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TORSION–FREE SHEAVES AND ACM SCHEMES

Dedicated to the memory of Gianfranco Casnati

Abstract. In this paper we study short exact sequences $0 \to \mathscr{P} \to \mathscr{N} \to \mathscr{I}_D(k) \to 0$ with $\mathscr{P}, \mathscr{N}$ torsion–free sheaves and D closed projective scheme. This is a classical way to construct and study projective schemes. In particular, we give homological conditions on $\mathscr{P}$ and $\mathscr{N}$ that force D to be ACM, without constrains on its codimension. As last result, we prove that if $\mathscr{N}$ is a higher syzygy sheaf of an ACM scheme X, the scheme D we get contains X.

1. Introduction

Homological methods have proved to be very useful in studying projective schemes (see [1,4,9,19] among the many papers where such methods have been applied). For example, many information on the geometry of a closed scheme $X \subseteq \mathbb{P}^r$ are encoded in the minimal free resolution of the saturated ideal I_X of X. Homological methods are used also to construct schemes with prescribed properties (see [2,12,14,16,20] in connection with liaison theory and its generalizations). For example, in [14], M. Martin–Deschamps and D. Perrin gave a homological construction of the ideal of a curve C in $\mathbb{P}^3$ with a prescribed Hartshorne–Rao module and of minimal degree. In more detail, given a graded Artinian $R := K[x, y, z, w]$–module M with minimal free resolution

$$0 \to L_4 \to L_3 \to L_2 \to L_1 \to L_0 \to M \to 0,$$

they show how to compute a free graded R–module P such that the cokernel of a general injective map $\gamma : P \to N := \ker(L_1 \to L_0)$ is isomorphic to the saturated ideal of a locally Cohen–Macaulay curve $C \subset \mathbb{P}^3$, up to a shift in grading, that is to say, they produce a short exact sequence

$$0 \to P \xrightarrow{\gamma} N \to I_C(k) \to 0. \tag{1}$$

An analogous sequence was used first by J.P.Serre in [19] to construct subcanonical curves in $\mathbb{P}^3$. To this end, he considered a rank 2 vector bundle $\mathscr{N}$, a global section s whose zero–set has codimension 2, and the corresponding map $\mathscr{O} \xrightarrow{s} \mathscr{N}$. The image of the dual map $\mathscr{N}^\vee \to \mathscr{O}$ is the ideal sheaf of a subcanonical curve $C \subset \mathbb{P}^3$. J.P. Serre's construction was generalized to construct codimension 2 schemes in $\mathbb{P}^r$ (see [7], [18], among others). For example, in [9], R. Hartshorne considered sections, whose zero–set has codimension 2, of reflexive rank 2 sheaves on

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$\mathbb{P}^3$. In this new and more general setting, the constructed schemes were generically locally complete intersection curves.

While studying the construction of minimal curves by M. Martin–Deschamps and D. Perrin given in [14], we applied it to syzygy modules of 0–dimensional schemes of $\mathbb{P}^3$ instead of syzygy modules of graded Artinian R–modules. The curves we produced were all arithmetically Cohen–Macaulay. To understand why the curves share this unexpected property, we were led to consider all the previous apparently different constructions from the same point of view, getting as result a quite general construction of arithmetically Cohen–Macaulay schemes of arbitrary codimension. For particular choices, we construct arithmetically Cohen–Macaulay schemes containing a given scheme with the same property but of larger codimension.

We outline the structure of the paper. In section 2, first of all we describe some properties of torsion–free coherent sheaves, and their cohomology. Then, we get some bounds on the projective dimensions of $\mathcal{N}$ and $\mathcal{P}$ in terms of the codimension of D and of the cohomology of its ideal sheaf $\mathcal{I}_D$. Finally, we recall the well known result of Martin–Deschamps and Perrin, described in [14], about maximal subsheaves which allows us to assure that the cokernel of a given injective map $\mathcal{P} \to \mathcal{N}$ is an ideal sheaf.

Section 3 is the heart of the paper. At first, we give some conditions on the coherent torsion–free sheaves $\mathcal{N}$ and $\mathcal{P}$ to assure that the short exact sequence (1) ends with the ideal sheaf of a closed arithmetically Cohen–Macaulay subscheme D of $\mathbb{P}^r$ of codimension $2+\mathrm{pd}(\mathcal{P})$, where $\mathrm{pd}(\mathcal{P})$ is the projective dimension of $\mathcal{P}$. Moreover, we show that the construction characterizes the couple $(D,\mathcal{P})$ in the sense that starting from an arithmetically Cohen–Macaulay scheme D and a torsion–free coherent sheaf $\mathcal{P}$, we can construct a sheaf $\mathcal{N}$ fulfilling our conditions.

In the codimension 2 case we give a geometrical description of our construction associating to any non–zero element of $H^0(D,\omega_D(c))$ an extension as (1). This is a new reading of the analogous result of [19], for coherent torsion–free sheaves, without bounds on the rank of $\mathcal{N}$. We show also that some schemes we obtain in our setting cannot be obtained with Hartshorne's construction, and conversely. So, the two constructions are not the same one.

Section 4 is devoted to solve the problem of finding a codimension s closed scheme D containing a given codimension $t(>s)$ scheme X, them both arithmetically Cohen–Macaulay. We end the section with some examples.

2. Preliminary results

Let K be an algebraically closed field, and let $R = K[x_0,\ldots,x_r]$ be the graded polynomial ring. Let $\mathbb{P}^r = \mathrm{Proj}(R)$ be the projective space of dimension r over K. If $X \subseteq \mathbb{P}^r$ is a closed scheme, we denote by $\mathcal{I}_X$ its ideal sheaf in $\mathcal{O}_{\mathbb{P}^r}$ and by I_X its saturated ideal in R, and it holds $I_X = H^0_*(\mathbb{P}^r,\mathcal{I}_X)$.

By R–module (sheaf, resp.) we mean "graded R–module" ("coherent $\mathcal{O}_{\mathbb{P}^r}$–module", resp.). If F is a R–module we denote by $\mathcal{F}$ the corresponding sheaf, namely $\mathscr{F} := \tilde{F}$.

We recall that a local ring A is Cohen–Macaulay if $\dim(A) = \operatorname{depth}(A)$. A ring A is Cohen–Macaulay if $A_{\mathfrak{M}}$ is Cohen–Macaulay for every maximal ideal $\mathfrak{M} \subset A$. A scheme X is Cohen–Macaulay if the ring $\mathscr{O}_{X,x}$ is Cohen–Macaulay for every closed point $x \in X$. A closed scheme $X \subseteq \mathbb{P}^r$ is arithmetically Cohen–Macaulay (ACM, for brief) if the coordinate ring $R_X = R/I_X$ is a Cohen–Macaulay ring. This is equivalent to say that $H^i_*(\mathscr{I}_X) = 0$ for $1 \le i \le \dim(X)$.

For any finitely generated R–module P we denote by $\operatorname{pd}(P)$ the projective dimension of P, i.e., the length of the minimal free resolution of P ([5], Theorem 19.1 and the previous Definition).

Let $D \subseteq \mathbb{P}^r$ be a closed scheme, and let $I_D \subseteq R$ be its saturated ideal. If

$$0 \to F_t \to \cdots \to F_2 \to F_1 \to I_D \to 0$$

is the minimal free resolution of I_D, with $t \le r$, and P is the kernel of $F_1 \to I_D$, then we have a short exact sequence

$$0 \to P \to F_1 \to I_D \to 0$$

which is equivalent to the minimal free resolution. The R–module P is a torsion–free finitely generated R–module with projective dimension $\operatorname{pd}(P) = \operatorname{pd}(I_D) - 1$. We can also consider the short exact sequence

$$0 \to \mathscr{P} \to \mathscr{F}_1 \to \mathscr{I}_D \to 0$$

obtained by considering the sheaves associated to the modules in the former sequence. Of course, $\mathscr{P}$ is a torsion–free sheaf, and $\mathscr{F}_1$ is dissocié, according to the following definitions.

DEFINITION 1. *A R–module M is torsion–free if every non–zero element of R is a non zero–divisor of M.*

A sheaf $\mathscr{F}$ on $\mathbb{P}^r$ is torsion–free if $\mathscr{F}(U)$ is a torsion–free $\mathscr{O}_{\mathbb{P}^r}(U)$–module for every open subset $U \subseteq \mathbb{P}^r$, or equivalently, $\mathscr{F}_x$ is torsion–free over $\mathscr{O}_{\mathbb{P}^r,x}$ for every $x \in \mathbb{P}^r$.

DEFINITION 2. *Let $\mathscr{F}$ be a sheaf on $\mathbb{P}^r$. We say that $\mathscr{F}$ is dissocié of rank s if*

$$\mathscr{F} = \oplus_{i=1}^{s} \mathscr{O}_{\mathbb{P}^r}(a_i)$$

for suitable integers $a_1, \ldots, a_s$.

Of course, if F is a free R–module, then $\mathscr{F} = \tilde{F}$ is dissocié. Conversely, if $\mathscr{F}$ is dissocié, then $H^0_*(\mathscr{F})$ is a free R–module.

Generalizations of the approach consist in relaxing the strong hypothesis “dissocié" on $\mathscr{F}_1$. Hence, let us consider the short exact sequence

$$0 \to \mathscr{P} \xrightarrow{\gamma} \mathscr{N} \to \mathscr{I}_D(k) \to 0 \tag{2}$$

with $\mathscr{P}$ torsion–free, and $k \in \mathbb{Z}$. Standard arguments allow us to prove that $\mathscr{N}$ is torsion–free, as well. So, the weakest hypothesis on $\mathscr{N}$ is torsion–free. On the other hand, short exact sequences are classified by $\mathrm{Ext}^1_{\mathscr{O}_{\mathbb{P}^n}}(\mathscr{I}_D, \mathscr{P})$.

As we are interested in sequences of sheaves, it will help to have the analogue for sheaves of the minimal free resolution and of projective dimension of a graded finitely generated module.

By ([8], Ch. II, Corollary 5.18), we have that any sheaf $\mathscr{P}$ admits a *dissocié resolution*, namely a resolution by dissocié sheaves. We need to be more precise on this point, and so we begin with some preliminaries.

REMARK 1. We recall some facts about associated points. For more details see e.g. ([15], Ch. 3), where the case of (ungraded) modules is dealt with. Extending to sheaves is straightforward.

(i) Let $\mathscr{F}$ be a sheaf. A (not necessarily closed) point $y \in \mathbb{P}^r$ is associated to $\mathscr{F}$ if there is an open affine $U = \mathrm{Spec}(A) \subseteq \mathbb{P}^r$ containing y such that the prime ideal of A corresponding to y is associated to the A-module $\Gamma(U, \mathscr{F})$; this is equivalent to say that $\mathrm{depth}_{\mathscr{O}_{\mathbb{P}^r,y}}(\mathscr{F}_y) = 0$.

(ii) The set $Ass(\mathscr{F})$ of the associated points to $\mathscr{F}$ is finite.

(iii) Any form f of degree n avoiding all elements of $Ass(\mathscr{F})$ induces by multiplication an injective morphism $\mathscr{F} \xrightarrow{\cdot f} \mathscr{F}(n)$. Hence a general form of degree n has this property.

(iv) ([3], Exercise 20.4.21) The graded R- module $H^0_*(\mathscr{F})$ is finitely generated if and only if $\mathrm{Ass}(\mathscr{F})$ contains no closed points, if and only if $\mathrm{depth}_{\mathscr{O}_{\mathbb{P}^r,x}}(\mathscr{F}_x) > 0$ for every (closed) $x \in \mathbb{P}^r$.

Now, we prove that every sheaf has a dissocié resolution of finite length. We recall that all sheaves we consider are coherent, as stated at the beginning of the present section.

LEMMA 1. *Let $\mathscr{F}$ be a sheaf and let M be a graded submodule of $H^0_*(\mathscr{F})$. Then*

(a) *any general linear form induces by multiplication an injective map $M \to M(1)$;*

(b) *if M is finitely generated then $pd(M) \leq r$;*

(c) *$\mathscr{F}$ admits a dissocié resolution of length $\leq r$.*

Proof. (a) follows easily from Remark 1(iii).

(b) By (a) we have $\mathrm{depth}(M) \geq 1$ and the conclusion follows by the Auslander-Buchsbaum formula ([5], Exercise 19.8).

(c) Since $\mathscr{F}$ is coherent we have $\mathscr{F} = \tilde{M}$, where M is a suitable finitely generated graded submodule of $H^0_*(\mathscr{F})$ ([8], Ch. II, proof of Theorem 5.19). The conclusion follows from (b), because we get a dissocié resolution of $\mathscr{F}$ by sheafifying the minimal free resolution of M. □

Following ([6], Section 2), we define the minimal dissocié resolution of a coherent sheaf.

DEFINITION 3. *Let $\mathscr{P}$ be a sheaf such that $P := H^0_*(\mathscr{P})$ is finitely generated. Let*

$$0 \to H_d \to \cdots \to H_0 \to P \to 0$$

be the minimal free resolution of the R–module P . We name minimal dissocié resolution *of $\mathscr{P}$ the exact sequence*

$$0 \to \mathscr{H}_d \to \cdots \to \mathscr{H}_0 \to \mathscr{P} \to 0$$

obtained by sheafifying the minimal free resolution of P. (Recall that $\tilde{P} = \mathscr{P}$ by ([8], Ch. II, Proposition 5.4)).

Moreover, we define the projective dimension *of $\mathscr{P}$ as* $\mathrm{pd}(\mathscr{P}) := \mathrm{pd}(P)$.

REMARK 2. It is known that there exist many submodules of $P = H^0_*(\mathscr{P})$ whose associated sheaf is $\mathscr{P}$: in fact, it is enough that such a submodule M agrees with P for some large degree on. Of course, the sheafification of a minimal free resolution of M is still a dissocié resolution of $\mathscr{P}$, and no map is split. However, the resolution of M is longer than the minimal one. In fact, from the short exact sequence of modules

$$0 \to M \to P \to P/M \to 0,$$

we get that $\mathrm{pd}(M) = r$, because P/M is an Artinian module. Hence, $\mathrm{pd}(M) \geq \mathrm{pd}(P)$, as we claimed.

REMARKS 1. (i) Clearly $\mathrm{pd}(\mathscr{P}) = 0$ if and only if $\mathscr{P}$ is dissocié.
(ii) $\mathrm{pd}(\mathscr{P}) \leq r$ whenever defined (Lemma 1(b) applied with $M = H^0_*(\mathscr{P})$).

The next Lemma gives a bound for the projective dimension of a torsion-free sheaf.

LEMMA 2. *Let $\mathscr{P}$ be a torsion-free coherent sheaf on $\mathbb{P}^r$, and let $P = H^0_*(\mathscr{P})$. Then:*

(a) *P is finitely generated;*

(b) *P torsion-free;*

(c) *$\mathscr{P}$ is a subsheaf of a coherent dissocié sheaf;*

(d) $\mathrm{pd}(P) = \mathrm{pd}(\mathscr{P}) \leq r - 1$.

Proof. (a) It follows easily from Remark 1(iv).

(b) Any non zero form $f \in R$ of degree n induces, by multiplication, an injective morphism $\mathscr{P} \xrightarrow{\cdot f} \mathscr{P}(n)$, and consequently an injective homomorphism $P \xrightarrow{\cdot f} P(n)$.

(c) By (a) and (b), P is a torsion–free R-module and hence it is a graded submodule of a free R–module L. Then, $\mathscr{P} = \tilde{P}$ is a subsheaf of $\mathscr{L} := \tilde{L}$ and the claim follows.

(d) By (c) there exist a sheaf $\mathscr{F}$ and an exact sequence $0 \to \mathscr{P} \to \mathscr{L} \to \mathscr{F} \to 0$, whence an exact sequence of R-modules:

$$0 \to P \to L \to M \to 0,$$

where M is a graded submodule of $H^0_*(\mathscr{F})$. By Lemma 1(b) we have $\text{pd}(M) \leq r$, whence $\text{pd}(P) \leq r-1$. By (a), Definition 3 applies and the proof is complete. □

From now on, every R–module will be finitely generated, and so we shall skip this assumption.

It is possible to describe the cohomology of a coherent sheaf, as we said before.

LEMMA 3. *Let $r \geq 3$, let P be a R–module and let $\mathscr{P} = \tilde{P}$ be its associated sheaf. Suppose $d = \text{pd}(P) < r$. Then:*

(a) $H^0_*(\mathscr{P}) = P$;

(b) $H^i_*(\mathscr{P}) = 0$ *for* $1 \leq i \leq r-d-1$;

(c) $H^{r-d}_*(\mathscr{P}) \neq 0$.

(d) *If $\mathscr{P}$ is any torsion-free sheaf with $d := \text{pd}(\mathscr{P})$, then (b) and (c) hold.*

Proof. We prove claims $(a),(b),(c)$ together, by induction on d.

If $d = 0$, the sheaf $\mathscr{P}$ is dissocié and the claims hold by ([8], Ch. III, Theorem 5.1).

If $d = 1$ we have a non–split exact sequence

$$0 \to L_1 \to L_0 \to P \to 0$$

with L_1 and L_0 free. By passing to sheaves, we get a non–split exact sequence

$$0 \to \mathscr{L}_1 \to \mathscr{L}_0 \to \mathscr{P} \to 0, \tag{3}$$

whence $\text{Ext}^1(\mathscr{P}, \mathscr{L}_1) \neq 0$. It follows easily that $\text{Ext}^1(\mathscr{P}, \mathscr{O}_{\mathbb{P}^r}(k)) \neq 0$ for some $k \in \mathbb{Z}$. On the other hand by duality and properties of Ext we get $H^{r-1}(\mathbb{P}^r, \mathscr{P}(-k-r-1)) \cong \text{Ext}^1(\mathscr{P}(-k-r-1), \omega_{\mathbb{P}^r}) \cong \text{Ext}^1(\mathscr{P}, \mathscr{O}_{\mathbb{P}^r}(k))$, whence (c). Since (a) and (b) are immediate from the exact sequence (3), the statement holds for $d = 1$ as well.

Assume now $d \geq 2$. We have an exact sequence

$$0 \to P_1 \to G \to P \to 0$$

where G is a free R–module and P_1 is a R–module with $\text{pd}(P_1) = d-1$. In fact, it is enough to consider the first short exact sequence that can be obtained from the minimal free resolution of P, as explained before. By taking the sheaves associated to each item, we get the short exact sequence of sheaves

$$0 \to \mathcal{P}_1 \to \mathcal{G} \to \mathcal{P} \to 0. \tag{4}$$

By induction, we may assume that $H^0_*(\mathcal{P}_1) = P_1$, $H^i_*(\mathcal{P}_1) = 0$ for $1 \le i \le r-(d-1)-1 = r-d$, and that $H^{r-d+1}_*(\mathcal{P}_1) \neq 0$.

By assumption, $d < r$, and so $r - d \ge 1$. In particular, $H^1_*(\mathcal{P}_1) = 0$.

By taking the cohomology sequence associated to (4) and using the assumptions on the cohomology of $\mathcal{P}_1$ we get the conclusion.

To prove (d), set $P := H^0_*(\mathcal{P})$. Then by definition and by Lemma 2 we have $d = \mathrm{pd}(P) < r$. Since $\tilde{P} = \mathcal{P}$ the conclusion follows by (b) and (c). □

The previous Lemma allows us to generalize Horrocks' splitting criterion ([18], Theorem 2.3.1) to torsion–free sheaves, with a completely different proof (see ([1] Corollary 1.3) for another generalization to locally–free sheaves).

COROLLARY 1. *A torsion–free sheaf $\mathcal{P}$ over $\mathbb{P}^r$ is dissocié precisely when $H^i_*(\mathcal{P}) = 0$ for $i = 1,\dots,r-1$.*

Proof. Assume $H^i_*(\mathcal{P}) = 0$ for $i = 1,\dots,r-1$, and set $d := \mathrm{pd}(\mathcal{P})$. By Lemma 3$(d)$ we have $H^i_*(\mathcal{P}) = 0$ for $i = 1,\dots,r-1-d$ and $H^{r-d}_*(\mathcal{P}) \neq 0$. This is possible only if $d = 0$, i.e. if $\mathcal{P}$ is dissocié. The converse is clear. □

Now, we consider the short exact sequence (2). Our first result relates the codimension of D and the projective dimension of $\mathcal{P}$.

PROPOSITION 1. *Let $D \subseteq \mathbb{P}^r$ be a closed scheme of codimension s, with $s \ge 2$, and let P a R–module with* $\mathrm{pd}(P) = d$. *If $s-2 > d$, then $\mathrm{Ext}^j_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{J}_D(k),\mathcal{P}) = 0$ for $j = 1,\dots,s-2-d$.*

Proof. We prove the claim by induction on d.

If $d = 0$, that is to say P is a free module, then there exist $a_1,\dots,a_n \in \mathbb{Z}$ such that $P = \oplus_{i=1}^n R(-a_i)$. By using standard properties of Ext groups, we have

$$\begin{aligned}\mathrm{Ext}^j{}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{J}_D(k),\mathcal{P}) &= \oplus_{i=1}^n \mathrm{Ext}^j_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{J}_D(k),\mathcal{O}_{\mathbb{P}^r}(-a_i)) = \\ &= \oplus_{i=1}^n \mathrm{Ext}^j_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{J}_D(k+a_i-r-1),\omega_{\mathbb{P}^r}) \cong \\ &\cong \oplus_{i=1}^n H^{r-j}(\mathbb{P}^r,\mathcal{J}_D(k+a_i-r-1)) = \\ &= \oplus_{i=1}^n H^{r-j-1}(D,\mathcal{O}_D(k+a_i-r-1)) = 0\end{aligned}$$

as soon as $r-j-1 > r-s$ by Grothendieck's vanishing Theorem ([8], Ch.III, Theorem 2.7), where $\omega_{\mathbb{P}^r} = \mathcal{O}_{\mathbb{P}^r}(-r-1)$ is the canonical sheaf of $\mathbb{P}^r$. Hence, $\mathrm{Ext}^j(\mathcal{J}_D(k),\mathcal{P}) = 0$ for $j = 1,\dots,s-2$ and for every $k \in \mathbb{Z}$, and the claim holds for $d = 0$.

Assume $d > 0$ and the claim to hold true for every R–module with projective dimension $d-1$. As in the proof of Lemma 3, we consider the short exact sequence

$$0 \to P_1 \to G \to P \to 0$$

with G free and P_1 of projective dimension $d-1$. By applying $\mathrm{Hom}(\mathscr{I}_D(k), -)$ to the sheafified sequence, we get the exact sequence

$$\mathrm{Ext}^i(\mathscr{I}_D(k), \mathscr{G}) \to \mathrm{Ext}^i(\mathscr{I}_D(k), \mathscr{P}) \to \mathrm{Ext}^{i+1}(\mathscr{I}_D(k), \mathscr{P}_1) \to \mathrm{Ext}^{i+1}(\mathscr{I}_D(k), \mathscr{G}).$$

From the first part of the proof, we get that $\mathrm{Ext}^i(\mathscr{I}_D(k), \mathscr{G}) = \mathrm{Ext}^{i+1}(\mathscr{I}_D(k), \mathscr{G}) = 0$ for every k and for $i = 1, \ldots, s-3$. From the induction assumption, $\mathrm{Ext}^{i+1}(\mathscr{I}_D(k), \mathscr{P}_1) = 0$ for every k and for $i = 0, \ldots, s-2-d$. Hence, $\mathrm{Ext}^i(\mathscr{I}_D(k), \mathscr{P}) = 0$ for every $k \in \mathbb{Z}$ and for $i = 1, \ldots, s-2-d$ as claimed. □

A direct consequence of the previous Proposition is that we can predict if N is the direct sum of P and I_D. In fact it holds:

COROLLARY 2. *Let $D \subseteq \mathbb{P}^r$ be a closed scheme of codimension $s \geq 2$, and let P be a R–module satisfying $s - 2 > \mathrm{pd}(P)$. Then, the only extension of $\mathscr{I}_D(k)$ with $\mathscr{P}$ is the trivial one, for every choice of $k \in \mathbb{Z}$. Consequently if there is a non–split exact sequence (2), we must have $s \leq \mathrm{pd}(P) + 2$.*

Proof. The previous Proposition shows that $\mathrm{Ext}^1(\mathscr{I}_D(k), \mathscr{P}) = 0$ and the claim follows. □

Now, we take into account the cohomology of D to get a bound on the projective dimension of $\mathscr{N}$.

PROPOSITION 2. *Let $D \subset \mathbb{P}^r$ be a closed scheme, and let $\mathscr{P}, \mathscr{N}$ be torsion–free sheaves such that the short sequence (2) is exact. If $\mathrm{pd}(\mathscr{N}) \geq \mathrm{pd}(\mathscr{P}) + 2$, then $H_*^{r-\mathrm{pd}(\mathscr{N})}(\mathscr{I}_D) \neq 0$.*

Conversely, if $H_*^j(\mathscr{I}_D) \neq 0$ for some $j \in \mathbb{Z}$ with $1 \leq j \leq r - 2 - \mathrm{pd}(\mathscr{P})$, then $\mathrm{pd}(\mathscr{N}) \geq \mathrm{pd}(\mathscr{P}) + 2$.

Proof. By Lemma 2, we have that $\mathrm{pd}(\mathscr{P})$ and $\mathrm{pd}(\mathscr{N})$ are strictly smaller than r. By taking the long exact cohomology sequence associated to (2), we get

$$H_*^i(\mathscr{P}) \to H_*^i(\mathscr{N}) \to H_*^i(\mathscr{I}_D) \to H_*^{i+1}(\mathscr{P}).$$

From Lemma 3, we know that $H_*^j(\mathscr{P}) = 0$ for $j = 1, \ldots, r - \mathrm{pd}(\mathscr{P}) - 1$, and so $H_*^i(\mathscr{N}) \cong H_*^i(\mathscr{I}_D)$ for $i = 1, \ldots, r - \mathrm{pd}(\mathscr{P}) - 2$.

If $\mathrm{pd}(\mathscr{P}) + 2 \leq \mathrm{pd}(\mathscr{N}) < r$, then $1 \leq r - \mathrm{pd}(\mathscr{N}) < r - \mathrm{pd}(\mathscr{P}) - 1$. It follows that $H_*^{r-\mathrm{pd}(\mathscr{N})}(\mathscr{N}) \cong H_*^{r-\mathrm{pd}(\mathscr{N})}(\mathscr{I}_D)$ and we get the claim by Lemma 3.

Assume now that $H^j(\mathscr{I}_D(k)) \neq 0$ for some $k \in \mathbb{Z}$ and some j such that $1 \leq j \leq r - \mathrm{pd}(\mathscr{P}) - 2$. Hence, $H_*^j(\mathscr{N}) \neq 0$. Again by Lemma 3, $r - \mathrm{pd}(\mathscr{N}) \leq j$ and so $\mathrm{pd}(\mathscr{N}) \geq \mathrm{pd}(\mathscr{P}) + 2$. □

REMARK 3. In the second part of the previous Proposition, the hypothesis on j implies that $\mathrm{pd}(\mathscr{P}) \leq r-3$. This last inequality is not automatically fulfilled. In fact, let $D \subseteq \mathbb{P}^r$ be a locally Cohen–Macaulay curve with $H^1_*(\mathscr{I}_D) \neq 0$. Let

$$0 \to G_r \to \cdots \to G_2 \to G_1 \to I_D \to 0$$

be the minimal free resolution of I_D and let $P = \ker(G_1 \to I_D)$. Then, $\mathrm{pd}(P) = r - 2 = \mathrm{pd}(I_D) - 1$, and $\mathrm{pd}(G_1) = 0$. Hence, we cannot apply the previous Proposition to the short exact sequence $0 \to P \to G_1 \to I_D \to 0$. Nevertheless, it could exist a different short exact sequence $0 \to Q \to N \to I_D \to 0$ with $\mathrm{pd}(Q) = r-3$. In this case, $\mathrm{pd}(N) = r-1$. Notice that $r-3$ is the smallest projective dimension allowed for the first item of the sequence, because of the codimension of D.

REMARK 4. The case considered in the previous Proposition, namely $\mathrm{pd}(\mathscr{N}) \geq \mathrm{pd}(\mathscr{P})+2$, occurs in the $\mathscr{N}$–type resolution of the ideal sheaf of a locally Cohen–Macaulay curve in $\mathbb{P}^3$ ([14], Ch. II, Section 4). In that case, $\mathscr{P}$ is dissocié and $\mathrm{pd}(\mathscr{N}) = 2$, where N is the second syzygy module of the Hartshorne–Rao module (graded Artinian R–module) of the curve, up to a free summand.

Now, we stress some consequences of the previous Proposition that we' ll use in next sections.

COROLLARY 3. *Consider an exact sequence* (2) *where D has codimension $s \geq 2$. If D is ACM and the sequence is non-split we have*

$$\mathrm{pd}(\mathscr{N}) \leq \mathrm{pd}(\mathscr{P}) + 1. \tag{5}$$

Proof. By Corollary 2 the non–splitting of the sequence (2) implies that $s \leq \mathrm{pd}(\mathscr{P}) + 2$. If $\mathrm{pd}(\mathscr{N}) \geq \mathrm{pd}(\mathscr{P})+2$, then $H^{r-\mathrm{pd}(\mathscr{N})}_*(\mathscr{I}_D) \neq 0$ by Proposition 2. On the other hand, $r - \mathrm{pd}(\mathscr{N}) \leq r-s$ and so $H^{r-\mathrm{pd}(\mathscr{N})}_*(\mathscr{I}_D) = 0$ because D is ACM. The contradiction proves that $\mathrm{pd}(\mathscr{N}) \leq \mathrm{pd}(\mathscr{P}) + 1$. □

REMARK 5. If $\mathrm{pd}(\mathscr{N}) \leq \mathrm{pd}(\mathscr{P}) + 1$, we can only prove that $H^i_*(\mathscr{I}_D) = 0$ for $i = 1, \ldots, r - \mathrm{pd}(\mathscr{P}) - 2$. Hence, D could not be an ACM scheme if $s < \mathrm{pd}(\mathscr{P}) + 2$.

A further problem related to the sequence (2) is the following: given the modules P and N, and an injective map $\mathscr{P} \to \mathscr{N}$, when is the cokernel an ideal sheaf? This problem was considered in [14], and we resume their results.

At first, we recall the definition and some properties of the maximal subsheaves, generalizing to $\mathbb{P}^r$ the one given for sheaves on $\mathbb{P}^3$ ([14], Ch. IV, Définition 1.1). In literature, maximal subsheaves are also named saturated sheaves (e.g., see [11]).

DEFINITION 4. *Let $\mathscr{M} \subset \mathscr{N}$ be $\mathscr{O}_{\mathbb{P}^r}$–modules. $\mathscr{M}$ is a maximal subsheaf of $\mathscr{N}$ if for all subsheaves $\mathscr{M}' \subset \mathscr{N}$ with rank$(\mathscr{M})$ = rank$(\mathscr{M}')$ such that $\mathscr{M} \subseteq \mathscr{M}' \subseteq \mathscr{N}$, we have $\mathscr{M} = \mathscr{M}'$.*

The interest in such subsheaves lies in the following properties.

PROPOSITION 3. *Let $\mathcal{M} \subseteq \mathcal{N}$ be $\mathcal{O}_{\mathbb{P}^r}$–modules. Consider the following properties:*

(a) *$\mathcal{M}$ is maximal;*

(b) *$\mathcal{N}/\mathcal{M}$ is torsion–free;*

(c) *$\mathcal{N}/\mathcal{M}$ is torsion–free in codimension* 1;

(d) *$\mathcal{N}/\mathcal{M}$ is locally free in codimension* 1;

(e) *$\mathcal{N}/\mathcal{M}$ has constant rank in codimension* 1;

(f) *$\mathcal{N}/\mathcal{M}$ is locally a direct summand of $\mathcal{N}$ in codimension* 1.

Then, $(a) \Leftrightarrow (b) \Rightarrow (c) \Leftrightarrow (d) \Leftrightarrow (e) \Rightarrow (f)$. *Furthermore, if $\mathcal{N}$ is torsion–free and $\mathcal{M}$ is locally free, they all are equivalent.*

Proof. The statement was proved for sheaves on $\mathbb{P}^3$ in ([14], Ch. IV, Proposition 1.2),but the proof works without changes also for sheaves on $\mathbb{P}^r$. □

Moreover, in the proof, the authors proved also the existence of maximal dissocié subsheaves of a sheaf $\mathcal{N}$.

As explained in ([14], Ch.IV, Remark 1.3(c)), in $\mathbb{P}^3$, if $\mathcal{N}$ is a rank $n+1$ vector bundle, and $\mathcal{M}$ is a rank n dissocié maximal subsheaf of $\mathcal{N}$, then $\mathcal{N}/\mathcal{M}$ is a rank 1 torsion–free sheaf, and so it is an ideal sheaf tensorized times $\det(\mathcal{N}) \otimes \det\left(\mathcal{M}^{-1}\right)$. Moreover, if $\mathcal{N}$ is not dissocié, then the ideal sheaf defines a curve.

3. A construction of ACM schemes

In this section, we consider two coherent torsion–free sheaves $\mathcal{P}$ and $\mathcal{N}$ and an injective map $\gamma : \mathcal{P} \to \mathcal{N}$, and we study the scheme D whose ideal sheaf is isomorphic to $\operatorname{coker}(\gamma)$, as in [14]. We limit ourselves to consider only the case D has the largest codimension to have a non–split exact sequence (2) (see Corollary 2) and $\mathcal{N}$ to have the largest projective dimension to allow D to be an ACM scheme (see Corollary 3). In more detail, we collect the hypotheses on $\mathcal{P}$ and $\mathcal{N}$ in the following

(H)

$(H.1)$ $\mathcal{P}$ is torsion-free and $s := \operatorname{pd}(\mathcal{P}) + 2 \le r$;

$(H.2)$ $\mathcal{N}$ is torsion-free and $\operatorname{pd}(\mathcal{N}) \le \operatorname{pd}(\mathcal{P}) + 1$;

$(H.3)$ the polynomial

$$p(t) := -\chi(\mathcal{N}(t-k)) + \chi(\mathcal{P}(t-k)) + \binom{t+r}{r}$$

has degree $r - s$ for some $k \in \mathbb{Z}$.

We remark that, in view of Definition 3 and Remark 1, the condition about the projective dimensions required in (H.1) and (H.2) means that $P := H^0_*(\mathscr{P})$ and $N := H^0_*(\mathscr{N})$ have, respectively, minimal free resolutions

$$0 \to G_{s-1} \xrightarrow{\Delta_{s-1}} G_{s-2} \xrightarrow{\Delta_{s-2}} \dots \xrightarrow{\Delta_2} G_1 \to P \to 0$$

and

$$0 \to F_s \xrightarrow{\delta_s} F_{s-1} \xrightarrow{\delta_{s-1}} \dots \xrightarrow{\delta_2} F_1 \to N \to 0.$$

REMARKS 2. (i) We allow $F_j = 0$ for some j in the minimal free resolution of N. In such a case, $F_{j+h} = 0$ for every $h \geq 0$.

(ii) Condition (H.3) implies that $\mathrm{rank}(\mathscr{N}) = \mathrm{rank}(\mathscr{P}) + 1$, because the rank of $\mathscr{F}$ is equal to $r!$ times the coefficient of t^r in $\chi(\mathscr{F}(t))$. Moreover, recalling that $\mathscr{O}_{\mathbb{P}^r}(a)$ has degree a and that the degree is additive on exact sequences, we have that $k = \deg(\mathscr{N}) - \deg(\mathscr{P})$.

Now, we describe the geometric properties of the schemes that can be obtained from such torsion–free sheaves.

THEOREM 1. *Let $\mathscr{P}$ and $\mathscr{N}$ be torsion–free coherent sheaves that fulfil the hypotheses (H). Assume that there exists an injective map $\gamma : \mathscr{P} \to \mathscr{N}$ whose image is a maximal subsheaf of $\mathscr{N}$. Then there exists a codimension $s = 2 + \mathrm{pd}(\mathscr{P})$ scheme D, closed and ACM, whose ideal sheaf fits into the short exact sequence (2) with $k = \deg(\mathscr{N}) - \deg(\mathscr{P})$. Moreover, the sequence $0 \to P \to N \to I_D(k) \to 0$ is exact.*

Proof. The cokernel of γ is a rank 1 torsion–free sheaf $\mathscr{F}$. Let $\mathscr{F}^{\vee\vee}$ be its double dual. Since $\mathscr{F}$ is torsion–free, the natural map $\mathscr{F} \to \mathscr{F}^{\vee\vee}$ is injective. By ([9], Corollary 1.2 and Proposition 1.9), $\mathscr{F}^{\vee\vee} \cong \mathscr{O}_{\mathbb{P}^r}(h)$ for some $h \in \mathbb{Z}$, and so $\mathscr{F} \cong \mathscr{I}_D(h) \subseteq \mathscr{O}_{\mathbb{P}^r}(h)$, i.e. we have an exact sequence

$$0 \to \mathscr{P} \xrightarrow{\gamma} \mathscr{N} \to \mathscr{I}_D(h) \to 0.$$

Clearly $h = \deg(\mathscr{N}) - \deg(\mathscr{P}) = k$, and hence the above sequence coincides with (2). Now, by Remark 2, (ii), k is the integer occurring in the polynomial $p(t)$ of (H.3), then $p(t)$ is the Hilbert polynomial of D, whence $\dim(D) = r - s$ by (H.3). Moreover, (H.2) and the second part of Proposition 2 imply that D is ACM. Finally, by (H.1) we have $\mathrm{pd}(\mathscr{P}) \leq r-2$, whence $r - \mathrm{pd}(\mathscr{P}) - 1 \geq 1$. Then Lemma 3$(d)$ implies that $H^1_*(\mathscr{P}) = 0$ and the last statement follows. □

REMARK 6. The map $\gamma : \mathscr{P} \to \mathscr{N}$ induces a map of complexes between the minimal free resolutions of P and N. Let $\gamma_i : G_i \to F_i$ be the induced map. Of course, $\gamma_i \circ \Delta_{i+1} = \delta_{i+1} \circ \gamma_{i+1}$, for each $i \geq 1$. Hence, a resolution of $I_D(k)$ can be obtained via mapping cone from (1), and it is

$$(6)\qquad 0\to \begin{array}{c}G_{s-1}\\ \oplus\\ F_s\end{array} \xrightarrow{\varepsilon_s} \begin{array}{c}G_{s-2}\\ \oplus\\ F_{s-1}\end{array} \xrightarrow{\varepsilon_{s-1}} \dots \xrightarrow{\varepsilon_3} \begin{array}{c}G_1\\ \oplus\\ F_2\end{array} \xrightarrow{\varepsilon_2} F_1 \to I_D(k)\to 0$$

where $\varepsilon_i : G_{i-1}\oplus F_i \to G_{i-2}\oplus F_{i-1}$ is given by

$$\begin{pmatrix} \Delta_{i-1} & 0\\ (-1)^i\gamma_{i-1} & \delta_i\end{pmatrix},\ \text{for } i\geq 2.$$

We remark that $\varepsilon_2 : G_1\oplus F_2\to F_1$ is represented by the matrix (γ_1,δ_2).

By general results on free resolutions, it is clear that the minimal free resolution of $I_D(k)$ can be obtained by cancelling the free modules corresponding to constant non–zero entries of any matrix representing the map ε_i, $i=2,\dots,s$.

REMARK 7. If there exists an injective map $\gamma:\mathscr{P}\to\mathscr{N}$ whose image is a maximal subsheaf of $\mathscr{N}$ of rank $\mathrm{rank}(\mathscr{P})=\mathrm{rank}(\mathscr{N})-1$, then the general map in $\mathrm{Hom}(\mathscr{P},\mathscr{N})$ has the same property.

Before studying further the properties of the construction, we give an example to illustrate it.

EXAMPLE 1. In $\mathbb{P}^4=\mathrm{Proj}(R=K[x,y,z,t,u])$, let C_1 and C_2 be plane curves of degrees d and e, with $e<d$, whose saturated ideals are

$$I_{C_1}=\langle x,y,f_1\rangle \qquad I_{C_2}=\langle t,u,f_2\rangle$$

where $f_1\in K[z,t,u]_d$, $f_2\in K[x,y,z]_e$. Moreover, we assume that $f_1-z^d, f_2-z^e\in\langle x,y,t,u\rangle$. Let $C=C_1\cup C_2$. It follows from the assumptions that C is a degree $d+e$ curve, not ACM because $H^1_*\mathscr{I}_C\cong R/\langle x,y,t,u,z^e\rangle$. We adapt an argument by [20] to choose the sheaves $\mathscr{N}$ and $\mathscr{P}$ so that our construction provides an ACM curve D containing C.

Let $\mathscr{N}$ be the sheaf associated to the first syzygy module of $H^0_*(\mathscr{O}_C)$. An easy computation shows that the minimal free resolution of $\mathscr{N}$ is

$$0\to \begin{array}{c}\mathscr{O}(-e-2)\\ \oplus\\ \mathscr{O}(-d-2)\end{array} \xrightarrow{\delta_3} \begin{array}{c}\mathscr{O}^2(-2)\\ \oplus\\ \mathscr{O}^2(-e-1)\\ \oplus\\ \mathscr{O}^2(-d-1)\end{array} \xrightarrow{\delta_2} \begin{array}{c}\mathscr{O}^4(-1)\\ \oplus\\ \mathscr{O}(-e)\\ \oplus\\ \mathscr{O}(-d)\end{array} \xrightarrow{\delta_1} \mathscr{N}\to 0,$$

where $\mathscr{O}$ is the structure sheaf of $\mathbb{P}^4$ and the maps are represented by the matrices

$$\delta_2=\begin{pmatrix} y&0&0&0&f_1&0\\ -x&0&0&0&0&f_1\\ 0&u&f_2&0&0&0\\ 0&-t&0&f_2&0&0\\ 0&0&-t&-u&0&0\\ 0&0&0&0&-x&-y\end{pmatrix}\qquad \delta_3=\begin{pmatrix}0&f_1\\ f_2&0\\ -u&0\\ t&0\\ 0&-y\\ 0&x\end{pmatrix}.$$

Furthermore, let $\mathscr{P}$ be defined by the exact sequence

$$0 \to \mathscr{O}(-3) \xrightarrow{\Delta_2} \mathscr{O}^2(-2) \xrightarrow{\Delta_1} \mathscr{P} \to 0$$

where Δ_2 is represented by the matrix

$$\Delta_2 = \begin{pmatrix} l_2 \\ l_1 \end{pmatrix}$$

with l_1, l_2 linear forms intersecting along a plane.

It is evident that both $\mathscr{P}$ and $\mathscr{N}$ are torsion–free, with projective dimensions 1 and 2, respectively. Hence, they fulfil the conditions $(H.1)$ and $(H.2)$ with $s = 3$. A straightforward computation shows that, for $k = 1$, the polynomial $p(t)$ is equal to $t(d+e+2)+1-\binom{e}{2}-\binom{d}{2}$, and so it has degree $r-s = 4-3$. Hence, condition $(H.3)$ is fulfilled, too. Now, we want to give a map $\gamma : \mathscr{P} \to \mathscr{N}$, such that its image is a rank 1 maximal sub–sheaf of $\mathscr{N}$. By lifting γ to the minimal free resolutions of the two sheaves, we get the two maps $\gamma_1 : \mathscr{O}^2(-2) \to \mathscr{O}^4(-1) \oplus \mathscr{O}(-e) \oplus \mathscr{O}(-d)$ and $\gamma_2 : \mathscr{O}(-3) \to \mathscr{O}^2(-2) \oplus \mathscr{O}^2(-e-1) \oplus \mathscr{O}^2(-d-1)$, that verify

$$\delta_1 \circ \gamma_1 = \gamma \circ \Delta_1 \quad \text{and} \quad \delta_2 \circ \gamma_2 = \gamma_1 \circ \Delta_2.$$

With some non difficult computations, it is possible to prove that for each choice of the maps $a : \mathscr{O}(-1) \to \mathscr{O}^4(-1)$ and $b : \mathscr{O}^2(-2) \to \mathscr{O}^2(-2)$, we get a map of complexes by setting

$$\gamma = \delta_{1|} \circ a, \quad \gamma_1 = a \circ \Delta_1 + \delta_{2|} \circ b, \quad \gamma_2 = b \circ \Delta_2$$

where $\delta_{1|}$ is the restriction of δ_1 to $\mathscr{O}^4(-1)$ and $\delta_{2|}$ is the restriction of δ_2 to $\mathscr{O}^2(-2)$. As $\mathscr{P}$ is the sub–sheaf of $\mathscr{O}(-1)$ spanned by $\mathrm{Im}(\Delta_1)$, for general a, $\mathrm{Im}(\gamma)$ is a maximal sub–sheaf of $\mathscr{N}$, because for a rank 1 sheaf the maximality condition is equivalent to the vanishing locus of the generator to have codimension at least 2. The curve D is then defined by the ideal

$$I_D = \langle xt, yt, xu, yu, f_2(a_1x + a_2y), f_1(a_3t + a_4u) \rangle$$

where a is represented by the transpose of the matrix (a_1, a_2, a_3, a_4).

Once we have constructed a closed ACM scheme D of codimension s as cokernel of a short exact sequence (2), we can construct the minimal free resolution of I_D and it is

$$0 \to H_s \xrightarrow{\sigma_s} \dots \xrightarrow{\sigma_2} H_1 \xrightarrow{\sigma_1} I_D \to 0$$

where $H_i = \oplus_{n\in\mathbb{Z}} R(-n)^{h_i(n)}$. Let $K = \ker(\sigma_1)$. Then, the ideal sheaf $\mathscr{I}_D$ is also the cokernel of the short exact sequence

$$0 \to \mathscr{K} \xrightarrow{j} \mathscr{H}_1 \xrightarrow{\sigma_1} \mathscr{I}_D \to 0. \tag{7}$$

Now we compare the two sequences (2) and (7).

PROPOSITION 4. *Let $D \subseteq \mathbb{P}^r$ be an ACM scheme of codimension s and let* (7) *be as above.*

(i) If there is a sequence (2) *with* $\mathrm{pd}(\mathscr{P}) = s-2$ *then there exists a map* $\psi : \mathscr{K} \to \mathscr{P}$ *such that* $\mathscr{N}$ *is the push–out of* $\mathscr{P}$ *and* $\mathscr{H}_1$.

(ii) Conversely, let $\mathscr{P}$ *be a torsion–free coherent sheaf with* $\mathrm{pd}(\mathscr{P}) = s-2$. *Then, for every map* $\psi : \mathscr{K} \to \mathscr{P}$ *there exists a short exact sequence (2) whose third item is* $\mathscr{I}_D$.

Proof. (i) Up to twisting the sequence (2), we can assume that $k = 0$. The minimal free resolution of I_D is

$$0 \to H_s \xrightarrow{\sigma_s} \dots \xrightarrow{\sigma_2} H_1 \xrightarrow{\sigma_1} I_D \to 0,$$

and so σ_1 maps the canonical bases of H_1 onto a minimal set of generators of I_D. The surjective map $\mathscr{N} \to \mathscr{I}_D$ induces a surjective map $N = H^0_*(\mathscr{N}) \to I_D$ because $\mathrm{pd}(\mathscr{P}) = s-2$ implies that $H^1_*(\mathscr{P}) = 0$ (see Lemma 3). Hence, we have a well defined map $H_1 \to N$ given on the canonical bases of H_1 and extended by linearity. So, there exists a map $\varphi : \mathscr{H}_1 \to \mathscr{N}$. It is straightforward to check that φ maps the kernel of σ_1 to the image of $\mathscr{P}$, and so φ induces a map $\psi : \mathscr{K} \to \mathscr{P}$. At the end, there exists a commutative diagram

$$\begin{array}{ccccccc} 0 \to \mathscr{K} & \xrightarrow{\ j\ } & \mathscr{H}_1 & \longrightarrow & \mathscr{I}_D \to 0 \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\varphi} & & \| \\ 0 \to \mathscr{P} & \longrightarrow & \mathscr{N} & \longrightarrow & \mathscr{I}_D \to 0 \end{array}$$

where the last map is the identity of $\mathscr{I}_D$. From the universal property of the push–out (see ([17], Ch. 3, Theorem 11) for the definition and the properties of the push–out), it follows that $\mathscr{N}$ is the push–out of $\mathscr{H}_1$ and $\mathscr{P}$ as claimed.

(ii) As soon as we fix a map $\psi : \mathscr{K} \to \mathscr{P}$, we can construct the same commutative diagram we considered in the first part of the proof. In more detail, let $q : \mathscr{K} \to \mathscr{H}_1 \oplus \mathscr{P}$ be defined as j on the first summand and as $-\psi$ on the second one. Then, the sheaf $\mathscr{N}$ satisfies $\mathscr{N} = \mathscr{H}_1 \oplus \mathscr{P}/\mathrm{im}(q)$, and is torsion–free of rank $\mathrm{rank}(\mathscr{P}) + 1$. The second row of the commutative diagram above gives the short exact sequence $0 \to P \to N \to I_D \to 0$ because $H^1_*(\mathscr{P}) = 0$ by Lemma 3. Hence, $\mathrm{pd}(\mathscr{N}) \leq \mathrm{pd}(\mathscr{I}_D) = \mathrm{pd}(\mathscr{P}) + 1$, and the proof is complete. □

REMARK 8. If $\psi = 0$, then $\mathscr{N} = \mathscr{P} \oplus \mathscr{I}_D$, and the sequence is not interesting. On the other hand, if ψ is an isomorphism, then $\mathscr{N} \cong \mathscr{H}_1$ and once again we get nothing new.

Summarizing the above discussed results, we have that if we start from two sheaves $\mathscr{N}$ and $\mathscr{P}$ satisfying our hypotheses, we can construct codimension s ACM schemes, and conversely, given a codimension s ACM scheme D and a torsion–free sheaf $\mathscr{P}$, we can construct a sheaf $\mathscr{N}$ fulfilling the conditions we ask.

Starting from two given torsion–free sheaves $\mathscr{N}$ and $\mathscr{P}$, there are constrains on the ACM schemes we can obtain.

PROPOSITION 5. *In the same hypotheses as Theorem 1, let $D \subset \mathbb{P}^r$ be a codimension s ACM closed scheme whose ideal sheaf fits into a short exact sequence*

$$0 \to \mathscr{P} \to \mathscr{N} \to \mathscr{I}_D(k) \to 0$$

for some $k \in \mathbb{Z}$. Then, the minimal number of generators of I_D is not larger than rank$(\mathscr{F}_1)$ while the free modules H_i that appear in the minimal free resolution of $I_D(k)$ are direct summands of $F_i \oplus G_{i-1}$.

Proof. We constructed a free resolution of $I_D(k)$ in Remark 6. The minimal free resolution of I_D can be obtained from this last one by cancelling suitable summands. □

As a consequence of the hypotheses (H), to construct ACM schemes of codimension $s \geq 3$, we have to consider a torsion–free sheaf $\mathscr{P}$ satisfying pd$(\mathscr{P}) > 0$, that is to say, $\mathscr{P}$ non–dissocié. On the other hand, if the codimension of D is 2, then $\mathscr{P}$ is dissocié. In this case, we have a more geometric interpretation of the construction, and it can be compared with Serre's construction (Hartshorne's one, respectively) when $\mathscr{N}$ is a rank 2 vector bundle (reflexive sheaf, respectively).

PROPOSITION 6. *Let $D \subset \mathbb{P}^r$ be a codimension 2 ACM closed scheme, and let c be an integer such that $H^0(D, \omega_D(c)) \neq 0$. Then, for every non–zero $\xi \in H^0(D, \omega_D(c))$ we can construct a short non-split exact sequence*

$$0 \to \mathscr{O}_{\mathbb{P}^r}(c-r-1) \to \mathscr{N} \to \mathscr{I}_D \to 0$$

with $\mathscr{N}$ torsion–free, of rank 2 and pd$(\mathscr{N}) \leq 1$.

Proof. By Serre's duality for $\mathbb{P}^r$ ([8], Ch. III, Theorem 7.1), we get $\mathrm{Ext}^1(\mathscr{I}_D, \mathscr{O}_{\mathbb{P}^r}(c-r-1)) \cong H^{r-1}(\mathbb{P}^r, \mathscr{I}_D(-c))'$. From the inclusion $D \hookrightarrow \mathbb{P}^r$, we get $H^{r-1}(\mathbb{P}^r, \mathscr{I}_D(-c))' \cong H^{r-2}(D, \mathscr{O}_D(-c))'$, and again by Serre's duality on D ([8], Ch. III, Theorem 7.6 and Proposition 6.3(c)), we have the further isomorphisms $\mathrm{Ext}^1(\mathscr{I}_D, \mathscr{O}_{\mathbb{P}^r}(c-r-1)) \cong \mathrm{Hom}(\mathscr{O}_D(-c), \omega_D) \cong H^0(D, \omega_D(c))$. Hence, every non–zero $\xi \in H^0(D, \omega_D(c))$ can be thought of as an extension of $\mathscr{I}_D$ with $\mathscr{O}_{\mathbb{P}^r}(c-r-1)$ and so as a non–split short exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^r}(c-r-1) \to \mathscr{N} \to \mathscr{I}_D \to 0.$$

The sheaf $\mathscr{N}$ has rank 2, and it is torsion–free. Moreover, if

$$0 \to \mathscr{H}_2 \xrightarrow{\varphi} \mathscr{H}_1 \to \mathscr{I}_D \to 0$$

is the minimal dissocié resolution of $\mathscr{I}_D$, there is a natural surjection $\mathrm{Hom}(\mathscr{H}_2, \mathscr{O}_{\mathbb{P}^r}(c-r-1)) \to \mathrm{Ext}^1(\mathscr{I}_D, \mathscr{O}_{\mathbb{P}^r}(c-r-1))$, and so there exists a map $\psi : \mathscr{H}_2 \to \mathscr{O}_{\mathbb{P}^r}(c-r-1)$ that does not factor through $\varphi : \mathscr{H}_2 \to \mathscr{H}_1$ whose image in $\mathrm{Ext}^1(\mathscr{I}_D, \mathscr{O}_{\mathbb{P}^r}(c-r-1))$ is equal to ξ. By using standard results from homological algebra, we get that $\mathcal{N}$ is the push–out of $\mathcal{H}_1$ and $\mathcal{O}_{\mathbb{P}^r}(c-r-1)$ via $(\varphi,-\psi)$. Hence, the resolution of $\mathcal{N}$ with dissocié sheaves is

$$0 \to \mathcal{H}_2 \xrightarrow{(\varphi,-\psi)} \mathcal{H}_1 \oplus \mathcal{O}_{\mathbb{P}^r}(c-r-1) \to \mathcal{N} \to 0$$

and so $\mathcal{N}$ has projective dimension less than or equal to 1. □

REMARK 9. From the proof of the previous Proposition, we get that $\mathrm{pd}(\mathcal{N}) = 0$ if and only if $\mathcal{H}_2 = \mathcal{O}_{\mathbb{P}^r}(c-r-1)$, i.e. D is a complete intersection scheme.

REMARK 10. We can easily modify the proof to get sheaves $\mathcal{N}$ of larger rank: it is enough to consider $c_1,\dots,c_n \in \mathbb{Z}$ such that $H^0(D,\omega_D(c_i)) \neq 0$ for at least a c_i. As in the proof of the previous Proposition, $\oplus_{i=1}^n H^0(D,\omega_D(c_i)) \cong \mathrm{Ext}^1(\mathcal{J}_D, \oplus_{i=1}^n \mathcal{O}_{\mathbb{P}^r}(c_i - r-1))$ and so a non–zero element $\xi \in \oplus_{i=1}^n H^0(D,\omega_D(c_i))$ can be considered as an extension of $\mathcal{J}_D$ with $\mathcal{P} = \oplus_{i=1}^n \mathcal{O}_{\mathbb{P}^r}(c_i - r-1)$, and we can construct $\mathcal{N}$ as in the proof.

REMARK 11. In comparing Proposition 6 with Serre's and Hartshorne's constructions mentioned above, it is evident that the hypothesis on $\mathcal{N}$ strongly affects the properties of the constructed scheme. For example, when $\mathcal{N}$ is a rank 2 reflexive sheaf, as in Hartshorne' s setting, the associated schemes are generically locally complete intersection. In fact, the locus where the reflexive sheaf $\mathcal{N}$ is not locally free has codimension ≥ 3 ([3], Corollary 1.4 and Theorem 4.1) for the case of curves in $\mathbb{P}^3$). The properties of the associated schemes show that the constructions are not the same one. In fact, following Proposition 6, it is possible to construct ACM schemes which are locally complete intersection at no point, while if $\mathcal{N}$ is reflexive and D is the associated scheme, the locus of the points of D where D is not locally complete intersection has codimension ≥ 1 in D. On the other hand, all the schemes constructed via Proposition 6 are ACM, while the ones associated to reflexive sheaves can have non–zero cohomology.

Now, we show how to construct ACM codimension 2 schemes which contain the first infinitesimal neighborhood of another ACM codimension 2 scheme. They are candidates to have no points at which the scheme is locally complete intersection.

PROPOSITION 7. *Let Y be an ACM codimension 2 scheme and let $\mathcal{N} = \mathcal{J}_Y \oplus \mathcal{J}_Y$. Then, every codimension 2 ACM scheme D we obtain from the construction above contains the first infinitesimal neighborhood of Y. Moreover, D is not locally complete intersection at any point of Y.In particular it is not generically locally complete intersection.*

Proof. For the first statement, it is enough to prove that $I_D \subset I_Y^2$.

Let $0 \to \mathcal{L}_1 \xrightarrow{\varphi} \mathcal{L}_0 \to \mathcal{I}_Y \to 0$ be the minimal dissocié resolution of $\mathcal{I}_Y$. Let φ be represented by a matrix A. Hence, the maximal minors of A generate the ideal I_Y.

Let $\mathscr{P} = \mathscr{O}_{\mathbb{P}^r}(-m)$ and let $\gamma : \mathscr{P} \to \mathscr{N}$ be a general map whose image is a maximal subsheaf of $\mathscr{N}$. Let $\gamma' : \mathscr{P} \to \mathscr{L}_0 \oplus \mathscr{L}_0$ be a lifting of γ.

The ideal I_D is generated by the maximal minors of the matrix

$$M = \begin{pmatrix} A & O & C' \\ O & A & C'' \end{pmatrix}$$

where the last column represents γ'. Every maximal minor of M can be computed by Laplace rule with respect to the last column, and so it is a combination of the maximal minors of the block matrix $\begin{pmatrix} A & O \\ O & A \end{pmatrix}$, whose maximal minors generate the ideal I_Y^2.

Let now $x \in Y$ and set $S := \mathscr{O}_{\mathbb{P}^r,x}$. We have an exact sequence of S-modules

$$0 \to S \to \mathscr{N}_x \to \mathscr{I}_{D,x} \to 0.$$

It is easy to see that $\mathscr{N}_x$ needs at least four generators whence $\mathscr{I}_{D,x}$ needs at least three generators. Since D has codimension 2 it cannot be a complete intersection at x. □

REMARK 12. The easiest case we can consider is when the scheme Y is the complete intersection of two hypersurfaces. In this case, the scheme defined by I_Y^2 is ACM of codimension 2 and it can be obtained from the previous construction.

For large m, and a general lifting γ, the scheme D constructed in the previous Proposition properly contains the first infinitesimal neighborhood of Y, with a residual part not supported at Y. For example, let Y be the line $x = y = 0$. For a general $\gamma : R(-2) \to R^2(-1) \oplus R^2(-1)$, D has degree 6, and is the union of the first infinitesimal neighborhood of Y and a twisted cubic curve meeting Y at two points.

A similar result holds both for the direct sum of $s(\geq 2)$ copies of $\mathscr{I}_Y$, and for non–trivial extensions of $\mathscr{I}_Y$ with itself or with twists of another ACM codimension 2 scheme Z, but we do not state them.

Now, we relate extensions associated to divisors that differ by hypersurface sections.

PROPOSITION 8. *Let $D \subset \mathbb{P}^r$ be a codimension 2 ACM scheme. Let us take $\xi \in H^0(D, \omega_D(c))$ and $\xi' \in H^0(D, \omega_D(c+d))$ both non–zero, with $d \geq 0$, and let*

$$0 \to \mathscr{O}_{\mathbb{P}^r}(c-r-1) \to \mathscr{N} \to \mathscr{I}_D \to 0$$

and

$$0 \to \mathscr{O}_{\mathbb{P}^r}(c+d-r-1) \to \mathscr{N}' \to \mathscr{I}_D \to 0$$

be the associated short exact sequences. Then, there exists a degree d hypersurface $S = V(f)$ that cuts D along a codimension 3 subscheme such that $\xi' = f\xi$ if, and only if, there exists a short exact sequence

$$0 \to \mathscr{N} \to \mathscr{N}' \to \mathscr{O}_S(c+d-r-1) \to 0$$

that induces the identity on $\mathcal{J}_D$.

Proof. In the proof of previous Proposition, we constructed the sheaf $\mathcal{N}$ as push–out

$$\begin{array}{ccc} \mathcal{H}_2 & \xrightarrow{\varphi} & \mathcal{H}_1 \\ \psi\downarrow & & \downarrow \\ \mathcal{O}_{\mathbb{P}^r}(c-r-1) & \longrightarrow & \mathcal{N} \end{array}.$$

Assume that $\xi' = f\xi$. The section $f\xi \in H^0(D,\omega_D(c+d))$ is the image of the map $f\psi \in \mathrm{Hom}(\mathcal{H}_2,\mathcal{O}_{\mathbb{P}^r}(c+d-r-1))$ in $\mathrm{Ext}^1(\mathcal{J}_D,\mathcal{O}_{\mathbb{P}^r}(c+d-r-1))$ and so the sheaf $\mathcal{N}'$ is the push–out of $\mathcal{H}_1$ and $\mathcal{O}_{\mathbb{P}^r}(c+d-r-1)$ via φ and $-f\psi$. From the universal property of the push–out (see [13], pp. 62), we get the following map of complexes

$$\begin{array}{ccccccc} 0\to & \mathcal{O}_{\mathbb{P}^r}(c-r-1) & \longrightarrow & \mathcal{N} & \longrightarrow & \mathcal{J}_D & \to 0 \\ & f\downarrow & & \varepsilon\downarrow & & \| & \\ 0\to & \mathcal{O}_{\mathbb{P}^r}(c+d-r-1) & \longrightarrow & \mathcal{N}' & \longrightarrow & \mathcal{J}_D & \to 0 \end{array}$$

and so ε is injective, and $\mathrm{coker}(\varepsilon) \cong \mathcal{O}_S(c+d-r-1)$, as claimed.

Assume now that the short exact sequence

$$0 \to \mathcal{N} \to \mathcal{N}' \to \mathcal{O}_S(c+d-r-1) \to 0$$

induces the identity on $\mathcal{J}_D$. Standard arguments allow us to lift ε to an injective map $\mathcal{O}_{\mathbb{P}^r}(c-r-1) \to \mathcal{O}_{\mathbb{P}^r}(c+d-r-1)$ whose cokernel is isomorphic to $\mathcal{O}_S(c+d-r-1)$. Hence, the map is the multiplication by f, and $\mathcal{N}'$ is the push–out of $\mathcal{H}_1$ and $\mathcal{O}_{\mathbb{P}^r}(c+d-r-1)$ via φ and $f\psi$. Hence, $\xi' = f\xi$, and the proof is complete. □

REMARK 13. Let $\xi,\xi' \in H^0(D,\omega_D(c))$. By applying the previous Proposition, we get that ξ and ξ' are linearly dependent if and only if the sheaves $\mathcal{N}$ and $\mathcal{N}'$ associated to them are isomorphic.

4. ACM schemes from ACM ones

Let $X \subset \mathbb{P}^r$ be a codimension t ACM scheme. For general choices, $s(< t)$ hypersurfaces of large degree containing X define a complete intersection codimension s ACM scheme containing X. In this section, we discuss the related problem of finding an ACM codimension s closed scheme $D \subset \mathbb{P}^r$ containing X. Of course, we make use of the construction described in the previous section.

The main result is the following.

PROPOSITION 9. *Let X be a codimension t ACM scheme in $\mathbb{P}^r$ with $3 \le t \le r$ and let*

(8) $$0 \to F_t \xrightarrow{\delta_t} F_{t-1} \to \cdots \to F_2 \xrightarrow{\delta_2} F_1 \xrightarrow{\delta_1} I_X \to 0$$

be the minimal free resolution of the saturated ideal that defines X. Let $N = \ker(\delta_{t-s})$ be the $(t-s)$–th syzygy module of X, for some $s \geq 2$, and let P be a torsion–free R–module of projective dimension $s-2$. Assume further that $\mathcal{N}$ and $\mathcal{P}$ satisfy the condition $(H.3)$, and that there exists an injective map $\gamma : \mathcal{P} \to \mathcal{N}$ such that $\gamma(\mathcal{P})$ is a maximal subsheaf of $\mathcal{N}$. Then, for every ACM codimension s closed scheme D constructed as in Theorem 1 there is a short exact sequence

$$0 \to \mathcal{E}xt^{s-2}(\mathcal{P}, \omega_{\mathbb{P}^r}) \to \omega_D(-k) \to \omega_X \to 0.$$

Moreover, D contains X.

Proof. The R–module N is torsion–free, and has no free summand, because it is computed from the minimal free resolution of I_X. Moreover,

$$0 \to F_t \xrightarrow{\delta_t} F_{t-1} \xrightarrow{\delta_{t-1}} \dots \xrightarrow{\delta_{t-s+2}} F_{t-s+1} \to N \to 0$$

is the minimal free resolution of N and so the projective dimension of N is $s-1$. Hence, $\mathcal{N}$ and $\mathcal{P}$ satisfy all the conditions (H).

Hence, by Theorem 1 there exists a codimension s ACM closed scheme $D \subset \mathbb{P}^r$, and an integer k such that

$$0 \to \mathcal{P} \to \mathcal{N} \to \mathcal{J}_D(k) \to 0$$

is a short exact sequence. By applying $\mathcal{H}om(-, \omega_{\mathbb{P}^r})$ we get

$$\mathcal{E}xt^{s-2}(\mathcal{N}, \omega_{\mathbb{P}^r}) \to \mathcal{E}xt^{s-2}(\mathcal{P}, \omega_{\mathbb{P}^r}) \to \mathcal{E}xt^{s-1}(\mathcal{J}_D(k), \omega_{\mathbb{P}^r}) \to$$
$$\to \mathcal{E}xt^{s-1}(\mathcal{N}, \omega_{\mathbb{P}^r}) \to \mathcal{E}xt^{s-1}(\mathcal{P}, \omega_{\mathbb{P}^r}).$$

$\mathcal{E}xt^{s-1}(\mathcal{P}, \omega_{\mathbb{P}^r}) = 0$ because $\mathrm{pd}(\mathcal{P}) = s-2$, while $\mathcal{E}xt^{s-j}(\mathcal{N}, \omega_{\mathbb{P}^r}) = \mathcal{E}xt^{t-j}(\mathcal{J}_X, \omega_{\mathbb{P}^r})$ by definition of N. Hence, $\mathcal{E}xt^{s-1}(\mathcal{N}, \omega_{\mathbb{P}^r}) = \omega_X$, and $\mathcal{E}xt^{s-2}(\mathcal{N}, \omega_{\mathbb{P}^r}) = 0$ because X is ACM of codimension t ([8], Ch. III, Proposition 7.5 and Theorem 7.1). Again by ([8], Ch. III, Proposition 7.5), $\mathcal{E}xt^{s-1}(\mathcal{J}_D(k), \omega_{\mathbb{P}^r}) = \omega_D(-k)$. Summarizing the above arguments, the construction induces a short exact sequence

$$0 \to \mathcal{E}xt^{s-2}(\mathcal{P}, \omega_{\mathbb{P}^r}) \to \omega_D(-k) \to \omega_X \to 0$$

that relates the dualizing sheaves of X and D. In particular, we can think of ω_X as a quotient of ω_D, up to a twist. The annihilator of ω_X is $\mathcal{J}_X$ (see, ([5], Corollary 21.3)), the one of ω_D is $\mathcal{J}_D$, and so we get the last claim because it is evident that the annihilator of $\omega_D(-k)$ is contained in the one of ω_X. □

The previous Proposition can be applied in the following case: let $\mathcal{N}$ be the first syzygy module of a zero–dimensional scheme X in $\mathbb{P}^3$ and so $\mathrm{pd}(N) = 1$. When applying to $\mathcal{N}$ the algorithm by M.Martin–Deschamps and D.Perrin for computing minimal curves in a biliaison class, we get a free module P ($\mathrm{pd}(P) = 0$) and a general injective map $\gamma : P \to N$ whose cokernel is, up to a twist, the ideal of a curve D (and so the codimension of D is 2). Hence, the hypotheses of Proposition 9 are fulfilled and the curve D is ACM and contains X.

We rephrase Proposition 5 in the case $\mathscr{N}$ is the $(t-s)$–syzygy sheaf of an ACM scheme X of codimension t. We recall that the Cohen–Macaulay type of an ACM scheme X of codimension t is equal to the rank of the free module F_t in a minimal free resolution of the saturated ideal I_X. An ACM scheme X with Cohen–Macaulay type 1 is said arithmetically Gorenstein (see [10] for equivalent definitions and properties).

COROLLARY 4. *Let X and D be schemes as in Proposition 9. Then, the Cohen–Macaulay type of X is not greater than the one of D. In particular, D is arithmetically Gorenstein if and only if X is such.*

Proof. The minimal dissocié resolution of $\mathscr{N}$ agrees with the one of $\mathscr{I}_X$, and so $F_t \oplus G_{s-1}$ appears in a free resolution of $I_D(k)$, as it follows from Remark 6. F_t cannot be cancelled because it maps to F_{t-1} and the resolution of I_X is minimal, and so the first claim follows. In particular, F_t is equal to the last free module in a minimal free resolution of $I_D(k)$ if and only if $\gamma_{s-1} : G_{s-1} \to F_{t-1}$ is split–injective, where γ_{s-1} is induced from $\gamma : P \to N$. The second statement is straightforward. □

For example, if $X \subset \mathbb{P}^3$ is a set of 5 general points, it is arithmetically Gorenstein with Pfaffian resolution

$$0 \to R(-5) \to R^5(-3) \to R^5(-2) \to I_X \to 0.$$

By applying the previous construction with $P = R^3(-3)$, we get that $k = -1$ and the minimal free resolution of I_D is

$$0 \to R(-5) \to R^2(-3) \to I_D(-1) \to 0,$$

so D is a complete intersection curve in $\mathbb{P}^3$.

REMARK 14. Among the ACM closed schemes D constructed in Proposition 9 we might not find the ones of minimal degree containing X. For example, let $X \subset \mathbb{P}^3$ be the degree 4 reduced scheme consisting of the vertices of the unit tetrahedron. With an easy computation, we get that I_X is generated by xy, xz, xw, yz, yw, zw, and its minimal free resolution is

$$0 \to R^3(-4) \to R^8(-3) \to R^6(-2) \to I_X \to 0.$$

An ACM curve C of minimal degree containing X is the union of the three lines $V(x, y), V(y, z), V(z, w)$. The minimal free resolution I_C is

$$0 \to R^2(-3) \to R^3(-2) \to I_C \to 0.$$

It follows that C cannot be obtained from Proposition 9 because the Cohen–Macaulay types of X and C are 3 and 2, respectively, and this is not possible by Corollary 4.

EXAMPLE 2. In this example, we construct two ACM curves with different Cohen–Macaulay types starting from the same X.

Let $r = 3$ and let X be a set of four general points in a plane. Of course, I_X is the complete intersection of a linear form and two quadratic forms, and so its minimal free resolution is

$$0 \to R(-5) \to R^2(-3) \oplus R(-4) \to R(-1) \oplus R^2(-2) \to I_X \to 0.$$

If we choose $P = R(-3)$, we get a complete intersection curve D whose minimal free resolution is

$$0 \to R(-5) \to R(-3) \oplus R(-4) \to I_D(-2) \to 0.$$

On the other hand, if we choose $P = R(-5)$, we get an ACM curve E whose minimal free resolution is

$$0 \to R^2(-5) \to R^2(-3) \oplus R(-4) \to I_E \to 0.$$

Both curves are constructed by choosing a general injective map from P to $R^2(-3) \oplus R(-4)$.

Summarizing the obtained results, we proved that it is possible to construct a codimension s ACM closed scheme D containing a given codimension t ACM scheme X as soon as $s < t$. Some of the restrictions are: the number of minimal generators of I_D cannot be larger than the number of minimal generators of the R–module N we used in the construction, and the last free module in a minimal free resolution of I_X is a direct summand of the last free module in a minimal free resolution of $I_D(k)$. A consequence of the restrictions is that there are ACM schemes containing X that cannot be constructed as explained in Proposition 9 (e.g., see Remark 14).

The last result we present in this section allows us to reconstruct an ACM scheme D from a subscheme X of D obtained by intersecting D with a complete intersection S.

PROPOSITION 10. *Let $D \subset \mathbb{P}^r$ be a codimension s ACM scheme with minimal free resolution*

$$0 \to H_s \xrightarrow{\varepsilon_s} H_{s-1} \xrightarrow{\varepsilon_{s-1}} \dots \xrightarrow{\varepsilon_2} H_1 \to I_D \to 0,$$

and let $S = V(f_1, \dots, f_t)$ be a codimension t complete intersection scheme that cuts D along a codimension $s + t \leq r$ scheme X. Then, D can be constructed from X as explained in Proposition 9.

Proof. Let $F = \oplus_{i=1}^{t} R(-deg(f_i))$. Then, the minimal free resolution of I_S is given by the Koszul complex

$$0 \to \wedge^t F \xrightarrow{\varphi_t} \wedge^{t-1} F \xrightarrow{\varphi_{t-1}} \dots \xrightarrow{\varphi_2} F \xrightarrow{\varphi_1} I_S \to 0$$

where $\varphi_i = \wedge^i\varphi$ and $\varphi : F \to R$ is defined as $\varphi(e_i) = f_i$ for each $i = 1,\dots,t$, where $e_1,\dots,e_t$ is the canonical basis of F.

Let $X = D \cap S$, and let $I_X \subset R$ be its saturated ideal. It is easy to prove that a free resolution of I_X can be constructed as tensor product of the resolutions of I_D and I_S (for the definition of the tensor product of complexes see Section 17.3 in [5]). Hence, it is equal to

$$0 \to G_{s+t} \to G_{s+t-1} \to \cdots \to G_1 \to I_X \to 0$$

where

$$G_h = \bigoplus_{i+j=h,\ i,j\geq 0} H_i \otimes \wedge^j F$$

for $h = 1,\dots,s+t$, and the map $\delta_h : G_h \to G_{h-1}$ restricted to $H_i \otimes \wedge^j F \to (H_{i-1} \otimes \wedge^j F) \oplus (H_i \otimes \wedge^{j-1} F)$ is defined as

$$\delta_i = \begin{pmatrix} \varepsilon_i \otimes 1 \\ (-1)^i\ 1 \otimes \varphi_j \end{pmatrix}.$$

In particular, X is ACM of codimension $s+t$.

Let N be the kernel of δ_t, and so a resolution of N is equal to

$$0 \to G_{s+t} \to \cdots \to G_{t+1} \to N \to 0.$$

Moreover, N is torsion–free.

Now, let $G'_{t+j} = (H_{j+1} \otimes \wedge^{t-1} F) \oplus \cdots \oplus (H_s \otimes \wedge^{t+j-s} F)$ for $j = 1,\dots,s-1$. Of course, $G_{t+j} = (H_j \otimes \wedge^t F) \oplus G'_{t+j}$. Let $\Delta_{t+j} : G'_{t+j} \to G'_{t+j-1}$ be the restriction of δ_{t+j} to G'_{t+j}, and let $P = \mathrm{coker}(\Delta_{t+2})$. A free resolution of P is

$$0 \to G'_{s+t-1} \to G'_{s+t-2} \to \cdots \to G'_{t+1} \to P \to 0.$$

In fact, it is easy to prove that it is a complex. Furthermore, it is exact, because it is a sub–complex of the resolution of I_X. It is obvious that the inclusion $G'_{t+j} \to G_{t+j}$ for $j \geq 1$, induces an inclusion $P \to N$. The resolution of the cokernel is

$$0 \to H_s \otimes \wedge^t F \to H_{s-1} \otimes \wedge^t F \to \cdots \to H_1 \otimes \wedge^t F \to N/P \to 0.$$

But $\wedge^t F \cong R(-\sum_{i=1}^t \deg(f_i))$ and the maps are $\varepsilon_i \otimes 1$. Hence, $\mathcal{N}/\mathcal{P} \cong \mathcal{J}_D(k)$ where $k = -\sum_{i=1}^t \deg(f_i)$. In particular, from Proposition 3 it follows that $\mathcal{P}$ is a maximal sub–sheaf of $\mathcal{N}$, and so the claim is proved. □

REMARK 15. In the previous Proposition, suppose D is a complete intersection. Then, X is a complete intersection too and I_X is generated by a regular sequence obtained by taking all the generators of I_D and I_S.

Reversing this observation, we consider a complete intersection scheme X generated by a regular sequence of forms $(f_0,\dots,f_i)$, with $i \leq r$. Starting from X we can obtain all the schemes D generated by a subset of generators of X. In particular, if we take $i = r$, and $f_j = x_j, j = 0,\dots r$, the $(r-1)$–syzygy sheaf $\mathcal{N}$ involved in the construction is a twist of the tangent sheaf $T_{\mathbb{P}^r}$.

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