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ORBIFOLD CLASSIFYING SPACES AND QUOTIENTS OF COMPLEX TORI

Dedicated to Gianfranco Casnati, in memoriam, a dear friend and a valuable collaborator

Abstract. In this paper we characterize the quotients $X = T/G$ of a complex torus T by the action of a finite group G as the Kähler orbifold classifying spaces of the even Euclidean crystallographic groups Γ, and we prove other similar and stronger characterizations.

Introduction

Complex tori are the simplest compact Kähler manifolds, they are the quotients $T = \mathbb{C}^n/\Lambda$ where $\Lambda \cong \mathbb{Z}^{2n}$ is a discrete subgroup of maximal rank $= 2n$.

To give the flavour of the results of this paper, we observe that complex tori are the cKM: = compact Kähler Manifolds X which are $K(\mathbb{Z}^{2n}, 1)$ spaces, that is, classifying spaces for the group $\mathbb{Z}^{2n}$ (Corollary 82 of [Cat15] shows more generally that they are exactly the compact Kähler manifolds X which are classifying spaces of a non trivial Abelian group). For non experts, this means that the fundamental group $\pi_1(X) \cong \mathbb{Z}^{2n}$ and the universal covering $\tilde{X}$ of X is a contractible topological space.

This follows from a stronger result, Proposition 4.8 of [Cat02] (= Proposition 2.9 of [Cat04]) showing that a compact complex manifold X whose integral cohomology algebra $H^*(X,\mathbb{Z})$ is isomorphic to the exterior algebra $\Lambda^*(\mathbb{Z}^{2n})$ is a torus if and only if X possesses n independent closed holomorphic 1-forms. If X is projective, then the first property alone suffices to guarantee that X is an Abelian variety (a complex tours which is a projective manifold).

The Generalized Hyperelliptic Manifolds, which we shall here simply call Hyperelliptic Manifolds, are the quotients $X = T/G$ of a complex torus T by the action of a finite group G acting freely (that is, no transformation in the group G has fixed points), and not consisting only of translations.

In dimension 2, these manifolds were introduced and classified by Bagnera and de Franchis and Enriques and Severi ([1] and [ES09]).

Recall also that a $K(\Gamma, 1)$ manifold is a manifold M such that its universal covering is contractible, and such that $\pi_1(M) \cong \Gamma$.

In [Cat-Cor17] it was shown that the Hyperelliptic Manifolds X are the cKM which are classifying spaces for torsion free even Euclidean crystallographic groups Γ, thus describing explicitly their Teichmüller spaces (Theorem 81 of [Cat15] uses a weaker assumption, similar to the one described above for complex tori, but does not describe the fundamental groups Γ).

The main purpose of this note is to extend these results to quotients $X = T/G$ by groups G which are not acting freely.

In the case where the action of G is **quasi-free**, namely, G acts freely outside of a closed algebraic set of codimension at least 2, this works simply by considering the normal complex space X; but, in the case where the set Σ of points x whose stabilizer is nontrivial has codimension 1, we must replace X by the complex orbifold $\mathscr{X}$ consisting of X and of the irreducible divisors $D_1, \dots, D_r$, whose union is the divisorial part of the image of Σ, each marked with the integer m_i which is the multiplicity of ramification over D_i (i.e., the ramification index is then $m_i - 1$).

As described in different ways in [Del-Mos93], [Cat00], [Cat08], [Cat15]*, we replace the fundamental group $\pi_1(X)$ by the Orbifold fundamental group $\Gamma := \pi_1^{orb}(\mathscr{X})$, which in the quasi-free case is the fundamental group $\pi_1(X^*)$ of the smooth locus X^* of X, while in general $\pi_1^{orb}(\mathscr{X})$ can be described as the group of lifts of the transformations of the group G to the universal cover $\tilde{T}$ of T.

In our special case Γ is a properly discontinuous group of affine transformations of $\mathbb{C}^n$, a so-called complex crystallographic group ([Bieb11, Bieb12], see also [Cat-Cor17]).

We have several options for the assumptions to be made, for instance this is a first result, in the projective case:

THEOREM 1. *Finite quotients of complex Abelian varieties are:*

(i) the Deligne-Mostow projective orbifolds which are orbifold classifying spaces for even Euclidean crystallographic groups Γ*,*

or more generally

(ii) the complex projective orbifolds with KLT singularities which are orbifold classifying spaces for even Euclidean crystallographic groups Γ*.*

In order to deal more generally with quotients of complex tori we need to use a Kähler assumption (since there are compact complex manifolds diffeomorphic to tori which are not complex tori, see [Som75], based on ideas introduced in [Blan56]).

The concept of a Kähler complex space was introduced by Grauert in [Gra62]: it means that it has a closed real form of type $(1,1)$ which at each point is induced from a positive definite one on the Zariski tangent space.

Fujiki in [Fuj78], see also [Fuj82] and [Ueno83] introduced the concept of a complex space in the class $\mathscr{C}$, which means that X is dominated by a holomorphic surjective map from a Kähler space X', equivalently, from a cKM X'. It was shown by Varouchas† [Var86], [Var89], that this is equivalent to requiring that the complex space X is bimeromorphic to a Kähler manifold.

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* [Del-Mos93] Section 14, [Cat00] definition 4.4 and Proposition 4.5, pages 25-26, [Cat08] definition 5.5, Proposition 5.8, pages 101-102, [Cat15] section 6.1, pages 316-318.

†Thanks to Thomas Peternell for providing the exact reference of this assertion, stated without further ado in [Cam91].

THEOREM 2. *Finite quotients of complex tori are:*

(i) the Deligne-Mostow orbifolds which are orbifold classifying spaces for even Euclidean crystallographic groups Γ*, and are moreover bimeromorphic to a Kähler manifold*

or more generally

(ii) the complex orbifolds with KLT singularities which are orbifold classifying spaces for even Euclidean crystallographic groups Γ*, and moreover are bimeromorphic to a Kähler manifold.*

THEOREM 3. *In Theorems 1 and 2 one can replace the assumption that they are orbifold classifying spaces by the conditions:*

1. *Their orbifold fundamental group is an even Euclidean crystallographic group* Γ,
2. *the integral cohomology algebra of the orbifold covering* Y *associated to the subgroup* $\Lambda < \Gamma$ *is a free exterior algebra* $\Lambda^*(\mathbb{Z}^{2n})$

We were inspired by the recent preprint [GKP23], which established a characterization for the quasi-free case under the assumptions of homotopy equivalence to such a torus quotient, KLT singularities, and being bimeromorphic to a Kähler manifold.

1. Complex orbifolds, Deligne-Mostow orbifolds, orbifold fundamental groups, orbifold coverings

DEFINITION 1. *(compare* 5.5 *in [Cat08], and section* 4 *of [Del-Mos93])*

Let Z *be a normal complex space, let* D *be a closed analytic set of* Z *and let* $\{D_i | i \in \mathcal{J}\}$ *be the irreducible components of* D *of codimension* 1 *in the case where* D *is compact.*

(1) Attaching to each D_i *a positive integer* $m_i \geq 1$*, we obtain a* **complex orbifold***, if moreover* $D = (\cup_{i \in \mathcal{J}} D_i) \cup Sing(Z)$.

(2) The **orbifold fundamental group** $\pi_1^{orb}(Z \setminus D, m_1, \dots, m_r, \dots))$ *is defined as the quotient*

$$\pi_1^{orb}(Z \setminus D, (m_1, \dots, m_r, \dots)) := \pi_1(Z \setminus D) / \langle\langle \gamma_1^{m_1}, \dots, \gamma_r^{m_r}, \dots \rangle\rangle$$

of the fundamental group of $(Z \setminus D)$ *by the subgroup normally generated by simple geometric loops* γ_i *going each around a smooth point of the divisor* D_i *(and counterclockwise).*

(3) The orbifold is said to be **quasi-smooth** *or geometric if moreover* D_i *is smooth outside of* $Sing(Z)$.

(4) The orbifold is said to be a **Deligne-Mostow orbifold** *if moreover for each point* $z \in Z$ *there exists a local chart* $\phi : \Omega \to U = \Omega / G$*, where* $0 \in \Omega \subset \mathbb{C}^n$*,* G *is a finite*

subgroup of $GL(n,\mathbb{C})$, $\phi(0) = z$, U is an open neighbourhood of z, and the orbifold structure is induced by the quotient map. That is, $D \cap U$ is the branch locus of Φ, and the integers m_i are the ramification multiplicities.

(5) We identify two orbifolds under the equivalence relation generated by forgetting the divisors D_i with multiplicity 1.

REMARK 1. (i) A D-M (= Deligne-Mostow) orbifold is quasi-smooth, and then Z has only quotient singularities, which are rational singularities.

(ii) In the case where there is no divisorial part, and we have an orbifold, then the orbifold fundamental group is just the fundamental group of $Z \setminus Sing(Z)$.

(iii) If $Z = M/\Gamma$ is the quotient of a complex manifold M by a properly discontinuous subgroup Γ, then Z is a D-M orbifold, because the stabilizer subgroups are finite, and, by Cartan's lemma ([Cart57]), the action of a stabilizer subgroup becomes linear after a local change of coordinates.

(iv) one could more generally consider a wider class of orbifolds allowing also the multiplicity $m_i = \infty$: this means that the relation $\gamma_i^{m_i} = 1$ is a void condition.

(v) replacing D by its intersection with the union of the singular locus with the divisorial components does not change the orbifold fundamental group.

Thanks to the extension by Grauert and Remmert [G-R58] of Riemann's existence theorem to finite holomorphic maps of normal complex spaces, we have that to a subgroup of the orbifold fundamental group corresponds a connected **orbifold covering** of orbifolds, that is (see for instance [Del-Mos93])

- a finite holomorphic map
$$f : \mathcal{Z} := (Z, D(m_1, \dots, m_r, \dots)) \to \mathcal{W} := (W, B(n_1, \dots, n_s, \dots))$$
such that
- f induces an étale (unramified) map $F : Z \setminus D \to W \setminus B$,
- for each D_i, $f(D_i) = B_j$ for some j, and locally $\gamma_i \mapsto \delta_j^{a_i}$, where $n_j = a_i m_i$, and δ_j is a simple loop around B_j.
- $f^{-1}(B_j)$ is set theoretically a union of divisors D_i (keep in mind here the equivalence relation explained in (5)).

To the trivial subgroup corresponds the orbifold universal cover
$$(\tilde{Z}, \tilde{D}, \{\tilde{m}_j\}).$$

DEFINITION 2. *We say that an orbifold $(Z, D(m_1, \dots, m_r))$ is an orbifold classifying space if its universal covering $(\tilde{Z}, \tilde{D}, \{\tilde{m}_j\})$ satisfies the properties*

(a) either $\tilde{Z}$ is contractible and the multiplicities $\tilde{m}_j$ are all equal 1, *or*

(b) there is a homotopy retraction of $\tilde{Z}$ to a point which preserves the subdivisor $\tilde{D}'$ consisting of the irreducible components with multiplicity $\tilde{m}_j > 1$.

REMARK 2. We end now this section showing two simple examples of an orbifold which is not a Deligne-Mostow orbifold.

(1) We just take as orbifold space $Z = \mathbb{C}^2$ and as divisors three distinct lines L_1, L_2, L_3 through the origin, marked with multiplicities m_1, m_2, m_3.

If this orbifold were an orbifold $\mathbb{C}^2/G$, since $\mathbb{C}^2$ is simply connected, then G would be isomorphic to the orbifold fundamental group π of Z.

But π (see for instance page 140 of [Cat06]) has a presentation

$$\pi := \langle \gamma_0, \gamma_1, \gamma_2, \gamma_3 | [\gamma_0, \gamma_i] = 1, \text{for } i = 1,2,3, \gamma_0 = \gamma_1\gamma_2\gamma_3 \rangle.$$

If $\pi \cong G$ were finite, also $\pi / \langle \gamma_0 \rangle$ would be finite. However this quotient is finite if and only if

$$\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} > 1,$$

since the corresponding orbifold covering of $\mathbb{P}^1$ branched in three points must be simply connected, hence equal to $\mathbb{P}^1$; and this is equivalent to the above inequality which is only satisfied for the Platonic triples $(2,2,n), (2,3,3), (2,3,4), (2,3,5)$.

(2) Another easy example is given by a normal surface singularity which is not a quotient singularity, for instance an elliptic surface singularity.

2. Euclidean crystallographic groups and Actions of a finite group G on a complex torus T

For the reader's benefit, we repeat some results on complex crystallographic groups, as exposed in [Cat-Cor17].

DEFINITION 3. *(i) A group Γ is an abstract Euclidean crystallographic group if there exists an exact sequence of groups*

$$(*) \quad 0 \to \Lambda \to \Gamma \to G \to 1$$

such that

1. *G is a finite group*
2. *Λ is free abelian (we shall denote its rank by r)*
3. *Inner conjugation $Ad : \Gamma \to Aut(\Lambda)$ has Kernel exactly Λ, hence Ad induces an embedding, called* **Linear part**,

$$L : G \to GL(\Lambda) := Aut(\Lambda).$$

(ii) An **affine realization defined over a field** $K \supset \mathbb{Z}$ *of an abstract Euclidean crystallographic group Γ is a homomorphism (necessarily injective)*

$$\rho : \Gamma \to Aff(\Lambda \otimes_{\mathbb{Z}} K)$$

such that

[1] Λ *acts by translations on* $V_K := \Lambda \otimes_{\mathbb{Z}} K$, $\rho(\lambda)(v) = v + \lambda$,

[2] for any γ *a lift of* $g \in G$ *we have:*

$$V_K \ni v \mapsto \rho(\gamma)(v) = Ad(\gamma)v + u_\gamma = L(g)v + u_\gamma, \text{ for some } u_\gamma \in V_K.$$

For the following Theorems, see [Cat-Cor17]: observe that the unicity of the affine realization was proven by Bieberbach in 1912 ([Bieb12])(who proved also many other deeper results).

THEOREM 4. *Given an abstract Euclidean crystallographic group there exists an affine realization, for each field* $K \supset \mathbb{Z}$, *and its class is unique.*

Assume that we have the action of a finite group G' on a complex torus $T = V/\Lambda'$, where V is a complex vector space, and $\Lambda' \otimes_{\mathbb{Z}} \mathbb{R} \cong V$.

Since every holomorphic map between complex tori lifts to a complex affine map of the respective universal covers, we can attach to the group G' the group Γ of (complex) affine transformations of V which are lifts of transformations of the group G'.

Again one easily sees ([Cat-Cor17]:

PROPOSITION 1. Γ *is an Euclidean crystallographic group, via the exact sequence*

$$0 \to \Lambda \to \Gamma \to G \to 1$$

where $\Lambda \supset \Lambda'$ *is the lattice in* V *such that* $\Lambda := Ker(Ad)$, $Ad : \Gamma \to GL(\Lambda')$, *and* $G \subset Aut(V/\Lambda)$ *contains no translations.*

Hence the datum of the action of a finite group G on a complex torus T, containing no translations, is equivalent to giving:

1. a crystallographic group Γ
2. a complex structure J on the real vector space $V_{\mathbb{R}}$ which makes the action of G complex linear.

The complex structure J exists if and only if Γ is even, according to the following definition:

DEFINITION 4. *A crystallographic group* Γ *is said to be* **even** *if:*

i) Λ *is a free abelian group of even rank* $r = 2n$

ii) considering the associated faithful representation

$$G \to Aut(\Lambda \otimes \mathbb{C}),$$

for each real irreducible representation χ *of* G, $(\Lambda \otimes \mathbb{C})_\chi$ **has even complex dimension**.

3. Proof of the Main Theorems

3.1. Properties of $X = T/G$ and proof of the easier implications

I) Let $X = T/G$ be as above the quotient orbifold of a torus by the action of a finite group. Since the universal covering of $T = V/\Lambda$ is the vector space V, which is contractible, and $X = V/\Gamma$, where $\Gamma := \pi_1^{orb}(X)$ we obtain that X is an orbifold classifying space. Moreover, that Γ is an Euclidean crystallographic group follows from proposition 1, and Γ is even since there is a G-invariant complex structure.

II) Consider now the normal subgroup $G^{pr} < G$ generated by the pseudoreflections, that is, the linear maps which have only one eigenvalue $\neq 1$.

We have in particular a factorization

$$T \to W := T/G^{pr} \to X = T/G,$$

where the second map is quasi-étale, that is, ramified only in codimension at least 2.

At any point $t \in T$ having a nontrivial stabilizer $G_t < G$, we have similarly a corresponding normal subgroup G_t^{pr}, generated by the pseudoreflections in G_t. By Chevalley's Theorem, the local quotient of T at t by G_t^{pr} is smooth, and then we have the further quotient by G_t/G_t^{pr}.

In particular, X is a Deligne-Mostow orbifold and its singularities are quotient singularities.

By [K-M98] (Prop. 5.15, page 158) quotient singularities (X, x) are rational singularities, that is, they are normal and, if $f : Z \to X$ is a local resolution, then $\mathcal{R}^i f_* \mathcal{O}_Z = 0$ for $i \geq 1$. They enjoy also the stronger property of being KLT (Kawamata Log Terminal) singularities.

Indeed Prop. 5.22 of [K-M98] (where dlt=KLT if there is no boundary divisor Δ, Δ') says that KLT singularities are rational singularities, while Prop. 5.20, page 160, says that if we have a finite morphism between normal varieties, $F : T \to X$, then X **is KLT if and only if** T **is KLT**).

III) If T is projective, then also X is projective, since, by averaging, we can find on T a G-invariant very ample divisor.

3.2. The Kähler property

IV) In the general case where the torus T is not projective, but only Kähler, to show that X enjoys some Kählerian properties, we need to recall some results by Fujiki and others (which could also be used to slightly vary the hypotheses in our results).

As already mentioned in the Introduction, Grauert [Gra62] defined the concept of a Kähler complex space, and later Varouchas [Var86] proved that if $X \to Y$ is a surjective holomorphic map with Y reduced, and X Kähler, then Y is bimeromorphic to a Kähler manifold.

Instead Fujiki in [Fuj78] (see also [Fuj82] and [Ueno83]), introduced the concept of a complex space X bimeromorphic to a Kähler manifold, and of a complex space in the class $\mathscr{C}$, meaning that X is dominated by a holomorphic surjective map from a Kähler space X', equivalently, from a cKM X'.

Here the first basic result in this line of thought: 1.5 of [Ueno83] implies that:

given a generally finite map $Y \to X$, X is in $\mathscr{C}$ if and only if $Y \in \mathscr{C}$

(in [Cat95], sections $17-1.9$ was observed the easier result that if a compact complex manifold Z has a generically finite map to a cKM M, then Z is bimeromorphic to a Kähler manifold: this evidently also holds if Z is a complex space).

Fujiki also asked (remark 4.4, page 35 of [Fuj78-b]) whether manifolds in the class $\mathscr{C}$ are just those which are bimeromorphic to a Kähler manifold: his conjecture was shown to be true by Varouchas [Var86], [Var89].

Whence,

(I) the quotients $X = T/G$ are bimeromorphic to a Kähler manifold.

Moreover

(II) the quotients $X = T/G$ are also Kähler complex spaces since [Var89] proved that if $W \to X$ is proper and open, with X normal, from the property that W is a Kähler space follows that also X is Kähler.

Since, again for instance by [Var86], property (II) implies property (I), in our Theorems we opt for assuming only Property (I).

Finally, a crucial result is that (see Prop. 1.3 of [Ueno83]) if a compact complex manifold M is in the class $\mathscr{C}$, then the cohomology of M admits a Hodge decomposition, and in particular every holomorphic form is d-closed, and there is an Albanese map $\alpha : M \to Alb(M)$, such that $\alpha^* : H^1(Alb(M), \mathbb{C}) \to H^1(M, \mathbb{C})$ is an isomorphism.

3.3. Proof of Theorem 1.

We need to show the converse implication. **Key argument: we consider the orbifold covering Y associated to the normal subgroup**

$$\Lambda < \Gamma := \pi_1^{orb}(X),$$

and we show that Y is a complex torus (here projective because X is projective and Y has a finite map to X).

LEMMA 1. *The orbifold Y is just a normal complex space, that is, there are no marked divisors with multiplicity $m_i \geq 2$.*

Proof. Consider the exact sequence

$$1 \to \Lambda \to \Gamma \to G \to 1.$$

Then the generators γ_i have finite order m_i, hence their image in G has order exactly m_i, because Λ is torsion free.

This means that the covering $Y \to X$ is ramified with multiplicity m_i at the divisor D_i, and therefore their inverse image in Y is a reduced divisor with multiplicity 1.

□

In case (i), since X is a Deligne-Mostow orbifold, then also Y is a D-M orbifold, hence it is a normal space with quotient singularities, and these are rational singularities.

Let Y' be a resolution of Y. Since Y has rational singularities, $\mathcal{R}^1 f_*(\mathbb{Z}_{Y'}) = 0$ and we have an isomorphism

$$H^1(Y', \mathbb{Z}) \cong H^1(Y, \mathbb{Z}) \cong \mathbb{Z}^{2n}.$$

Hence the Albanese variety of Y' is a complex torus of dimension n, and the Albanese map $\alpha' : Y' \to T := Alb(Y')$ factors through Y, and $\alpha : Y \to T$ is a homotopy equivalence, in particular it has degree 1 (because it induces an isomorphism of $H^{2n}(T, \mathbb{Z}) \cong H^{2n}(Y, \mathbb{Z})$).

We follow a similar argument to the one used in [Cat02], proof of Proposition 4.8: it suffices to show that α is finite, because α, being finite and birational, is then an isomorphism $Y \cong T$ by normality.

Now, since α is birational, by Zariski's Main Theorem (the Hartogs property in the case of normal complex spaces) α is an isomorphism unless there is a divisor D which is contracted by α. And, since $H^j(T, \mathbb{Z}) \cong H^j(Y, \mathbb{Z})$, the class of D is trivial in $H^2(Y, \mathbb{Z})$, and a fortiori its pull back D' to Y' is trivial.

This is a contradiction since, Y' being Kähler, the class of D' cannot be trivial.

In case (ii) the proof is identical, we need only to establish that Y has rational singularities.

As already discussed, if X has KLT singularities, by Prop. 5.20 of [K-M98] also Y has KLT singularities, which are rational singularities.

3.4. Proof of Theorems 2 and 3

The proof is essentially the same.

Indeed, X is assumed to be bimeromorphic to a Kähler manifold, that is, in the class $\mathcal{C}$, and by the results of Fujiki, Ueno and Varouchas also Y is bimeromorphic to a Kähler manifold Y', that we can assume to dominate Y.

By our assumption Y has again rational singularities, and we may consider again the Albanese map $\alpha' : Y' \to T := Alb(Y')$, which again factors through a birational map $\alpha : Y \to T$. We derive the same contradiction.

REMARK 3. One may ask whether one can replace the condition of KLT singularities for X by the condition that X has rational singularities, proving then that also Y has rational singularities.

4. Parametrizing Families

We simply observe now, as in [Cat-Cor17], how these orbifolds are parametrized by a finite union of connected complex manifolds, which are just products of Grassmann manifolds.

The connected component $\mathcal{T}_n$ of the Teichmüller space of n-dimensional complex tori (see [Cat02], [Cat04] and [Cat13]) is the open set $\mathcal{T}_n$ of the complex Grassmann Manifold $Gr(n,2n)$, image of the open set of matrices

$\mathscr{F} := \{\Omega \in \mathrm{Mat}(2n,n;\mathbb{C}) \mid i^n det(\Omega\,\overline{\Omega}) > 0\}$.

Over $\mathscr{F}$ lies the following tautological family of complex tori: consider a fixed lattice $\Lambda := \mathbb{Z}^{2n}$, and associate to each matrix Ω as above the subspace V of $\mathbb{C}^{2n} \cong \Lambda \otimes \mathbb{C}$ given as

$$V := \Omega\mathbb{C}^n,$$

so that $V \in Gr(n,2n)$ and $\Lambda \otimes \mathbb{C} \cong V \oplus \bar{V}$.

To V we associate then the torus

$$T_V := V / p_V(\Lambda) = (\Lambda \otimes \mathbb{C})/(\Lambda \oplus \bar{V}),$$

$p_V : V \oplus \bar{V} \to V$ being the projection onto the first summand.

The crystallographic group Γ determines an action of $G \subset SL(2n,\mathbb{Z})$ on $\mathscr{F}$ and on $\mathcal{T}_n$, obtained by multiplying the matrix Ω with matrices $g \in G$ on the right.

We define then $\mathcal{T}_n^G$ as the locus of fixed points for the action of G. If $V \in \mathcal{T}_n^G$, then G acts as a group of biholomorphisms of T_V, and we associate then to such a V the orbifold

$$X_V := T_V / G.$$

We see as in [Cat-Cor17] that $\mathcal{T}_n^G$ consists of a finite number of components, indexed by the Hodge type of the Hodge decomposition.

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