1 \le i_1 < i_2 \le h_1 + 1 < l_1 < l_2 \le h_1 + h_2 + 2$, and we get

$$\begin{aligned}
&\mu\,\nu\left[(\alpha\hat{p}_{i_1l_1} + \beta\hat{q}_{i_1l_1})(\alpha\hat{p}_{i_2l_2} + \beta\hat{q}_{i_2l_2}) - (\alpha\hat{p}_{i_1l_2} + \beta\hat{q}_{i_1l_2})(\alpha\hat{p}_{i_2l_1} + \beta\hat{q}_{i_2l_1})\right]\\
&\quad\left[(\alpha\hat{p}_{i_1l_1} + \beta\hat{q}_{i_1l_1})(\alpha\hat{p}_{i_2l_2} + \beta\hat{q}_{i_2l_2}) - (\alpha\hat{p}_{i_2l_1} + \beta\hat{q}_{i_2l_1})(\alpha\hat{p}_{i_1l_2} + \beta\hat{q}_{i_1l_2})\right] -\\
&-(\alpha\hat{p}_{i_1l_1} + \beta\hat{q}_{i_1l_1})(\alpha\hat{p}_{i_2l_2} + \beta\hat{q}_{i_2l_2}) + (\alpha\hat{p}_{i_1l_2} + \beta\hat{q}_{i_1l_2})(\alpha\hat{p}_{i_2l_1} + \beta\hat{q}_{i_2l_1}) =\\
&= \Big(\mu\,\nu\left[(\alpha\hat{p}_{i_1l_1} + \beta\hat{q}_{i_1l_1})(\alpha\hat{p}_{i_2l_2} + \beta\hat{q}_{i_2l_2}) - (\alpha\hat{p}_{i_1l_2} + \beta\hat{q}_{i_1l_2})(\alpha\hat{p}_{i_2l_1} + \beta\hat{q}_{i_2l_1})\right] - 1\Big)\\
&\quad\left[(\alpha\hat{p}_{i_1l_1} + \beta\hat{q}_{i_1l_1})(\alpha\hat{p}_{i_2l_2} + \beta\hat{q}_{i_2l_2}) - (\alpha\hat{p}_{i_1l_2} + \beta\hat{q}_{i_1l_2})(\alpha\hat{p}_{i_2l_1} + \beta\hat{q}_{i_2l_1})\right] = 0
\end{aligned}$$

For general choices of α, β, we have the only relation

$$\mu\, \nu \left[(\alpha \hat{p}_{i_1 l_1} + \beta \hat{q}_{i_1 l_1})(\alpha \hat{p}_{i_2 l_2} + \beta \hat{q}_{i_2 l_2}) - (\alpha \hat{p}_{i_1 l_2} + \beta \hat{q}_{i_1 l_2})(\alpha \hat{p}_{i_2 l_1} + \beta \hat{q}_{i_2 l_1}) \right] = 1 \tag{13}$$

and so, for given $(\alpha : \beta) \in \mathbb{P}^1$, we get a conic from the parametrization of the Plücker coordinates described above. We have then proved the following result.

PROPOSITION 8. *Let $\hat{q} \in \mathbb{G}rass(2, V)$ be a general point, and let $\hat{p} \in \mathcal{D}$ another general point. Then, the intersection of the linear space $L(\hat{p}, \hat{q})$ with the Grassmannian $\mathbb{G}rass(2, V)$ is a surface.*

REMARK 6. In all numerical experiments, the surface we get as intersection of $L(\hat{p}, \hat{q})$ and $\mathbb{G}rass(2, V)$ is a scroll over the two conics for $\alpha = 0$ and $\beta = 0$ or a degeneration of a scroll.

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