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A QUESTION ON EFFECTIVE STRICTLY NEF DIVISORS (WITH AN APPENDIX BY ANDREAS HÖRING)

Abstract. We introduce and motivate the following question: Is every effective strictly nef Cartier divisor on a projective variety big? In the appendix, Andreas Höring produces a counterexample, thus providing a negative answer.

1. Introduction

Let X be a complex projective variety of dimension n. A Cartier divisor D on X is called *strictly nef* if it has strictly positive intersection product with every curve on X. Every ample divisor is indeed strictly nef, but after the classical examples by Mumford and Ramanujam (see [7], Chapter I., Examples 10.6 and 10.8) it is well known that the converse does not hold. On the other hand, a deep conjecture by Serrano predicts that every strictly nef divisor on a projective manifold becomes ample after a suitable deformation in the direction of the canonical divisor K_X:

CONJECTURE 1. ([9]) *If D is a strictly nef divisor on a projective manifold X then $K_X + tD$ is ample for every $t > n+1$.*

Serrano's Conjecture 1 holds for surfaces (see [9]), for threefolds with the unique possible exception of Calabi-Yau's with $D.c_2 = 0$ (see [9] and [2]), for K-trivial fourfolds (see [7]), and for projective manifolds of Kodaira dimension at least $n-2$ (see [2]). Otherwise, Conjecture 1 is still widely open.

A weaker version, involving only effective strictly nef divisors, was independently formulated by Beltrametti and Sommese in [1], p. 15:

CONJECTURE 2. ([1]) *Let D be an effective strictly nef divisor on a projective manifold X. If $D - K_X$ is nef then D is ample.*

On the other hand, if the strictly nef divisor D is also big, then Conjecture 1 holds for D, just by applying [9], Lemma 1.3:

PROPOSITION 1. *If D is a big strictly nef divisor on a projective manifold X then $K_X + tD$ is ample for every $t > n+1$.*

Furthermore, if $D - K_X$ is nef then from the ampleness of $K_X + tD$ it follows that also D is ample, hence Conjecture 2 holds for big strictly nef divisors as well:

PROPOSITION 2. *If D is a big strictly nef divisor on a projective manifold X and $D - K_X$ is nef then D is ample.*

Finally, also the singular version of Conjecture 1 (see [2], Conjecture 1.3, and [8], Question 1.4) holds for big strictly nef $\mathbb{Q}$-Cartier divisors. Namely, by applying [8], Lemma 5.2 and Lemma 5.3, we deduce:

PROPOSITION 3. *If (X, Δ) is a projective klt pair of dimension n and D is a big strictly nef $\mathbb{Q}$-Cartier divisor on X then $K_X + \Delta + tD$ is ample for every $t >> 0$.*

From this point of view, it is remarkable that all examples known so far of strictly nef divisors (see in particular [1]) have either negative or maximal Iitaka dimension. This experimental observation suggests the following question:

QUESTION 1. *Is every effective strictly nef Cartier divisor on a projective variety big?*

Even though this is trivially true for surfaces, in higher dimension the answer turns out to be far less obvious, so it seems wise to adopt a fully agnostic attitude.

As a starting point, we recall that from the case of surfaces it follows a partial positive result in any dimension (see [3], Proposition 22): if D is a strictly nef Cartier divisor on a projective variety of dimension n with Iitaka dimension $\kappa(D) \geq n-2$ then D is big. Hence the first instance to be addressed is the one of a strictly nef divisor D with $\kappa(D) = 0$ on a threefold. We point out that, once this case were ruled out in arbitrary dimension, then the whole picture would become clear. Namely, we formulate the following a priori weaker question:

QUESTION 2. *Does every effective strictly nef Cartier divisor D on a projective variety X satisfy $h^0(X, mD) \geq 2$ for some $m \geq 1$?*

We show that an affirmative answer to Question 2 would imply an affirmative answer to Question 1:

THEOREM 1. *Assume that every effective strictly nef Cartier divisor D on every projective variety X of dimension* $\dim(X) \leq n$ *satisfies $h^0(X, mD) \geq 2$ for some $m \geq 1$. Then every effective strictly nef Cartier divisor on a projective variety of dimension n is big.*

We also present an unconditional result pointing towards the same direction:

THEOREM 2. *If D is an effective strictly nef Cartier divisor on a projective variety and the schematic base locus of $|mD|$ becomes constant for large m then D is big.*

In the opposite direction, by closely following [7], Chapter I., Example 10.8 and [1], Example 1.2, we adapt Ramanujam construction and define an inductive procedure to build a strictly nef divisor which is effective but not big (see Example 1). This approach works by induction on the dimension of the ambient projective manifold and what is needed to obtain a negative answer to Question 1 is indeed the base of the induction. This is provided in Appendix 3 by Andreas Höring, hence both Question 1 and Question 2 have a negative answer in any dimension $n \geq 3$.

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2. The proofs

Proof of Theorem 1. We argue by induction on n, the case $n = 1$ being obvious. If D is an effective strictly nef divisor on a projective variety V of dimension n, by assumption we have $h^0(V, mD) \geq 2$ for some $m \geq 1$. Let $E = \sum a_i E_i$ be an effective divisor linearly equivalent to mD. Since $h^0(V, mD) \geq 2$ there exists i such that D restricts to an effective strictly nef Cartier divisor on E_i. From the inductive assumption applied to the projective variety E_i of dimension $n-1$ we obtain that the restriction of D to E_i is big. On the other hand, if D is not big then

$$0 = D^n = mD^n = D^{n-1}.E = \sum a_i D^{n-1}.E_i$$

Since D is nef and E is effective it follows that $D^{n-1}.E_i = 0$ for every i, in particular the restriction of D to E_i is not big. This contradiction ends the proof.

□

Proof of Theorem 2. The complement U of the support of the effective strictly nef Cartier divisor D does not contain complete curves, hence from [7], Chapter II., Theorem 5.1 and the Remarks following its statement, we deduce that U is affine. Now the claim is a direct consequence of Goodman's criterion ([7], Chapter II., Theorem 6.1), see for instance the statement of Theorem 3.1 in [10]: indeed, it follows easily from the remarks after the statement of Theorem 2.1 on p. 803 and at the beginning of the proof of Theorem 3.1 on p. 808.

□

EXAMPLE 1. Let Y be a projective manifold of dimension $n-1 \geq 3$ with an effective strictly nef divisor D such that $D^{n-1} = 0$.

Define $X := \mathbb{P}(\mathcal{O}_Y(D) \oplus \mathcal{O}_Y)$. If X_0 is the zero section of the projection $\pi : X \to Y$, then X_0 corresponds to the tautological line bundle on X and by arguing as in [1], Example 1.2, we compute

$$X_0^i = D_{X_0}^{i-1} \tag{2.1}$$

for every i with $2 \leq i \leq n$.

Define $E := X_0 + \pi^* D$. From (2.1) with $i = 2$ we deduce

$$E_{X_0} = (X_0 + \pi^* D)_{X_0} = 2D_{X_0}. \tag{2.2}$$

Then the following holds:

(i) E is effective because D is effective.

(ii) E is strictly nef. Indeed, let C be a curve on X.

If C is a fibre of π, then by the projection formula we have $E.C = X_0.C + \pi^* D.C = X_0.C + D.\pi_* C = 1 + 0 > 0$.

If $C \subset X_0$, then by (2.2) we have $E.C = E_{X_0}.C_{X_0} = 2(D.C)_{X_0} > 0$.

If $C \not\subset X_0$ and $\pi(C)$ is a curve in Y, then by the projection formula we have $E.C = X_0.C + \pi^* D.C = X_0.C + D.\pi_* C > 0$ since $X_0.C \geq 0$ and $D.\pi_* C > 0$.

(iii) E is not big. Indeed, we have $E^n = (X_0 + \pi^* D)^n = 0$ because $\pi^* D^n = 0$, $X_0.(\pi^* D)^{n-1} = D^{n-1}_{X_0} = 0$, and $X_0^i.(\pi^* D)^{n-i} = 0$ for every i with $2 \leq i \leq n$ by (2.1).

3. Appendix by Andreas Höring

Let $\pi : \mathbb{P}(V) \to C$ be a Mumford example, i.e. C is a smooth projective curve of genus $g > 1$ and V is a rank two vector bundle on C such that $c_1(V) = 0$ and

$$H^0(C, S^m V) = 0 \qquad \forall\ m \in \mathbb{N}.$$

Then it is known [6, Ex.1.5.2] that the tautological class $c_1(\mathscr{O}_{\mathbb{P}(V)}(1))$ is strictly nef, but not big. Observe that by Serre duality and Riemann-Roch

$$h^1(C, V^*) = h^0(C, K_C \otimes V) \geq \chi(C, K_C \otimes V) = c_1(K_C \otimes V) + 2\chi(\mathscr{O}_C) = 2(g-1) > 0,$$

so there exists an extension

$$0 \to \mathscr{O}_C \to W \to V \to 0$$

such that the extension class $\eta \in H^1(C, \mathscr{O}_C \otimes V^*)$ is not zero. Let $p : X := \mathbb{P}(W) \to C$ be the projectivisation, and observe that it contains

$$Y := \mathbb{P}(V) \subset \mathbb{P}(W) = X$$

as a prime divisor such that $[Y] = c_1(\mathscr{O}_{\mathbb{P}(W)}(1))$.

PROPOSITION 4. *The prime divisor $Y \subset X$ is effective, strictly nef, but not big.*

Proof. Let $C \subset X$ be an irreducible curve. If $C \subset Y$ then

$$Y \cdot C = c_1(\mathscr{O}_{\mathbb{P}(W)}(1))|_Y \cdot C = c_1(\mathscr{O}_{\mathbb{P}(V)}(1)) \cdot C > 0$$

since $c_1(\mathscr{O}_{\mathbb{P}(V)}(1))$ is strictly nef. Thus Y is nef and strictly nef on all curves contained in its support. Clearly Y is not big, since

$$Y^3 = c_1(\mathscr{O}_{\mathbb{P}(W)}(1))|_Y^2 = c_1(\mathscr{O}_{\mathbb{P}(V)}(1))^2 = 0.$$

Assume now that there exists an irreducible curve such that $Y \cdot C = 0$. Since $C \not\subset Y$ this implies that $C \subset (X \setminus Y)$. Yet $\eta \neq 0$, so by [5, Lemma 3.9] the complex manifold $X \setminus Y$ contains no positive-dimensional compact subvarieties of positive dimension, a contradiction.

□

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