Rendiconti Sem. Mat. Univ. Pol. Torino
Vol. 82, 1 (2023), 71 – 109

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# A GLIMPSE INTO THE EARLY HISTORY OF ELLIPTIC INTEGRALS AND FUNCTIONS

*To the memoir of Gianfranco Casnati, generous friend called away too soon.*

**Abstract.** This paper is a historical account about the early studies on elliptic integrals and functions aimed at analyzing in some depth the evolution of ideas about the "new transcendents" from the pioneering works of the Bernoullis and Fagnano to Legendre's *Esercices de calcul intègral*, passing through Euler's addition theorem and Lagrange's fundamental principle. Moreover, we will see that the first works of Abel and Jacobi on the topic, were greatly inspired by Legendre's achievements.

## 1. Introduction.

New transcendental functions –that is not depending on trigonometric, exponential, or logarithmic functions known for a long time– appeared in the late seventeenth century in connection with mechanical problems, and geometrical questions about quadratures and rectifications of plane curves. These integrals were mainly of the form

$$u(x) = \int_0^x \frac{P(x)dx}{\sqrt{p(x)}} \tag{1.1}$$

where $P(x)$ is a rational function and $p(x)$ a polynomial of degree 3 or 4, with distinct roots. Much time later A.-M. Legendre gave to these integrals the name of "elliptic function", because those measuring arcs of ellipses were among them. Consider the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $a \geq b > 0$; then the arc element is

$$ds = \sqrt{\frac{a^2 - g^2x^2}{a^2 - x^2}}dx$$

with $g^2 = (a^2 - b^2)/a^2$, and the integral

$$\int_0^x \sqrt{\frac{a^2 - g^2x^2}{a^2 - x^2}}dx = \int_0^x \frac{(a^2 - g^2x^2)dx}{\sqrt{(a^2 - x^2)(a^2 - g^2x^2)}}$$

expresses the arc length.

Today, the functions $u(x)$ are called *elliptic integrals*, while the term "elliptic functions" is reserved to functions which appear by inverting the simplest elliptic integrals (see below in this section).

As soon as integrals like (1.1) entered the mathematical scene attempts were made to express them by means of elementary functions, i.e. rational, trigonometric and logarithmic ones, but these attempts failed. Therefore, the mathematicians became convinced that these functions constituted an essentially new class of transcendents.[1] Then mathematicians began to compare elliptic integrals among them, trying to reduce their computation to those they believed the simplest ones; that is, arcs of conic sections, and particularly of ellipses.

In the study of elliptic integrals, four fundamental stages can be identified: G. Fagnano's discovery of addition and multiplication formulas for arcs of lemniscate; L. Euler's addition theorem for integrals of the form $\int \frac{dx}{\sqrt{p(x)}}$ [21]; J. Lagrange's transformation of elliptic differentials in [41]; Legendre's classification of elliptic integrals into three canonical kinds, and the study of their properties he carried on over forty years culminated in the publication of the first volume of his *Traité sur les fonctions elliptiques* in 1825. These were the main achievements on the "new transcendents" when, around the mid-1820s, N. H. Abel and C. G. J. Jacobi began to work in this field of mathematics.

Abel considered even more general integrals than those expressed by (1.1), namely integrals as

$$\int_0^x R(x,y)\,dx \tag{1.2}$$

where $y$ is an algebraic function of $x$, i.e. $y(x)$ is implicitly defined by an irreducible polynomial equation

$$f(x,y) = y^n + f_1(x)y^{n-1} + \cdots + f_n(x) = 0$$

Later these integrals were called *Abelian integrals* in his honour; and for such integrals he proved the important theorem, today known as *Abel addition theorem*,[2] which generalized to Abelian integrals the one given by Euler and those that Legendre provided for the other kinds of elliptic integrals [47, 21-41] (see Sections 3 and 5.2). In the course of his investigations Abel discovered that the functions defined by inverting elliptic integrals of first kind were doubly periodic once extended to the complex field. He called these functions "elliptic functions", and in his memoir [2], published in 1827, he expounded several properties of this class of functions.

In that same year, Jacobi put the basis for his theory of transformation of elliptic differentials, which two years later culminated in *Fundamenta nova theoriae functionum ellipticarum* [37].

Some twenty years after Abel's discovery of the double periodicity of elliptic functions, Liouville found in it the property characterizing a new class of functions,

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[1] However, this statement was actually proved only in 1833 by Joseph Liouville (1809–1882) in his memoir [49].

[2] On the meaning of this theorem and its importance in Algebraic geometry see [38].

the one of doubly periodic complex functions having only "infinities" as singularities (i.e. doubly periodic meromorphic functions) [50]; and today it is to these that the name of *elliptic functions* is reserved.

Several young mathematicians from all Europe attended the course that Liouville gave on this topic at the Sorbonne in 1847, and they contributed to disseminate the theory that Liouville had developed by using Fourier's series, without resorting to Cauchy's theory of complex integration [50]. Then the theory of elliptic functions was carried on by Charles Hermite (1822-1901), who put the basis for the use of Cauchy's theory of residues in the study of these functions (see [31] and the historical account [6]). Finally Karl Weierstrass (1815-1897) gave to the theory of elliptic function the form we know today, shaping it on the concept of meromorphic functions of one complex variable "satisfying an algebraic addition theorem", see [10], [16].

In synthesis, the history of elliptic integrals and functions from the origin up to the late nineteenth century can be roughly divided into four periods: the one which goes from the first encounters with elliptic integrals to Euler's addition theorem (1650-1770 c.); that of identification of elliptic integrals as a subject of study in itself, which culminated with the publication of Legendre's *Traité* (1780-1825); that of the discovery and exploration of the main properties of elliptic functions, due to the epoch making works of Abel and Jacobi (1825-1839); and that of development and first systematization of the theory of elliptic function properly said (1840-1880 c.).

This paper deals with the first two periods and marginally with the third; a time frame that can be headed "Early history of elliptic integrals and functions". Over time several historical pieces of information on this subject have been given in the introduction of treatises, or scattered throughout the text, and even in real historical essays devoted to the topic.[3] Moreover, articles on the contributions of a single author, or specific arguments, have become part of the historiography of the last century on the subject.[4] However, despite this crop of works we think that some aspects of this long story deserve to be further enlightened and discussed. Thus, in the present paper describing the process that from the discovery of the new transcendets led to that of doubly periodic functions, we will analyze in depth the first results on elliptic integrals, especially those regarding the fundamental contributions of Fagnano, Euler, Lagrange and Legendre; and we explain in detail their specific achievements with particular attention to those of Legendre which later inspired Abel and Jacobi in their early researches.

Here is the content of the other sections of the present paper.

In Section 2, we start by recalling the first results by Jakob and Johann Bernoulli on the arc length of certain mechanical curves. Then we focus on Fag-

[3] We mention: [47], [48], [11], [17], [54] (which was written in Swedish and then translated into English in 1923), [24], [39], and closer to our time [32], [13]. Recently U. Bottazzini and J. Gray have reserved part of the first chapter of their important book [10] to the early history of elliptic functions.

[4] See for instance [52], [55], [5], [57], [58].

nano's works on the lemniscate. Here we show in some detail how he discovered the duplication formula for the arc length of this curve, results which later inspired Euler. In the last part of the section we describe the two roads of investigation that, in the first decades of the eighteenth century, were opened regarding elliptic integrals, and the transformation that John Landen discovered in the 1770s which much impressed Legendre. Section 3 is devoted to Euler's addition theorem; we start by describing Euler's discovery of Fagnano's results and his immediate reception, and how he arrived to the addition theorem through various stages. In the next Section 4, we expound, in enough detail, Lagrange's general principle for reducing elliptic integrals to simpler forms, which led him to an important transformation between certain elliptic integrals. Then in Section 5, Legendre's fourteen-year work on elliptic integrals is analyzed focusing on those results that stimulated the first researches of Abel and Jacobi in the field. Notably, regarding Abel, the addition theorems for the three kinds of elliptic integrals, and the theory of multiplication and division for those of the first kind; and regarding Jacobi the discovery of the double scale of moduli for integrals of the first kind. In the last Section 6, we introduce and discuss in some detail the very first researches carried out by Abel and Jacobi on elliptic integrals. In particular: it is presented Abel's addition theorem; his discovery of the double periodicity of the inverse functions of an elliptic integrals of the first kind when they are extended to the field of complex numbers; and Jacobi's transformation theory. In doing this we will show how Legendre's *Exercices de calcul intègral*, appeared in 1811, resulted fundamental in formation and the development of Abel and Jacobi's thinking on the subject of elliptic functions.

## 2. The emergence of elliptic integrals

In the years immediately following the publication of Newton's *Principia mathematica Philosophiae naturalis* in 1687, and Leibniz's papers on calculus, several mathematicians applied the new-found analytical tools to solve mechanical and geometrical problems. Among them there were the brothers Jakob (1665-1705) and Johann (1667-1749) Bernoulli, who published a series of articles concerning integration by arcs of curves,[5] often connected with mechanical problems.

Johann Bernoulli studied a problem that Leibniz had posed in 1687, namely: To determine the curve (supposed lying in a vertical plane) described by an heavy body $P$ that falling, subject only to gravity, comes close or moves away from a fixed point $O$ so that the distance $OP$ is proportional to time. Leibniz called such a trajectory "paracentric isochron". Johann Bernoulli showed that to solve the problem one has to integrate the differential equation

$$(a\,dx + y\,dy)\sqrt{y} = (x\,dy - y\,dx)\sqrt{a}$$

where $a$ is a constant.

[5] That is to express integrals like (1.1) by means of fixed types of arcs of curves.

In 1691, Jakob Bernoulli proposed the following question: To determine the curve formed by a thin elastic beam under a force applied to one extremity being constrained at the other one. In June 1694, in his work *Curvatura laminae elasicae etc.* [7] published in *Acta Eruditorum*, he expounded the solution of a special case which had led him to consider the differential equation

$$dy = \frac{x^2 dx}{\sqrt{a^4 - x^4)}}$$

where $a$ is the length of the elastic beam. He also gave the geometrical construction of such a curve, that he called the *elastic* ("elasticae"), by means of the formula

$$y = \int_0^x \frac{x^2 dx}{\sqrt{a^4 - x^4}}$$

that is by "quadratures".

This article was followed immediately by [8] in which Jakob Bernoulli presented his solution to Leibniz's problem. In it he connected the construction of the paracentric isochron, called by him "curve of uniform go and back" to that of his "elastic" via the rectification of an algebraic curve.

By applying the transformation $ay = tz$, $ax = t\sqrt{a^2 - z^2}$ to the differential equation that his brother had found, Jakob was able to separate the indeterminate and to express the arc length of the paracentric isochron by means of the one of the elastic curve.

Then, he observed that the isochron could be expressed by the rectification of an algebraic curve by putting $x = \sqrt{at + t^2}$, $y = \sqrt{at - t^2}$. In fact, in this way, the arc of the elastic coincides with that of the curve obtained by eliminating $t$ between the two equations above, i.e.

$$(x^2 + y^2)^2 = a^2(x^2 - y^2)$$

The above mentioned article was followed by an *addenda* in the September issue of the same journal. Here to the curve defined by the above equation, "For its shape like the symbol $\infty$ as bandage folded in a knot, or lemnis, by a French ribbon knot",[6] he gave the name of *lemniscate* ("lemniscatae"). Then, he commented [8, 337]:

> Thus, as infinitely many other mechanical curves depend on the rectification of a circle, a parabola, or logarithmic spirals, also the curve of uniform go and back [paracentric isochron], and, as we shall see in the following, the elastic curve, derive from the rectification of the aforementioned algebraic curve of dimension [degree] four.

Let us observe that the lemniscate is a particular case of the Cassini ovals;[7] that is, the curves whose points $P$ satisfy the condition $PO \times PO' = c$, were $O$, $O'$

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[6] "constituta formam refert jacentis notae octonarii $\infty$, seu complicatae in nodum fasciae, sive lemnici, *d'un noeud de ruban Gallis*", (Bernoulli 1694b, 337).

[7] Giovanni Domenico Cassini (1625-1712).

are fixed points and $c$ is a constant. In fact, if the mid-point of $OO'$ belongs to the curve, the curve itself is a lemniscate, but we do not know if Jakob Bernoulli was aware of this.

Jakob Bernoulli also observed that for computational reasons the best representation of a transcendental curve was that by the rectification of an algebraic curve.

We notice that, independently of his brother, also Johann Bernoulli faced the same problems obtaining the same results, which he published in the October and November issues of *Acta Eruditorum.* Three years later, Johann Bernoulli proved that the sum, or the difference, of two arcs of the same cubic parabola can always be represented by logarithms and arcs of circle [9]. He judged this result very elegant, and on page 465 he wrote, “it would be not useless to see if other functions [the new transcendents] enjoy properties similar to those of logarithm and trigonometric ones”. We observe that the differential equation

$$f(x) = \pm f(y)dy$$

where $\int f(x)dx$ is a logarithm or an inverse of a trigonometric function, has as an integral an algebraic function of $x, y$ in spite of the fact that the single differentials have do not have an algebraic integral. The question that Johann Bernoulli addressed was whether this property holds for other transcendents different from logarithms and the inverse of circular functions. The first answer to the question was given by Fagnano.

### 2.1. Fagnano’s works on the arc of lemniscate

Giulio Fagnano (1682-1766),[8] dealt with arcs of curves at least since 1714, and since then he was aware of the results obtained by the Bernouilli brothers: his article on the integrability of the difference of two arcs of a biquadratic parabola, published in volume 19 of the *Giornale de’ letterati d’Italia*, witnesses to this. In 1715, in the same journal, Fagnano published *Nuovo methodo per rettificare la differenza di due archi*, and gave a first answer to Bernoulli’s question [54, 274]; and in the years immediately following, he also proved that the difference of two arcs of the same ellipse, or of the same hyperbola, can be algebraically expressed. But it was in 1718 that he obtained the results that, as we shall see, opened the way to a new kind of investigations.

In the three papers devoted to the “method for measuring the lemniscate”, which appeared in issues 29, 30 and 34 of the aforementioned journal, Fagnano expounded new properties of the arc of lemniscate.[9] He considered this curve as the Cassini oval when the fixed points are $(\pm b, 0)$, and $c = b = a/\sqrt{2}$; that is, he put

[8] Giulio Carlo de’ Toschi di Fagnano, was born in Senigallia, near to Ancona, on the Adriatic sea. He was an aristocratic self-taught mathematician.

[9] These papers can be found in [23], vol. 2.

$z^2 = x^2 + y^2$ as described by the equation

$$z^4 - 2a^2x^2 + a^2z^2$$

So that the arc length in the first quadrant is given by the integral

$$s(z) = \int_0^z \frac{a^2 dz}{\sqrt{a^2 - z^4}} \quad (0 \le z \le a)$$

The salient theorems are found in the first and second articles. In the first, Fagnano proved the following theorem:

Theorem 1. *If the equation*

$$u = a\frac{\sqrt{a^2 - z^2}}{\sqrt{a^2 + z^2}} \tag{2.1}$$

*holds true, then*

$$\int \frac{a^2 dz}{\sqrt{a^4 - z^4}} = \int -\frac{a^2 du}{\sqrt{a^4 - u^4}} \tag{2.2}$$

To show this he differentiated (2.1); then, by taking into account the (2.1) with some computation he obtained (2.2). We observe that this means that $a^2x^2 + a^2y^2 + x^2y^2 - a^4 = 0$ is an integral of the differential equation

$$\frac{a^2 dz}{\sqrt{a^4 - z^4}} = \frac{a^2 du}{\sqrt{a^4 - u^4}} \tag{2.3}$$

In the second article Fagnano obtained a formula for doubling a given arc of lemniscate. He knew that the sum or the difference of arcs of the same curve can be algebraically expressed, and this fact suggested to him the strategy to follow: to link two differentials by means of an algebraic equation between their variables. Of course this is equivalent to subject the differentials to a transformation, but Fagnano apparently seemed completely unaware of this. To reach his goal, he developed several theorems, that he numbered in Roman numerals. We focus here on those we think central to our aims, namely Theorem V (the duplication theorem) and the preparatory Theorems I, III, IV.

Theorem I. *If the equation*

$$x = \frac{\sqrt{1 \mp \sqrt{1 - z^4}}}{z} \tag{2.4}$$

*holds true, then*

$$\frac{\pm dz}{\sqrt{1 - z^4}} = \frac{\sqrt{2} dx}{\sqrt{1 + x^4}} \tag{2.5}$$

By differentiating (2.4) and some computation he obtained

$$dx = \frac{\pm dz\sqrt{1 \mp \sqrt{1-z^4}}}{z^2\sqrt{1-z^4}}$$

and by taking into account (2.4) itself he deduced

$$\frac{\sqrt{x^4+1}}{\sqrt{2}} = \frac{\sqrt{1 \mp \sqrt{1-z^4}}}{z^2}$$

then by dividing the formula of $dx$ above found by this last equation he proved the claim.

Theorem III. *If the equation*

$$x = \frac{u\sqrt{2}}{\sqrt{1-u^4}} \tag{2.6}$$

*holds true, then*

$$\frac{du}{\sqrt{1-u^4}} = \frac{1}{\sqrt{2}} \cdot \frac{dx}{\sqrt{1+x^4}} \tag{2.7}$$

Theorem IV. *If the equation*

$$x = \frac{1-t^4}{t\sqrt{2}} \tag{2.8}$$

*holds true, then*

$$\frac{-dt}{\sqrt{1-t^4}} = \frac{1}{\sqrt{2}} \cdot \frac{dx}{\sqrt{1+x^4}} \tag{2.9}$$

To prove these two he proceeded in the same way as in Theorem I. Finally he stated

Theorem V. *If the following equation*

$$\frac{u\sqrt{2}}{\sqrt{1-u^4}} = \frac{1}{z}\sqrt{1-\sqrt{1-z^4}} \tag{2.10}$$

*holds true, then*

$$\frac{dz}{\sqrt{1-z^4}} = \frac{2du}{\sqrt{1-u^4}} \tag{2.11}$$

To get this Fagnano argued as follows. He chose the sign "−" in equation (2.4) and replaced $x$ by $u\sqrt{2}/\sqrt{1-u^4}$ as determined in equation (2.6), and obtained equation (2.10); then, to get equation (2.11) he replaced $\sqrt{2}dx/\sqrt{1+x^4}$ by $2du/\sqrt{1-u^4}$ as in equation (2.5).

We do not know if Fagnano arrived at Theorem V by trial and error or through a complete reasoning. Carl L. Siegel (1896–1981), in [56], conjectured that Fagnano was probably driven by what happens for the integral

$$\int_0^x \frac{dx}{\sqrt{1-x^2}}$$

which can be rationalized by putting

$$x = \frac{2t}{\sqrt{1+t^2}}$$

and he used the transformation

$$x^2 = \frac{2t^2}{1+t^4}$$

which does not rationalize but leads to

$$\int_0^x \frac{dx}{\sqrt{1-x^4}} = \sqrt{2}\int_0^t \frac{dt}{\sqrt{1-t^4}};$$

and then by putting

$$t^2 = \frac{2u^2}{1-u^4}$$

he obtained

$$\int_0^x \frac{dx}{\sqrt{1-x^4}} = 2\int_0^u \frac{du}{\sqrt{1-u^4}}$$

However it happened, Fagnano was aware of having obtained a formula for the duplication of arcs of lemniscate; and that this could be achieved algebraically by

$$2\int_0^\alpha \frac{dx}{\sqrt{1-x^4}} = \int_0^{\frac{2\alpha\sqrt{1+\alpha^4}}{1+\alpha^4}} \frac{dx}{\sqrt{1-x^2}}$$

Thus the arc of lemniscate shared with the arc of circle the property of being doubled algebraically. We recall that for the arc of circle one has

$$2\int_0^u \frac{dx}{\sqrt{1-x^2}} = \int_0^{2u\sqrt{1-u^2}} \frac{dx}{\sqrt{x^2}};$$

but, was Fagnano really aware of this? In fact the arcs of circle enjoy an algebraic addition theorem, and, as it seems, he never tried to get a similar theorem for arcs of lemniscate.

Afterwards Fagnano showed how to multiply by 3 and 5, or to divide into 3 and 5 equal parts, an arc of lemniscate. In subsequent papers he proved similar theorems also for other kinds of arcs of curves.

The duplication and division theorems of Fagnano gave the first deeper insight into the nature of elliptic integrals, but this result went unnoticed, and did not have influence on the research in the field until Fagnano's collected works *Produzioni matematicche*, published in 1754, came into Euler's hands (see Sect. 3).

### 2.2. Others contributions

From what has been said, we see that as soon as elliptic integrals made their appearance, two roads were open to investigations:

1) Try to reduce the new transcendents to a minimum; for instance by using arcs of conic sections and in particular of ellipse which was believed the simplest one;

2) To investigate what might be their intrinsic properties.

The first road was followed by Colin Maclaurin (1698–1746), Jean Baptiste Le Rond d'Alembert (1717–1783), and John Landen (1719–1790) among others.

In 1742, in his *Treatise of Fluxions* [51], Maclaurin defined a programme for integrating elliptic differential through arcs of conic sections by applying geometrical reasoning. Four years later d'Alembert in [14], published by the Academy of Sciences of Berlin, continued on the same route but by means of algebraic computation on differentials and without resorting to "figures" [27], [28].

Landen presented his researches on arcs of conic sections in two memoirs published in the *Philosophical Transaction of the Royal Society of London* in 1771 and 1775 respectively. In the first he perfected the methods developed by Maclaurin and d'Alembert [42, 298]:

> Mr Mac Laurin, in his Treatise of Fluxions, has given sundry very elegant Theorems for computing the Fluents of certain Fluxions by means of Elliptic and Hyperbolic Arcs; Mr. D'Alembert, in the Memoirs of the Berlin Academy, has made some improvement upon what had been written on that subject. But some of the Theorems given but these Gentlemen being in part expressed by the difference between an Arc of a Hyperbola and its Tangent, and such difference being not directly attained, when such Arc and its Tangent both become infinite [...] The supplying that defect I considered as a point of some importance in Geometry, and therefore I earnestly wished, and endeavoured, to accomplish that business.

At the very end of his memoir Landen announced the rectification of the arc of hyperbola by means of two arcs of ellipses [42, 309]:

> Since writing the above, I have discovered a general theorem for the rectification of the Hyperbola by means of two Ellipsis; the investigation whereof I purpose to make the subject of another paper.

The announced result appeared four years later in [43]. Here, Landen deduced this theorem from three geometrical lemmas.[10] Here we record that once reached the proof Landen wrote [43, 285]:"Thus, beyond my expectation, I find, that the hyperbola may in general be rectified by means of two ellipsis". In fact, he obtained this remarkable result with the simple goal of perfecting those of Maclaurin and d'Alembert. Landen concluded his memoir by saying [43, 289]:

> Before Mr Maclaurin published his excellent Treatise of Fluxions, some very eminent mathematicians imagined that the *elastic curve* could not be constructed by quadratures or

[10]For some detail about the proof we refer to [60] and also [58].

rectifications of conic sections. But that Gentleman has shewn, in that treatise, that the said curve may in every case be constructed by rectification of the hyperbola and ellipsis; and he has observed that, by the same means, we may construct the curve along which, if a heavy body moved, it would recede equally in equal time from a given point. Which last mentioned curve Mr. James [Jakob] Bernoulli constructed by the rectification of the elastic curve, and Mr. Leibniz and Mr. John [Johann] Bernoulli by the rectification of a geometrical curve of a higher kind than the conic sections. It is observable, that Mr. Maclaurin's method of construction just now adverted to, though very elegant, is not without a defect. The difference between the hyperbolic arc and its tangent [...]. The contents of this paper, properly applied, will evince, that both *elastic curve* and the *curve of equable recess from a give point* (with many others) may be constructed by rectification of the ellipsis only, without failure in any point.

These words clarify the tasks Landen had set himself, and what the aims of research on elliptic integrals were in the 1770s.

The important fact in Landen's discovery is that the two ellipses to which the problem was ultimately brought back are related to one another so that the semi-axes $a', b'$ of one are the arithmetic and the geometric mean of the semi-axes $a, b$ of the other, that is

$$a' = (a+b)/2, \qquad b' = \sqrt{ab}$$

A pair of ellipses so related are nowadays called *connected by Landen's transformation.* Landen did not perceive the value of his discovery, and the first acknowledgement came from Legendre, who, as we shall see in Sect.5, was an attentive reader of Landen works. From Landen's result Legendre elaborated an efficient method for the approximate numerical calculation of elliptic integrals. However, what is more important is that Landen's transformation led to more general ones which, after Jacobi, opened the way to the theories of modular equations and complex multiplication. In passing we note that the overall importance of the arithmetic-geometric mean for the theory of elliptic integrals was revealed only after the publication of Gauss's posthumous works [29].

The second road of investigation pointed out above, was much harder to take, and this mostly because of the still imperfect understanding of complex analysis, but after Fagnano an important step was done by Euler.

## 3. Euler's addition theorem

The election of Fagnano as a corresponding member of the Berlin Academy was a fortunate circumstance for the development of the theory of elliptic integrals. In December 1751, Euler was asked to judge Fagnano's *Produzioni matematiche*, the two volumes of collected works that for the occasion Fagnano had printed the year before. Undoubtedly Euler judged them good enough as Fagnano was elected. However, for us the most interesting aspect is that Euler found Fagnano's results on the lemniscate very stimulating, for on May 30 1752 he wrote to Christian Goldbach [25]:

Recently I came across some decidedly curious integrals. As of the equation

$$\frac{dx}{\sqrt{1-xx}} = \frac{dy}{\sqrt{1-yy}}$$

the integral is $yy+xx = cc+2xy\sqrt{1-cc}$, also of this equation

$$\frac{dx}{\sqrt{1-x^4}} = \frac{dy}{\sqrt{1-y^4}}$$

the integral is $yy+xx = cc+2xy\sqrt{1-c^4}-ccxxyy$. Moreover, the integral of the equation

$$\frac{dx}{\sqrt{1-x^3}} = \frac{dy}{\sqrt{1-y^3}}$$

is $xx+yy+ccxxyy = 4c-4cc(x+y)+2xy-2cxy(x+y)$.

It seems that Euler had grasped something that Fagnano had missed, namely that certain elliptic integrals could enjoy of an algebraic addition theorem, like the inverse functions of trigonometric ones. Euler published his first article on this topic in 1756. In it he began by saying [18]:

When for the first time I saw the equation discovered by the Illustrious Count Fagnano and the algebraic relation between the variables $x$ and $y$ satisfying it, I realized that this relation could not be the complete integral, because it did not contain an arbitrary constant.

Euler was referring to Theorem 1 of Fagnano. He observed that the complete integral of the differential equation

$$\frac{dx}{\sqrt{1-x^4}} = \frac{dy}{\sqrt{1-y^4}}$$

had to be reduced to the integral discovered by Fagnano for particular values of the arbitrary constant contained in it, and also to $y = x$ which is another particular integral. Then Euler claimed that the complete integral was given by $x^2+y^2+c^2x^2y^2 = c^2+2xy\sqrt{1-c^2}$; and to show this he proceeded as Fagnano had done, that is by differentiating the last equation above and making the appropriate substitution suggested by that equation. From this Euler deduced the addition theorem for arcs of lemniscate (in the first quadrant): *If*

$$\int_0^x \frac{dt}{\sqrt{1-t^4}} + \int_0^y \frac{dy}{\sqrt{1-t^4}} = \int_0^z \frac{dt}{\sqrt{1-t^4}}$$

*then*

$$z = x\sqrt{1-y^2}+y\sqrt{1-x^2}$$

Regarding the method he had applied Euler wrote [18, 40], "I was led to this integral by no secure method, but rather experimenting, or by divining, I found it out". From these words we perceive that he was not satisfied of how he had obtained this.

As the title of his paper suggests, Euler's real goal was the determination of the complete integral of the more general differential equation

$$\frac{m\,dx}{\sqrt{1-x^4}} = \frac{n\,dy}{\sqrt{1-y^4}}$$

Then, by iterating the procedure adopted (see also [54] for details), he proved that if $m/n$ is rational the complete integral of the above differential equation is an algebraic function of the variables $x$, $y$ and of an arbitrary constant $c$.

Euler also pointed out that, "though not derived from the nature of the problem, this method which indirectly leads to the solution is of wide application"; and he proved that it worked with the differential equation

$$\frac{dx}{\sqrt{1+mxx+nx^4}} = \frac{dy}{\sqrt{1+myy+ny^4}}$$

where $m, n$ are arbitrary constants.

In a dozen of years Euler wrote several papers concerning this kind of equations and integration by arcs. For instance, in the subsequent article [19] he compared arcs of parabola, ellipse, hyperbola and lemniscate, studied when the difference of two arcs of these types is algebraically expressible; and proved the division theorem in $2^2(2^m+1)$ parts of an arc of lemniscate. The following year, in the memoir [20], Euler showed that also the differential equation

$$\frac{dx}{\sqrt{A+Cx^2+Ex^4}} = \frac{dy}{\sqrt{A+Cy^2+Ey^4}}$$

($A \neq 0$), has an algebraic complete integral, namely $a(c^2-x^2-y^2+Ec^2x^2y^2+2xy\sqrt{A(A+Cc^2+Ec^4})$; and obtained the addition theorem for such differentials: *If*

$$\int_0^x \frac{dt}{\sqrt{A+t^2+Et^4}} + \int_0^y \frac{dt}{\sqrt{A+t^2+Et^4}} = \int_0^z \frac{dt}{\sqrt{A+t^2+Et^4}} +$$

*then*

$$z = \sqrt{A}\frac{x\sqrt{A+y^2+Ey^4}+y\sqrt{A+x^2+Ex^4}}{A-Ex^2y^2}$$

Finally, in [21], Euler dealt with the differential equation

$$\frac{dx}{\sqrt{P(x)}} = \frac{dy}{\sqrt{P(y)}}$$

where $P$ is a general polynomial of the degree 4. He began by saying [21, 3]:

I had once arrived at the integration of this differential equation, whose complete integral is an algebraic equation involving $x, y$, by a very singular and oblique method.

This seems all the more surprising because the last integral cannot even be expressed by quadrature of circle or hyperbola. But, and it is worth noting, there was no direct method of dealing with the question at the time. So, for bringing forth the ends of analysis there seems to be no better occasion than, after having obtained it by the method already employed, to investigate the same problem by a direct method.

In this memoir Euler proved that by means of the transformation

$$x = \frac{mz + a}{nz + b}$$

($mb - na \neq 0$) the equation above can be reduced to the form

$$\frac{dx}{\sqrt{\alpha + \beta x^2 + \gamma x^4}} = \frac{dy}{\sqrt{\alpha + \beta y^2 + \gamma x^4}}$$

and from this he easily deduced the general addition theorem for integral of the form

$$\int \frac{dx}{\sqrt{P(x)}}$$

where $P(x)$ is a general polynomial of degree 3 or 4.

Although Euler was aware of the importance of his results, in several occasions he regretted that these were more the fruit of chance ("potius fortuito quasi detecta" he wrote) than of the application of the general principles of analysis. This sounded like an invitation to mathematicians to overcome such a flaw, and this was soon taken up by Lagrange.

## 4. Lagrange's general principle

The basic idea permeating Lagrange's investigation in [40] is expressed in following observations that we find in paragraph n.7:

> If a differential equation of the first order is given, of which the solution is unknown, one can differentiate it and to see whether, by combining the differential equation so obtained and the old one, one is led to a complete integral of the first order which does not embrace, as a particular case, the original equation; if this works, then the sought-for integral is obtained by eliminating the derivative between the two differential equation of first order. If this does not work, one passes to third order differentials and sees if the second and third differentials can be eliminated by means of the given equation and its differential; and so on.

After having tested this principle in some simple cases, Lagrange applied it to find the complete integral of the differential equation

$$\frac{dx}{\sqrt{\alpha + \beta x + \gamma x^2 + \delta x^3 + \epsilon x^4}} = \frac{dy}{\sqrt{\alpha + \beta y + \gamma y^2 + \delta y^3 + \epsilon y^4}}$$

We summarize his argument. First, to simplify computation, he put $X = \alpha+\beta x+\gamma x^2+\delta x^3+\epsilon x^4$ and $Y = \alpha+\beta y+\gamma y^2+\delta y^3+\epsilon y^4$; and considered the system of the two differential equations $dt/T = dx/\sqrt{X}$, $dt/T = dy/\sqrt{Y}$ where $T$ is a function of $x$ and $y$. Multiplying, squaring and differentiating he obtained

$$\frac{2tdtdx + 2t2d^2x}{dt^2} = \beta + 2\gamma x + 3\delta x^2 + 4\epsilon x^3$$

$$\frac{2tdtdy + 2t2d^2y}{dt^2} = \beta + 2\gamma y + 3\delta y^2 + 4\epsilon y^3$$

Then he replaced the old variables $x, y$ by the new ones $p = x + y$, $q = x - y$, set $dT = Mdp + Ndq$, and with some calculation of great elegance he obtained the differential equation

$$\frac{T^2(Mdp^2 + Td^2p)}{dt^2} = (\beta + \gamma p)(T - Nq) + \frac{\delta}{4}(p^2 + q^2) - N(3p^2q + q^3) +$$

$$+ \frac{\epsilon}{2}[T(p^2q + 3pq^2) - N(p^3q + pq^3)]$$

Since $T$ is arbitrary, by putting $T = Nq$ and some computation he arrived to

$$\frac{2NdNdp^2 + 2N^2dpd^2p}{dt^2} = \delta dp + 2\epsilon pdp$$

which is integrable, and

$$\frac{N^2dp^2}{dt^2} = G^2 + \delta p + \epsilon p^2$$

where $G$ is an arbitrary constant is its general integral. From this he obtained

$$\frac{Ndp}{dt} = \sqrt{G^2 + \delta p + \epsilon p^2}$$

At this point he performed the suitable substitutions, and found that

$$\sqrt{\alpha + \beta x + \gamma x^2 + \delta x^3 + \epsilon x^4} + \sqrt{\alpha + \beta y + \gamma y^2 + \delta y^3 + \epsilon y^4} =$$

$$= (x - y)\sqrt{G^2 + \delta(x + y) + \epsilon(x + y)^2}$$

is the general integral of the given differential equation.

Lagrange also showed that, except for some particular cases, this method did not apply to similar differential equations when the polynomial under the square-root has degree greater than 4.

Ten years later Euler, in his memoir [22], commented upon Lagrange's method; and, in particular, he gave a new proof of his general addition which this time he stated as follows: *Denoting* $\Pi(x) = \int_0^x \frac{dx}{\sqrt{X}}$, *where* $X = A + Bx + Cx^2 + Dx^3 + Ex^4$ *if*

$$\Pi(x) + \Pi(y) = \Pi(k)$$

*then $x, y, k$ satisfy the following algebraic relation*

$$\frac{2A + B(x+y) + 2Cxy + Dxy(x+y) + 2Exxyy \mp 2\sqrt{XY}}{(x-y)^2} =$$

$$= \frac{2A + Bk \mp 2\sqrt{AK}}{kk}$$

In 1784 Lagrange took over an idea already emerged in Euler's [21]; that is, to transform an elliptic differential $\frac{dx}{\sqrt{X}}$ into a simpler one by means of an algebraic transformation. Actually, Lagrange programme in [41], was more ambitious as he tackled the question of reducing to the simplest possible form an elliptic integral

$$\int P(x,R)dx$$

where $P(x,R)$ is a rational function of $x$, $R = \sqrt{a + bx + cx^2 + dx^3 + fx^4}$, and the polynomial under the square-root has distinct roots. His goal, however, was not to arrive at some sort of more general addition theorem, but, as he stated introducing his work, to find a method for accelerating the approximate calculations of elliptic integrals. We give word to him [41, 218-219]:

> We know that every differential containing a square radical in which the variable does not exceed degree two is integrable by logarithm and circular functions, because it is always possible to reduce it to a rational form making disappear the radical by suitable substitution. But this reduction no longer succeeds, in general, when the variable appears to a power greater than 2, and integration escapes to known methods. If the greatest power does not exceed the fourth degree, we can in several cases to construct the integral by arcs of conic sections. The search for these cases has occupied Geometers a lot; their work is of advantage to the progress of integral calculus, since it serves to bring back to determined classes a large number of differentials of different forms; but it is not of use in the actual integration of these differentials, because the rectification of conic sections is still only imperfectly known given the little convergence of the series found so far for this. Actually the series are the only means for solving this problem, and in general to arrive to the integration of all essentially irrational differentials; but this method is really useful only as long as we can make the series always convergent, and also the error which comes from the neglected terms decreases at will. The method which I give in this memoir adds to this advantage that of being general for every differential containing a square radical in which the degree of the variable does not exceed four.

Lagrange began explaining his method by observing that whatever the function $P(x,R)$ is, it can be put in the form

$$\frac{A+BR}{C+DR}$$

where $A, B, C, D$ are rational functions of $x$. Thus

$$P = \frac{A+BR}{C+DR} = \frac{(A+BR)(C-DR)}{C^2 - D^2R^2} = M + \frac{N}{R}$$

where $M$ and $N$ are rational functions of $x$.

He first supposed $R=\sqrt{a+cx^2+fx^4}$, and observed that $N$ can be always written in the form $(F+Gx)/(H+Lx)$ where $F,G,H,L$ are polynomials in $x^2$; and obtained

$$N=\frac{F+Gx}{H+Lx}=\frac{(F+Gx)(H-Lx)}{H^2-L^2x^2}=T+Vx$$

where $T$ and $V$ are rational functions in $x^2$. Now, $T/R$ does not contain odd powers of $x$, so by setting $x^2=y$ the differential $(Vx/R)dx$ is transformed into

$$\frac{V(y)dy}{2\sqrt{a+cy+fy^2}},$$

with $V(y)$ a rational function of $y$, which is integrable by means of elementary functions.

Afterwards Lagrange supposed $R=\sqrt{a+bx+cx^2+dx^3+fx^4}$, and put the polynomial under the root in the form $a(m+nx+x^2)(m'+n'x+x^2)$. By setting

$$y^2=\frac{f(m'+n'x+x^2)}{m+nx+x^2} \tag{4.1}$$

he was able to show that the differential $Ndx/R$ is transformed into

$$\frac{2Ndy}{\sqrt{\alpha+\beta y^2+\gamma y^4}}$$

where $\alpha=f^2(n'^2-4m')$, $\beta=-2f(nn'-2m-2m')$, $\gamma=n^2-4m$.

Then by substituting to $x$ its value

$$\frac{ny^2-n'f+\sqrt{\alpha+\beta y^2+\gamma y^4}}{\alpha(f-y^2)^2}$$

and eliminating the radical in the denominator of $N$, the differential decomposes into a rational part (which can be integrated without difficulty) and one of the form

$$\frac{Qdy}{\sqrt{\alpha+\beta y^2+\gamma y^4}}$$

where $Q$ is a rational function of $y^2$.

Summing up Lagrange had showed that the differential to be integrated can be reduced to the sum of a rational differential and one of the form

$$\frac{Ndx}{\sqrt{a+bx^2+cx^4}}$$

By means of skilful algebraic calculations, Lagrange transformed this last differential into the following

$$\frac{Ndx}{\sqrt{(a+bx^2)(m+nx^2)}}$$

where $N$ is a rational function of $x^2$ and $a, b, m, n$ are arbitrary real constants. At this point, by applying the transformation

$$y = \frac{x}{a}\sqrt{\frac{a+bx^2}{m+nx^2}} \tag{4.2}$$

and pushing his algebraic calculations further (that for brevity we continue to omit), he proved that the differential above can be further reduced to the form

$$Ldy + \frac{Mdy}{\sqrt{(1 \pm p^2y^2)(1 \pm q^2y^2)}}$$

where $L, M$ are rational functions of $y^2$, and

$$p = \sqrt{\pm bm} + \sqrt{\pm(bm - an)}$$

$$q = \sqrt{\pm bm} - \sqrt{\pm(bm - an)}, \qquad \text{or} = \pm\sqrt{\pm(bm - an)} - \sqrt{\pm bm}$$

are positive real numbers with $p > q$.

"Now", he wrote, "all difficult consists in the integrations of the new differential"

$$\frac{Mdy}{\sqrt{(1 \pm p^2y^2)(1 \pm q^2y^2)}} \tag{4.3}$$

and added:

> Once we have arrived at a differential of this form, all we have to do is repeat the substitutions and the transformations that we have just taught, and for this we can make use of the previous formulas by setting
>
> $$a = 1, m = 1, b = \pm p^2, n = \pm q^2$$
>
> and so on. Here is the table of operations
>
> $$p' = p + \sqrt{p^2 - q^2} \qquad q' = p - \sqrt{p^2 - q^2}$$
>
> $$p'' = p' + \sqrt{p'^2 - q'^2} \qquad q'' = p' - \sqrt{p'^2 - q'^2}$$
>
> $$\ldots\ldots \qquad \ldots\ldots$$
>
> $$y' = \frac{yR}{1 \pm y^2R}$$
>
> $$y'' = \frac{y'R'}{1 \pm y'^2R'}$$
>
> $$\ldots\ldots$$
>
> where $R = \sqrt{(1 \pm p^2y^2)(1 \pm q^2y^2)}$, $R' = \sqrt{(1 \pm p'^2y'^2)(1 \pm q'^2y'^2)}$, etc.

We see that by this procedure the differential from which we started is transformed into the following sum

$$Ldy + L'dy' + \cdots + \frac{Zdz}{\sqrt{(1 \pm r^2 z^2)(1 \pm s^2 z^2)}}$$

where $L, L'$, etc. are rational functions of $y^2, y'^2$, etc. and $Z, z, r, s$ the last terms of the sequences $M, M', \ldots;\ y, y', \ldots;\ p, p', \ldots;\ q, q', \ldots$.[11]

Lagrange observed that in this way we can arrive at a term $s$ so small that (according the degree of approximation one wants) $s^2 z^2$ can be supposed to be 0, and the last differential in the formula above is reduced to

$$\frac{Zdz}{\sqrt{(1 \pm r^2 y^2)}}$$

It is convenient to enucleate the main result from Lagrange's entire argument: *The differential*

$$\frac{dx}{\sqrt{(1 \pm p^2 x^2)(1 \pm q^2 x^2)}}$$

*is carried into a new differential of the same form*

$$\frac{dy}{\sqrt{(1 \pm p_1^2 y^2)(1 \pm q_1^2 y^2)}}$$

*by means of the transformation*

$$y = x\sqrt{\frac{1 - p^2 x^2}{1 - q^2 x^2}}$$

*where* $p_1 = p + \sqrt{p^2 - q^2}, q_1 = p - \sqrt{p^2 - q^2}$.

In the second part of his memoir Lagrange gave two detailed applications of his method, that he introduced by saying [41, 250-251]:

> By the general method we have just expounded it is assured that we can integrate, so exactly as we want, differentials containing a square radical in which the variable does not appear to a degree greater than four; this is the case of a large number of geometrical and mechanical problems which until now could be solved only in an incomplete and limited manner. Since this method is of a very new kind, and some difficulties can be met in its use, we go to apply it in detail to the rectification of elliptic and hyperbolic arcs.

While these words reaffirm what Lagrange's intentions were in writing this memoir, they also suggest that he was unaware of Landen's achievements; otherwise he would definitely have mentioned him. As we shall see shortly, Legendre was aware of Landen's results; in fact he cited him although he could not but to remark that "he [Landen] did not get much out of them".

---

[11] Let us observe that $p' = \frac{p+q}{2}, q' = \sqrt{pq};\ p'' = \frac{p'+q'}{2}, q'' = \sqrt{p'q'}$; and so on.

## 5. Legendre's classification of ellitic integrals and beyond

From what has been said in the previous section, it is clear that, as late as the mid-1780s, the interest in elliptic integrals was more directed to the development of effective methods for approximate calculation rather than of a speculative character; and this was the climate when, in 1786, Legendre began to work on the new transcendents. The first two papers on the topic he published [44, 45], appeared in the *Mémoires de l'Académie des Sciences*, concerned the integration by means of arcs of ellipses, which were believed to be the simplest ones among the new transcendents. In the first memoir, in full accordance with the research of the time, Legendre offered a method capable of facilitating the construction of numerical tables useful in the applications.

He started by considering the arc of ellipse expressed in trigonometrical form:[12]

$$E = E(c,\varphi) = \int d\varphi \sqrt{1 - c^2 \sin^2 \varphi}$$

$$F = F(c,\varphi) = \int d\varphi \sqrt{1 - c^2 \cos^2 \varphi}$$

where the angle $\varphi$ is called *amplitude* of $E$.

Legendre developed $E(c,\varphi)$ and $F(c,\varphi)$ in power series of $\sin\varphi$ and, taking into account the eccentricity $c$, he studied the convergence of the series obtained. This gave him a method for approximate computation, which he applied to the arc of hyperbola and to other cases.[13] At the end of the paper, he showed how it was possible to treat the integral $\int dz/\sqrt{A + Bz^2 + Cz^4}$ and proved that the integral $\int dz/\sqrt{A + Bx + Cz^2 + Dz^3 + Ez^4}$, by means of a transformation of type $x = (m + nz)/(1+z)$ which allows to make disappear the odd powers of the variable, could be brought back the the previous one.

In the Introduction to his second memoir, Legendre wrote that only after having read the first one at the Academy of Sciences he became aware that Landen had successfully dealt with the integration by means of two arcs of ellipses, and expressed any arc of hyperbola as the difference of two arcs of ellipses. Legendre emphasized that Landen's result was even more interesting because, as he observed on p. 645, obtained by "a very skillful transformation, which had been missed by all who had dealt with the topic". Legendre praised the elegance of Landen's method, however he defended his own as the most suitable for the formation of numerical tables. Then Legendre stated that the aim of his second memoir was to deduce Landen's result from his own formulas. In particular, by varying the eccentricity according to a fixed law, he determined an infinite family of ellipses, "from the circle

---

[12] The ellipse is taken with centre $C$ in the origin of the coordinates, transverse semi-axis $CA = a = 1$, conjugate semi-axis $CB = b$, half focal distance $c$; in this case $c$ coincides with the eccentricity $e = \sqrt{(a^2 - b^2)/a^2}$. Let $DA$ be the arc of the circle with centre in $C$ and radius 1 in the first quadrant, $Z$ be a point on it, $\varphi$ the angle subtended by the arc $DZ$, and $M, P$ the intersections of the perpendicular to the $x$-axis with the arc ellipse $BA$ and the $x$-axis, respectively; in this case if $M = (x, y)$ we have $x = \sin\varphi$, $y = b\cos\varphi$. Then Legendre called $E$ and $F$ the arcs $BM$ and $MA$ of the ellipse, respectively.

[13] For more information we refer the interested reader to [57].

to the straight line", such that the rectification obtained by means of two of them immediately gives that of all others. On p. 646 he wrote:

> I cannot but notice about the singular concordance between two results obtained by completely different methods. In the Mémoires de Turin, *Tome V*, M. de la Grange has considered differentials that can be integrated by arcs of conic sections; in certain cases these are integrable exactly by ordinary means, that is by means of arcs of circles and logarithms; for this a relation among the constants is required. In all other cases Mr. de la Grange arrives, by means of successive substitutions, to increasingly finer approximations to made integration possible. It is clear that the result of this method perfectly agrees with ours.

It is clear that he had read and meditated Lagrange's memoirs. We do not comment further on Legendre's second memoir, except to say that in a way similarly to the one he used in the case of the ellipse, he also introduced the trigonometrical form of the arc of hyperbola

$$H = \int \frac{d\varphi}{\cos^2\varphi\sqrt{1-c^2\sin^2\varphi}}$$

### 5.1. Legendre's memoir of 1792

Legendre returned on the topic in *Mémoire sur les transcendates elliptiques* [46], read at the Academy of Sciences in April 1792. The continuation of the title, "*in which we give some easy methods to compose and evaluate these transcendents, which enclose the arcs of ellipse, and are of frequent use in the applications of integral calculus*", well explains what his aim still was. Nevertheless, this memoir marked a turning point in the history of elliptic integrals. Legendre, after a brief preamble on the usefulness of the new transcendents, introduced his work by observing:

> [Although] the arcs of ellipse add a lot to the tools of analysis, especially after it has been observed how the arcs of hyperbola can be deduced from them [Landen's result], they are still insufficient to solve more complicated questions, as the determination of the surface of an oblique cone, the movement of rotating bodies not subject to any accelerating force, etc. These questions, and many others, depend, in general, on the integral $\int P dx/R$ where $P$ is a rational function of $x$ and $R$ is a radical of the form $\sqrt{\alpha+\beta x+\gamma x^2+\delta x^3+\epsilon x^4}$ and $\alpha, \beta$, etc. are constants. Now, by carefully examining the nature of this integral, we find that it gives rise to three distinct species of transcendents. The first and the second can be expressed by means of arcs of ellipse; the third is more composed, although it has many analogies with the other two.

Legendre continued by saying (p. 4):

> The first two species could be considered a single one, as they can be expressed by arcs of an ellipse, nevertheless we have believed to distinguish them because if the first can be reduced to the second, the second cannot be reduced to the first; which is analytically simpler than the other, and so the arcs of an ellipse are not the simplest after the arcs of circle

and logarithms. Moreover, the relations among the three species of these transcendents are such that we believe they have to be called *Elliptic transcendents*.

Therefore, he debunked what had been thought until then about the arcs of ellipse, and gave the name to the transcedents enclosed in the integral $\int Pdx/R$. Then Legendre explained what he would have done:

In this memoir we aim at investigating the nature and the properties of these transcendents, in order to make their use in the applications of the integral calculus easier. Several among the methods and results which we are going to expound are already known to the geometers: we have assembled under the same point of view all what has been published on the theory until now; but, at the same time, we have tried to further refine it.

Hence Legendre outlined the programme he then developed in *Exercices de calcul integral* and perfected in the *Traité sur les fonctions elliptiques*.

Legendre considered the elliptic integral

$$I = \int \frac{Pdx}{R}$$

where $P$ is a rational function of $x$ and $R = \sqrt{\alpha + \beta x + \gamma x^2 + \delta x^3 + \epsilon x^4}$. By dividing the numerator by the denominator of $P$ and developing the rest in partial fractions he could write

$$P = \sum a_n x^n + \sum \frac{a_{ik}}{(x - a_{ik})^{ik}}$$

then, by substituting this expression in the integrand and by integrating term by term he obtained $I$ as a sum of integrals of the following types

$$\Pi^m = \int \frac{x^m dx}{R}; \qquad \Gamma^{-k} = \int \frac{dx}{(1+nx)^k R}$$

By differentiating $x^{m-3}R$ and then by integrating the expression obtained, he showed that $x^{m-3}R$ is a linear combination of the integrals $\Pi^m$, $\Pi^{m-1}$, $\Pi^{m-2}$, $\Pi^{m-3}$, $\Pi^{m-4}$. From this he deduced that any integral $\Pi^m$, $m \geq 3$ can be expressed by means of $\Pi^0$, $\Pi^1$, $\Pi^2$.

Next Legendre put $\omega = (1+nx)$. By differentiating $\omega^{-k+1}R$ and then integrating the expression obtained, he found that when $k > 1$ the integrals $\Gamma^{-k}$ can be reduced to a linear combination of $\Pi^m$, $m = 0,1,2$, and $\Gamma^{-1}$. Then, putting $(1+nx) = 1/z$ he showed that also $\Gamma^{-1}$ can be expressed by means of the $\Pi^m$, $m = 0,1,2$.

At this point Legendre proved that by means of the substitution $x = (p + qy)(1+y)^{-1}$ the odd powers of $x$ in $R$ can be eliminated. So he supposed the polynomial under the square root to be $\alpha + \beta x^2 + \gamma x^4$, and wrote it as a product of two real factors like $f + gx^2$. Then, Legendre showed that, whatever the factorization was, by putting

$$x^2 = \frac{A + B\sin^2\phi}{C + D\sin^2\phi}$$

where $A, B, C, D$ are constants, it is always possible to reduce the integral $\int P dx/R$ to the form

$$\int \frac{Q d\phi}{\sqrt{1-c^2 sin^2\phi}}$$

where $c$ is a constant $0 < c < 1$, and $Q$ is an even rational function of $\sin\phi$ containing $\sin\phi$ at the same powers at which $x$ appears in $P$. Thus he had brought back the integral $I$ to the sum of an algebraic part and an integral of the form

$$G = \int \frac{(A + B\sin^2\phi)d\phi}{\Delta}$$

where $\Delta = \sqrt{(1-c^2\sin^2\phi)}$, which can represent as particular cases both elliptic and hyperbolic arcs, and integrals of the form

$$\int \frac{N d\phi}{(1+n\sin^2\phi)\Delta}$$

where the constants $N$ and $n$ can assume any real or imaginary value.

Finally, Legendre gathered in the following formula

$$H = \int \frac{A + B\sin^2\phi}{1+n\sin^2\phi} \cdot \frac{d\phi}{\Delta}$$

the two latter types.

Legendre noticed that neither $E$ nor $G$ can be expressed by $F$, while $F$ can be expressed by $E$, thus he recognized $F$ as “the simplest among elliptic functions”. He called $\phi$ the *amplitude*, $c$ the *modulus*, and $b = \sqrt{1-c^2}$ the *complementary modulus*.

For the next twenty years Legendre intensely worked to develop the programme outlined in the memoir of 1792, and finally, in 1811, the first volume of the *Exercices de calcul intégral* saw the light of the day. This book will have great importance in Abel’s mathematical education.

### 5.2. Legendre’s first volume of the *Exercices*

In the Introduction to the first part, titled “On elliptic functions”, of the three in which Legendre divided the content of the volume, he quickly reviewed the history of the new transcendents. He cited Maclaurin [51], D’Alembet [14], and remarked that after their results mathematicians were convinced that the new transcendents were of extreme importance in the development of calculus. He also quoted Fagnano, “an Italian geometer of great sagacity”, and his *Produzioni matematiche*, which, he pointed out, “opened the way to the most profound investigations”. Afterwards Legendre mentioned Euler (p.2):

> who although by a lucky combination, but these coincidences only happen to those who know how to give birth to them, found the complete algebraic integral of a differential

equation formed by two separate terms of the same type, each of which is only integrable by arcs of conic sections.[...] This important discovery gave the way to compare among them not only arcs of the same curve but also all transcendents enclosed in the formula $\int Pdx/R$ where $P$ is a rational function of $x$ and $R = \sqrt{\alpha + \beta x + \gamma x^2 + \delta x^3 + \epsilon x^4}$

Immediately after he claimed:

The integral found by Euler was too remarkable not to attract the interest of geometers. Lagrange wanted to bring this integration into the ordinary processes of analysis; he succeeded by a very ingenious method, whose application gradually rises from the easier transcendents to the Eulerian ones;[14] but he tried in vain to arrive at a more general result than that of Euler. Shortly after the English geometer Landen demonstrated that the arc of hyperbola can be measured by two arcs of ellipse, memorable discovery, that reduces to only arcs of ellipse all those integrals that until then needed arcs of both curves to be expressed. Finally, Lagrange gave a general method to reduce by successive transformations the integral $\int Pdx/R$ to a similar one, but easier to find by approximation.

Legendre observed that the listed results were the main discoveries in the theory of the new trasncendents when he published his first memoir on the topic [44]. Believing that this matter deserved to be treated in a more systematic and in-depth way with respect what done in the memoir of 1792, on p. 5 he wrote, “Now, resuming this research after a long time, I succeeded to perfect this theory and to give it greater development”.

One of the great merits of Legendre’s investigations was to show that all integrals enclosed in the formula $\int Pdx/R$ can be reduced to three fixed canonical types, which he wrote in trigonometrical form:

$$F(\varphi) = \int_0^{\varphi} \frac{d\varphi}{\Delta}$$

$$E(\varphi) = \int_0^{\varphi} \Delta d\varphi$$

$$\Pi(\varphi) = \int_0^{\varphi} \frac{d\varphi}{(1 + n\sin^2\varphi)\Delta}$$

where $\Delta = \sqrt{1 - c^2 \sin^2 \varphi}$. Accordingly his memoir of 1792, he called $\varphi$ the *amplitude*, $c$ the *modulus* and $n$ the *parameter*; moreover, he put $b = \sqrt{1 - c^2}$ and called it the *supplementary modulus*.

At least to our knowledge it seems that Legendre did not analyze the singularities of the three species of integrals, and therefore he missed to detect the intrinsic nature which distinguishes the three canonical functions $F, E, \Pi$.

Afterwards, Legendre compared the integrals of each of the three species among them; and in doing this he discovered the addition theorems for $F, E, \Pi$.

[14] Here Legendre meant the case in which $R = \sqrt{\alpha + \beta x + \gamma x^2 + \delta x^3 + \epsilon x^4}$.

For the first (p. 21) he showed that: *If*

$$F(c,\varphi)+F(c,\psi)=F(c,\mu)$$

*then* $\mu$ *is an algebraic function of* $\varphi,\psi$.

Three pages below he commented:

> Since we know that the amplitude of the sum, or of the difference, of two given functions is an algebraic expression of the amplitudes of the summands, it is clear that we can algebraically find a function multiple of a given one; and, in general, we can solve the multiplication and division of the elliptic functions [elliptic integrals] of the first species, the same problem that has been solved for the arcs of circles.

Afterwards (p.36), he observed "Since the function $F$ can be multiplied and divided at will, this property gives a very simple method for approximate calculation". He illustrated this method and then applied it to the case of the lemniscate.

On page 41, Legendre began the comparison of elliptic functions of second species, and in a few pages he arrived to the addition theorem for the function $E$: *If*

$$E(c,\varphi)+E(c,\phi)=E(c,\mu)+A(\varphi,\psi)$$

*then* $A(\varphi,\psi)$ *is an algebraic function of the arguments.*

He considered the arc of hyperbola, developed the functions $E,F$ into series, compared among them the function of third species, and proved the addition theorem for them, that he stated as follows: *If*

$$\Pi(c,\varphi)+\Pi(c,\phi)=\Pi(c,\mu)+T(\varphi,\psi)$$

*then* $T(\varphi,\psi)$ *is a trigonometrical or logarithmic function of the arguments.*

We cannot help but imagine that these theorems were of stimulus to Abel, and driven him towards his more general addition theorem (see Sect. 6).

Legendre went on continuing the comparison among elliptic functions, and accomplished this he wrote (p. 81):

> Up to this point we have compared elliptic functions of first species having the same modulus, which can be thought of as forming different arcs of the same curve; and this comparison has been extended to functions of the second and third species. Now we will show that, by means of a very simple law, an infinite sequence of elliptic functions of the first species which differ from each other for both modulus and amplitude can be formed, and also have the interesting property of being in constant ratios among them.

Legendre considered two elliptic integrals of first species $F(c,\varphi)$ and $F'(c',\varphi')$, and showed that if $c'=2\sqrt{c}/(1+c)$ and $\sin\big(2\varphi'-\varphi\big)=c\sin\varphi$, then

$$F'(c',\varphi')=\frac{1+c}{2}F'(c,\varphi)$$

Afterwards he showed how by iterating such a transformation an infinite sequence of modules $c, c', c'', c''', \ldots$ it could constructed with

$$c' = \frac{2\sqrt{c}}{1+c}, \quad c'' = \frac{2\sqrt{c'}}{1+c'}, \quad c''' = \frac{2\sqrt{c''}}{1+c''}, \ldots$$

and consequently an infinite sequence of elliptic integrals $F(c,\varphi)$, $F(c',\varphi')$, $F(c'',\varphi''), \ldots$, such that any two of them are always in a constant ratio. The infinite series of modules, together with that of complementary $b, b', b'', \ldots$ allowed Legendre to develop an effective method for approximate computation of elliptic integrals. He had put Landen's transformation to good use.

We do not go any further with our quick foray into the first volume of the *Exercises*, this would take us too far.

Years later Legendre followed the first volume with two more. They appeared under the title *Exercices de calcul intégral sur divers ordres des transcendantes et sur le quadratures* in 1817 and 1819 respectively; the second one was to definite integrals and integration by series, and the third one to the construction of elliptic tables.

### 5.3. The first volume of the *Traité*.

Legendre continued to work tirelessly, completing and perfecting the material of the *Exercices*, and, in 1825, he published the first volume of his *Traité des fonctions elliptiques et des intégrales Eulériennes*. In the *Avertissement*, Legendre started by saying:

> The more extended and at the same time the more important part of the work the Author has published under the name of *Execices de calcul intégral*, is, as known, that which deals with elliptic functions, their application to various geometrical and mechanical problems, and the construction of tables useful for the use of these functions. This part, and that concerning the definite integrals, to which the Author has given the name of *Eulerian integrals*, are reproduced in this new treatise with many additions aiming at to perfect the theory of the transcendents and to extend their applications.

But he also added new results, which were included in Chapters 28-30, and chiefly in Chapter 31 where he expounded the construction of a new scale of modules.

Legendre considered two elliptic integrals of first species, $F(c,\varphi)$ and $F(\alpha,\omega)$, and showed that the constants $m, h, k$, $0 < m < 3$, in

$$\sin\omega = \frac{\cos\varphi(m + h\sin^2\varphi)}{1 + k\sin^2\varphi} \tag{5.1}$$

can be determined so that by performing this transformation one has

$$F(\alpha,\omega) = mF(c,\varphi)$$

Then, on p. 226, Legendre remarked “This allows the construction of a new scale of modules, essentially different from that already known, but with similar properties for multiplying at infinity the comparison of the functions $F$”.

In an *Addendum* to Chapter 31 (p. 325-328) he showed how the two scales allowed the formation of "a sort of analytic checkerboard" whose squares correspond to infinitely iterated transformations that the elliptic functions of the first type can undergo without ceasing to be similar to itself.

For what follows it is useful to observe that in the transformation (5.1) the numerator is of degree 3 in $\sin\varphi$ and does not contain even powers, while the denominator is of degree 2 and does contain odd powers. Although Legendre did not address the general question of finding all possible transformations in which the new variable is a rational function of arbitrary degree of the first one, this was an important step toward Jacobi’s general theory of transformation. We shall return on this topic in the next section.

It is worth remarking that Legendre made some changes in the historical introduction, with respect to that of the *Exercices*. The more significant were the deletion of certain passages concerning Landen and Lagrange, and the addition of others concerning Maclaurin and d’Alembert. Probably as observed in [58, 389], he wanted to erase the traces of the elaboration process, begun in 1786, from the culminating text of his forty years work on elliptic integrals.

The second volume of Legendre’s *Traité*, printed in 1826, contained the construction of elliptic tables, a treatise on Eulerian integrals i.e, definite ones, and an Appendix mainly devoted to quadratures; substantially the material of the second and third volumes of the *Exercices*. Soon after the publication this volume things changed dramatically in the panorama of the new transcendents, as the breakthrough works of Abel and Jacobi entered the scene. As we shall see in the next section, Legendre immediately understood the great value of the works of the two young geometers, and wanted to enrich his own with part of their achievements. To this end Legendre published three supplements, which later he collected to form the third volume of his treatise.

## 6. From an era to another, from integrals to functions

Niels Henrik Abel (1802-1829) entered the University of Christiania (Oslo) in 1821, probably already with a great knowledge in mathematics as his teacher at the Cathedral School, B. M. Holmboe, had taught him all he could. Abel went further studying by himself works of Newton, Euler, Lagrange, Gauss and Legendre, borrowed from the university library [59].

His first research interest concerned algebraic equations, but he must soon have turned towards elliptic transcendents. In fact, in the summer of 1823, when he was in Copenhagen visiting the local university, Abel showed to Professor Ferdinand Deger some of his early works among which possibly one concerning the elliptic transcendents, because in a letter addressed to Holmboe from that city he referred to “that little Treatment, that You must remember, dealt with the inverse Functions of Elliptic Transcendents” [59, 295]. An undated manuscript, but most likely dating back to the years 1823-24, of about one hundred and fifty in-folio pages titled *Théorie des transcendates elliptiques* was found after Abel death among his papers.[15] In this work chapters devoted to the reduction of elliptic integrals, their comparison, etc. are found, with several references to Legendre’s first volume of his *Exercices*. This witness to an in-depth study of Legendre’s work.

In the Autumn of 1825, funded by the university, Abel left Christiania for a long travel of study through Europe that led him to Berlin, Vienna, Geneva, and finally to Paris, where he arrived in June 1826. The scope of the travel was to make his name known to the scientific European milieu, and above all to present personally to the *Académie des Sciences* an important memoir on an “extended class of transcendental functions”, he had been working on before his departure from Christiania. The manuscript that he submitted later became known as the “Parisian memoir”. In it Abel proved the addition theorem bearing his name. We state it in its original form:

Abel addition theorem. *If one has several functions whose derivatives are roots of the same algebraic equation, whose coefficients are rational functions of the same variable, it is always possible to express the sum of any number of such functions by an algebraic and a logarithmic function, provided that a certain number of algebraic relations among the variables is established.*

Then he pointed out:“The number of these relations does not depend on the number of functions but only on the nature of the particular functions that are considered. Thus, for instance, for elliptic functions this number is 1, for the functions whose derivatives contain only square roots of a polynomial of degree 5 or 6, is 2, and so on”.

In modern terms all this is translated as follows (see [38, 421]): *Let $y(x)$ be an algebraic function, $p$ its genus, and $\psi x$ an associated Abelian integral. Let $\alpha$ be a positive integer, and $b_1, \dots, b_\alpha$ rational numbers. Then*

$$h_1 \psi x_1 + \cdots + h_\alpha \psi x_\alpha = v + \psi x_1' + \cdots + \psi x_p'$$

*where $v$ is an elementary function of $x_1, \dots, x_\alpha$ and where $x_1, \dots, x_p$ are algebraic functions of them.*

Legendre and Cauchy were charged to evaluate Abel’s memoir, but for one reason or another they delayed their report. Abel, unable to wait any longer, in December left Paris to return home.[16] On the way back he stopped in Berlin. Here, wanting to make his achievements known at least in part, he wrote down the memoir *Recherches sur les fonctions elliptiques*, that he left to August Crelle for publication in his journal.

[15] See [1], vol. 2, pp. 87–188

[16] The “Parisian memoir” was published only in 1841 [4]. For the events connected to this delay and the history of the manuscript see [15].

Presenting his work Abel wrote [2, 101]:

> Since a long time the logarithmic, exponential and circular functions have been the only functions which have attracted the attention of geometers. It is only recently that other functions have been considered. Among these there are those called elliptic functions, both for the beauty of their analytic properties and their applications in various branches of mathematics. The first result on these functions has been given by the immortal *Euler*, who proved that the equation
>
> $$\frac{dx}{\sqrt{\alpha+\beta x+\gamma x^2+\delta x^3+\epsilon x^4}}=\frac{dy}{\sqrt{\alpha+\beta y+\gamma y^2+\delta y^3+\epsilon y^4}}$$
>
> is algebraically integrable. After *Euler*, *Lagrange* added something by giving an elegant theory of transformation of the integral
>
> $$\int\frac{Rdx}{\sqrt{(1-p^2x^2)(1-q^2x^2)}}$$
>
> where $R$ is a rational function of $x$. But the first and the only, if I am not wrong, who has investigated the nature of these functions is Mr. *Legendre*, who, first with a memoir on the elliptic functions,[17] and later in his excellent Exercices de Mathématique,[18] has developed several elegant properties of these functions, and shown their applications. After the publication of this work nothing has been added to the theory of Mr. *Legendre*. I believe that, not without pleasure, further researches on these functions will be seen here.

After having recalled Legendre classification and said that the integral of the first species were those having the most remarkable and simple properties, Abel wrote (p. 102):

> In this memoir I put myself the task of considering the inverse function [of the integral of first kind], that is the function $\varphi\alpha$ determined by the equations
>
> $$\alpha=\int\frac{d\theta}{\sqrt{1-c^2\sin^2\varphi}};\qquad \sin\theta=\varphi\alpha=x$$
>
> The first gives
>
> $$d\theta\sqrt{1-\sin^2\theta}=d(\varphi\alpha)=dx$$
>
> then
>
> $$\alpha=\int_0\frac{dx}{\sqrt{(1-x^2)(1-c^2x^2)}}$$
>
> Mr. *Legendre* supposed $c^2$ positive, but I observe that formulas become simpler supposing $c^2$ negative, equal to $-c^2$. Similarly, for more simmetry I write $\sqrt{1-c^2x^2}$ instead of $\sqrt{1-x^2}$, so that the function $\varphi\alpha=x$ will be given by the equation
>
> $$\alpha=\int_0\frac{dx}{\sqrt{(1-c^2x^2)(1+e^2x^2)}}$$

[17] Abel was referring to [46].

[18] Abel clearly intended [47].

or

$$\varphi'\alpha = \sqrt{(1-c^2x^2)(1+e^2x^2)}$$

For further simplicity I also introduce the two functions of $\alpha$

$$f\alpha = \sqrt{1-c^2x^2}; \qquad F\alpha = \sqrt{1-e^2x^2}$$

Abel went on by putting

$$\frac{\omega}{2} := \int_0^{1/c} \frac{dx}{\sqrt{(1-c^2x^2)(1+e^2x^2)}},$$

and observed that the function $\varphi\alpha$ is increasing and positive in $[0,\omega/2]$; and that $\varphi(0)=0$ and $\varphi(\omega/2)=1/c$. Since $\alpha$ changes sign putting $-x$ in place of $x$, the same happens for $\varphi\alpha$ with respect to $\alpha$, so $\varphi(-\alpha)=-\varphi\alpha$, i.e. $\varphi\alpha$ is an odd function defined in $[-\omega/2,\omega/2]$. Moreover, he remarked that if in $\alpha$ one replaces $x$ by $ix$ we have

$$\int_0^{ix} \frac{idx}{\sqrt{(1+c^2x^2)(1-e^2x^2)}}.$$

Hence

$$\beta := \int_0^{x} \frac{dx}{\sqrt{(1+c^2x^2)(1-e^2x^2)}}$$

is an increasing positive function of $x$ in $[0,1/e]$. So, by putting

$$\frac{\widetilde{\omega}}{2} := \int_0^{1/e} \frac{dx}{\sqrt{(1+c^2x^2)(1-e^2x^2)}}$$

it is readily seen that $x$ is an increasing positive function of $\beta$ in $[0,\widetilde{\omega}/2]$. Legendre defined $\varphi(i\beta) := ix$, with $\varphi(i\widetilde{\omega}/2) = i/e$. Therefore the function $\varphi\alpha$ is defined for all values in the set $[-\omega/2,\omega/2] \cup [-i\widetilde{\omega}/2, i\widetilde{\omega}/2]$. Then he observed that the same is for $f\alpha$ and $F\alpha$.

Afterwards, Abel remarked that

$$\varphi'\alpha = \sqrt{(1-c^2\varphi^2\alpha)(1+e^2\varphi^2\alpha)} = f\alpha \cdot F\alpha$$

and that deriving $f^2\alpha$ and $F^2\alpha$ on gets

$$f\alpha \cdot f'\alpha = -c^2\varphi\alpha \cdot \varphi'\alpha, \qquad F\alpha \cdot F'\alpha = e^2\varphi\alpha \cdot \varphi'\alpha$$

Hence, by substituting in these last equations $\varphi'\alpha$ by the expression previously obtained for it, we have

$$f'\alpha = -c^2\varphi\alpha \cdot F\alpha, \qquad F'\alpha = e^2\varphi\alpha \cdot f\alpha$$

Then Abel claimed that if $\alpha$ and $\beta$ are indeterminate one has

$$\varphi(\alpha+\beta) = \frac{\varphi\alpha \cdot f\beta \cdot F\beta + \varphi\beta \cdot f\alpha \cdot F\alpha}{1 + c^2e^2\varphi\alpha^2 \cdot \varphi\beta^2}$$

and on page 105 he wrote, "this formula can be easily deduced from the known properties of elliptic functions (*Legendre*, Exercices de calcul intégral); but we can also verify it as follows". In this context we only recall that for doing that he differentiated with respect to $\alpha$ and then used the previous formulas. What is worth noting here is that referring to Legendre's *Exercices* he meant the additions theorem for elliptic integrals of the first species.

The above suggested him to put

$$\varphi(\alpha+i\beta)=\frac{\varphi(\alpha)f(i\beta)F(i\beta)+\varphi(i\beta)f(\alpha)F(\alpha)}{1+c^2e^2\varphi(\alpha)^2\varphi(i\beta)^2}. \tag{6.1}$$

It follows that the function $\varphi$ is defined in $[-\omega/2,\omega/2]\times[-i\widetilde{\omega}/2,i\widetilde{\omega}/2]\subset\mathbb{C}$.

With a similar procedure he also extended the functions $f$ and $F$ over this set.

Summing up Abel had defined the functions $\varphi$, $f$ and $F$ over $\mathbb{C}\backslash Z$, where $Z=\left\{(m+\frac{1}{2})\omega+(n+\frac{1}{2})i\widetilde{\omega}\right\}_{m,n\in\mathbb{Z}}$ constitutes the set of points of "infinity" of these functions.

Moreover Abel proved that

$$\frac{\partial\varphi}{\partial\alpha}=-i\frac{\partial\varphi}{\partial\beta}$$

which means that $\varphi$ is holomorphic in $\mathbb{C}\backslash Z$; but, of course, he did not use this terminology.

In his work Abel went on by expounding several properties of the function $\varphi\alpha$, $f\alpha$ and $F\alpha$.

For instance, he showed that these functions satisfy an algebraic addition theorem; that is, their values for $\alpha+\beta$ are algebraic expression of the values for $\alpha$ and $\beta$ of the same function. Thus Abel accomplished what Johann Bernoulli perhaps had foreseen.

Then the addition theorem allowed Abel to get formulas for multiplication and division of the argument; these express $\varphi(n\alpha)$, $f(n\alpha)$ and $F(n\alpha)$ rationally in terms of $x=\varphi\alpha$, $y=f\alpha$ and $z=F\alpha$. In particular, he showed that if $n$ is odd then

$$\varphi(n\alpha)=x\frac{P_n(x)}{Q_n(x)}$$

where $P_n$ and $Q_n$ are even polynomials of degree $n^2-1$; while if $n$ is even one has

$$\varphi(n\alpha)=xyz\frac{P_n(x)}{Q_n(x)}$$

where $P_n$ and $Q_n$ are also polynomials but the first of degree $n^2-4$ and the second of degree $n^2$. From these formulas it follows that the problem of dividing the argument by a positive integer $n$, that is to find the value of the function for $\alpha$ knowing that for $n\alpha$, requires to solve an equation of degree $n^2$ if $n$ is odd and of degree $2n^2$ if $n$ is even. This fact was noticed by Legendre as early as 1792,[19] but it was Abel to find an explanation in the double periodicity.

We do not longer insist on this matter, referring to [33], or directly to Abel's paper [2], all those eager of more information.

In June of 1827, two months before Abel's *Recherches* appeared in print, Jacobi wrote a letter to H. C. Schumacher, editor of *Astronomische Nach-richten*, in which he announced his discoveries:

> Please Sir! insert in your journal the notices about elliptic transcendents that I have the honour of sending you, because I flatter myself that I have made some very interesting discoveries in this theory which I submit to the judgement of geometers. The integrals of the form $\int \frac{d\varphi}{\sqrt{1-cc\sin\varphi^2}}$ according to the diversity of the modul $c$ belong to different transcendets. Only one system of modules is known that can be reduced to one another, and Mr. *Legendre* in his Exercices even said that this is the only one. But actually there are as many of these systems as there are of prime numbers; that is, there are infinitely many such systems independent one to another, and that already known corresponds to the prime number 2.

It is clear that, as in a footnote the Editor pointed out, at this time Jacobi had studied Legendre's *Exercices* but had not read the *Traité* yet.

Carl Gustav Jacob Jacobi (1804-1851) enrolled at the University of Berlin in 1821 where he graduated in 1825. He studied the works of Euler, Lagrange, Laplace, and Legendre's *Exercices* from which he learned the basics of the investigations carried out on elliptic transcendents [34]. As he wrote in [35], Jacobi had to take a more specific interest in elliptic transcendents by reading Gauss's celebrated memoir of celestial mechanics [26], in which Gauss had shown that by means of a certain transformation a given elliptic differential could be reduced to a simpler form. Impressed by this, Jacobi started to study the problem of rational transformation of elliptic integrals of the first kind with the utmost generality.

In the aforementioned communication to Schumacher, Jacobi outlined the terms of his research and stated two theorems. For any prime number $n$, Jacobi considered the transformation $\sin\varphi = u/v$ where $u, v$ are polynomials in $\sin\psi$, $u$ containing only odd powers of $\sin\phi$ up to degree $n$ and $v$ containing only even powers up to $n-1$, and stated that it is always possible to find the coefficients of such a transformation so that

$$\int \frac{d\varphi}{\sqrt{1-c^2\sin^2\varphi}} = m\int \frac{d\psi}{\sqrt{1-k^2\sin^2\psi}}$$

He also claimed that each of these substitutions gave rise to a new system of modules; and that the same could be done for any positive integer, pointing out that its

[19] We recall that for trigonometric functions it requires to solve an equation of degree $n$ and $2n$ respectively.

factorization would have produced a combination of such systems. Moreover, he wrote that by a further similar substitution one could express $\sin\psi$ by $\sin\theta$ so to get

$$\int \frac{d\varphi}{\sqrt{1-c^2\sin^2\varphi}} = n\int \frac{d\theta}{\sqrt{1-c^2\sin^2\theta}}$$

Then, the composition of the two transformation would express $\sin\varphi$ as a rational function of $\sin\theta$ whose numerator contains only odd powers of $\sin\theta$ up degree $n^2$, and the denominator only even powers of $\sin\theta$ up to $n^2-1$. Afterwards, Jacobi stated two theorems concerning the cases $n = 3,5$ respectively. For $n = 3$ the transformation was

$$\sin\varphi = \frac{\sin\psi(ac + (\frac{a-c}{2})^2\sin^2\psi)}{c^2 + (\frac{a-c}{2})(\frac{u+3c}{2})\sin^2\psi}$$

whose similarity with (5.1) will not escape the reader.

In the second letter to Schumacher, dated August 2, Jacobi gave the general analytic formulas for the transformation of any order $n$.

Two days later Jacobi also wrote to Legendre. In his letter Jacobi recognized to the French old mathematician the great merit of having found and elevated to high degree the theory of "elliptic functions", and also his intellectual lineage from him; then he shortly presented his discoveries. Legendre answered on November 30th: he said to be already informed about them for having read n. 123 of *Astronomische Nachrichten*; that Jacobi's first theorem coincided with that he had given in the *Traité*; and praised Jacobi for the second. An epistolary exchange started between the two.[20].

Legendre and the mathematical community were eager for more detailed proofs and insights. In November Jacobi sent to Schumacher the paper *Demonstratio theorematis ad theoriam functionum ellipticarum spectantis* [Jacobi 1827b], in which he provided a proof of the main result he had previously announced.

Jacobi opened declaring his scope:[21] to find all rational functions $y = U/V$, where $U, V$ are polynomials of degree $p$, $m$ respectively, with $m \le p$, such that

$$\frac{dy}{\sqrt{A' + B'y + C'y^2 + D'y^3 + E'x^4}} = \frac{a\,dx}{\sqrt{A + Bx + Cx^2 + Dx^3 + Ex^4}} \tag{6.2}$$

where $a$ may depends on $A, B, C, D; E$ but not on $x$.

After having recalled the transformation used by Legendre, by a count of constant Jacobi showed that wanting to be free to have $U$, $V$ of degree 1 such a transformation is possible only if $m = p, p-1$, and also that the case $m = p-1$ could

[20]The correspondence in its whole is published in the first volume of Jacobi's *Gesammelte Werke*; editing it C. W. Borchardt wrote,"This in one of the most remarkable in the whole history of exact sciences"; in fact, it allows to reconstruct the evolution of Jacobi' ideas about the theory of transformation, and the history of their interrelation with those of Abel

[21]In the following we adopt Jacobi's notation.

be brought back to the case $m = p$ by a transformation of type $x = (\alpha + \beta x)/(1 + \gamma x)$. So he had proved the following: *For every p the form*

$$y = \frac{a + a'x + a''x^2 + \cdots + a^{(p)}x^p}{b + b'x + b''x^2 + \cdots + b^{(p)}x^{pt}}$$

*can be determined so that equation* (6.2) *is satisfied*. He called this the *fundamental principle* of the theory of transformation, and $p$ the *degree*, or *order*, of the transformation.[22].

It is to these works of Abel and Jacobi that Legendre referred in the first supplement to his treatise.

When Abel heard about Jacobi's work he felt somewhat cheated and hastened to publish [3], in which Abel claimed priority over Jacobis's discovery.

In 1829, the same year of Abel death, Jacobi published his main work on elliptic functions the *Fundamenta nova functionum ellipticarum* [37]. In the *Proemium* he started by saying:

> Almost two years ago, when I decided to examine the theory of elliptic functions in more detail, I ran into some deep questions which seemed to create a new shape for that theory being capable to advance the entire analysis significantly. Having brought them to an happy outcome, and because of the difficulty of the matter which was harder than expected, I communicated the main results, first briefly and without proofs and soon later, since they were seen suspiciously and proofs desired, I communicated them to the geometers together with new results. At the same time I urged to make public the complete system of questions I had undertaken. In order to satisfy at least in part this desire, I decided to publish the foundations on which my investigations were built. I now commend these foundations of the new theory to the indulgence of geometers.

In the *Fundamenta nova* the theory of transformation was presented in a new and extended way with respect to [36]. Surely, the reading of Abel works, as the epistolary exchange with Legendre witnesses, much influenced Jacobi. In this process of reformulation the inversion of elliptic integrals of first species had a central role.

Jacobi started with

$$u = \int_0^{\varphi} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2\varphi}} = \int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}};$$

as usual he called $\varphi$ the *amplitude*, defined the inverse function $x = \sin \mathrm{am}\, u$ and also the functions

$$\cos \mathrm{am}(u) = \sqrt{1 - \sin^2 \mathrm{am}\, u}$$

$$\Delta \mathrm{am}\, u = \sqrt{1 - k^2 \sin^2 \mathrm{am}\, u}$$

[22] See [12] for some details

which correspond to Abel's functions $\varphi$, $f$ and $F$, respectively.[23]

To extend his functions to the complex field, Jacobi used the transformation $\sin\mathrm{am}\,u = i\tan\mathrm{am}\,v$, that he called "Abel's fundamental theorem", and which gives

$$\frac{d\mathrm{am}\,u}{\sqrt{1-k^2\sin^2\mathrm{am}\,u}} = \frac{d\mathrm{am}\,v}{\sqrt{1-(k')^2\sin^2\mathrm{am}\,v}}$$

where $k' = 1-k^2$.

Jacobi went on to study the problem of transformation of elliptic functions, multiplication and division of the arguments, the relations between elliptic integrals of second and third kinds, and so on. The most striking consequence of the new systematization of the transformation theory was the representation of the functions $\sin\mathrm{am}(u)$ and $\cos\mathrm{am}(u)$ as ratios of Jacobi's *fundamental theta functions*, $\Theta(u)$ and $H(u)$:

$$\sin\mathrm{am}(u) = \frac{H(x)}{\sqrt{k}\Theta(u)}; \qquad \cos\mathrm{am}(u) = \sqrt{\frac{k'}{k}}\cdot\frac{H(u+K)}{\Theta(u)}$$

where $K$ is equal to the complete integral $\int_0^1 \frac{dx}{\sqrt{1-k^2\sin^2\varphi}}$, see [53].

## Conclusion

There are no doubts that the works of Abel and Jacobi marked a turning point in the theory of elliptic integrals and functions, and the entry into a new era of development and fruitful applications of it in the realms of analysis, algebraic geometry and number theory. In the previous sections, by explaining in quite detail certain salient aspects of the theory of elliptic integrals and in particular of Legendre's forty-year work on the topic, we believe we have shed even more light on the genesis of Abel and Jacobi's research on elliptic functions. Finally, by illustrating in some detail the very first results obtained by Abel and Jacobi on elliptic integrals, we have shown how Legendre's *Exercices* was fundamental in their education as students; and this is especially true for Abel, a fact not yet highlighted enough.

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[2] ABEL N. H., *Recherches sur les fonctions elliptiques*, J. für die reine und ang. Mathematik, 2 (1827), 101–197; also in [1, I] pp. 263–388.

[23] Jacobi's notation was later simplified by C. Gudermann (1798-1851) who, in [30, 14–15], for these functions introduced the symbols *snu*, *cnu* and *dnu*, respectively.

[3] ABEL N. H. , *Solution d'une problème générale concernant la transformation des fonctions elliptiques*, Astronomische Nachrichten, 6 (1828), 381–388; also in [1, I] pp. 403–428

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**AMS Subject Classification:** Primary 01A45, 01A50, 01A55

**Keywords:** Elliptic integrals, Elliptic functions, Integration by arcs, Addition theorem for elliptic integrals, Multiplication and Division theory, Transformations theory
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*Lavoro pervenuto in redazione il 03.04.2024.*