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SOME REMARKS ON BRAUER CLASSES OF $K3$-TYPE

Abstract. An element in the Brauer group of a general complex projective $K3$ surface S defines a sublattice of the transcendental lattice of S. We consider those elements of prime order for which this sublattice is Hodge-isometric to the transcendental lattice of another K3 surface X. We recall that this defines a finite map between moduli spaces of polarized K3 surfaces and we compute its degree. We show how the Picard lattice of X determines the Picard lattice of S in the case that the Picard number of X is two.

Introduction

Let S be a complex projective $K3$ surface and let $T(S)$ be its transcendental lattice. The Brauer group $Br(S)$ of S can be identified with ([Hu16, §18.1])

$$Br(S) = \mathrm{Hom}(T(S), \mathbb{Q}/\mathbb{Z}) \ .$$

An element α of order p, for a prime number p, in the Brauer group $Br(S)$ then defines a Hodge substructure $T_\alpha(S)$ of index p in the transcendental lattice $T(S)$, it is the kernel of α. The isometry classes of these sublattices were determined in [vG05], [vGK23] for $p = 2$ and in [MSTV17] for $p > 2$. In particular, if the Picard rank of S is one and $Pic(S) = \mathbb{Z}h$ with $h^2 = 2d$, then for any prime number p there is one class whose lattices are isometric to the transcendental lattice of a K3 surface of degree $2p^2d$. We will call these *Brauer classes of K3-type*.

There is a finite map

$$\kappa : \mathcal{M}_{2p^2d} \longrightarrow \mathcal{M}_{2d}$$

between the moduli spaces of polarized K3 surfaces of degree $2p^2d$ and degree $2d$ whose fiber over S, with Picard rank one, consists of all X with $T(X) \cong T_\alpha(S)$ for some $\alpha \in Br(S)_p$ of K3-type. We determine the degree of this map in Proposition 2. Since the Hodge structure $T(X)$ does not determine X uniquely in general due to the presence of Fourier-Mukai partners, this degree is not simply the number of cyclic subgroups in $Br(S)_p$ of K3-type.

In case the Picard rank of X is two, its Picard lattice is determined by two integers b, c and we write $X_{b,c}$ for such a surface. We use the description of κ to determine the Picard lattice of $S_{b,c} := \kappa(X_{b,c})$. In the Picard rank two case it can (and does) happen that $X_{b,c} \cong S_{b,c}$, if so, the determinants of the Picard lattices are the same. This is an easy necessary, but not sufficient, criterion for the existence of an isomorphism.

The equality of the determinants occurs exactly when the transcendental lattice of $X_{b,c}$, which is $T_\alpha(S_{b,c})$, is equal to $T(S_{b,c})$. More precisely, consider a family of K3 surfaces over a disc with special fiber $S_{b,c}$ and general fiber a polarized K3 surface S of degree $2d$ of Picard rank one. Then we can identify $H^2(S_{b,c},\mathbb{Z}) = H^2(S,\mathbb{Z})$ and we have $T(S_{b,c}) \subset T(S)$. Therefore there is a restriction map on the Brauer groups and on their p-torsion subgroups. The kernel of this map is the subgroup of vanishing Brauer classes, any non-zero element in it is called a *vanishing Brauer class* ([GvG24]):

$$\langle \alpha_{van} \rangle = \ker(Br(S)_p \longrightarrow Br(S_{b,c})_p) \ .$$

Let X be a K3 surface of degree $2p^2d$ with Picard rank one and let $S = \kappa(X)$ so that $T(X)$ has index p in $T(S)$. We denote by $\alpha_X \in Br(S)_p$ an element such that $\ker(\alpha_X) = T(X)$. A specialization of S to $S_{b,c}$ then induces a specialization of X to $X_{b,c}$ and we have two cyclic subgroups in $Br(S)_p$, one is $\langle \alpha_X \rangle$ and the other is $\langle \alpha_{van} \rangle$. These two subgroups coincide exactly when α_X is trivial on $T(S_{b,c})$, so when $T_{\alpha_X}(S_{b,c}) = T(S_{b,c})$. In Theorem 2 we make these subgroups explicit.

We intend to use these results to study degree eight K3 surfaces S_β associated to certain cubic fourfolds in [GvG24, Proposition 5.1.4]. In [KS18], Kuznetsov and Shinder study the classes generated by K3 surfaces in the Grothendieck ring of $K_0(\mathrm{Var}/\mathbb{K})[\mathbb{L}^{-1}]$. They use the geometry of the conic bundles associated to a Brauer class of K3-type, in particular in the case of a specialization in which this class vanishes. The very basic results in this paper might give some more insight into these cases.

1. Brauer groups, vanishing classes and invariants

1.2. Brauer classes and B-fields

We recall, following [GvG24], the main definitions but now for the case of an arbitrary prime number p rather than only $p = 2$.

Let S be a $K3$ surface, its transcendental lattice is $T(S) := Pic(S)^\perp$ in $H^2(S,\mathbb{Z})$. The *Brauer group* of S can be identified with (cf. [Hu16, 18.1])

$$Br(S) = \mathrm{Hom}(T(S), \mathbb{Q}/\mathbb{Z}) \ .$$

Since $H^2(S,\mathbb{Z})$ is a selfdual lattice, any such homomorphism α can be defined by an element $B = B_\alpha \in H^2(X,\mathbb{Q})$, called a B-field representative of α:

$$\alpha : T(S) \longrightarrow \mathbb{Q}/\mathbb{Z}, \qquad t \longmapsto B\cdot t \mod \mathbb{Z} \ .$$

Let $Br(S)_p$ be the p-torsion subgroup of $Br(S)$ and let $\alpha \in Br(S)_p$. Then the homomorphism α takes values in $\frac{1}{p}\mathbb{Z}/\mathbb{Z}$. A B-field $B_\alpha \in \frac{1}{p}H^2(S,\mathbb{Z})$ is unique up to $\frac{1}{p}Pic(S) + H^2(S,\mathbb{Z})$:

$$B'_\alpha = B_\alpha + \tfrac{1}{p}D + c, \qquad D \in Pic(S), \quad c \in H^2(S,\mathbb{Z}) \ ,$$

1.3. Brauer classes, sublattices and invariants: the case $p = 2$

In case the Picard rank of S is one, the lattices $T_\alpha(S)$ for $S \in Br(S)_p$ are classified up to isometry by their discriminant groups. This leads to the following classification, for $p = 2$ in Lemma 1 and for $p > 2$ in Lemma 2. The case $p = 2$ is presented in a format similar to the one for $p > 2$.

LEMMA 1. *([vGK23, Theorem 2.3]) Let S be a K3 surface such that $Pic(S) = \mathbb{Z}h$, $h^2 = 2d > 0$. Let $\alpha \in Br(S)_2$, $\alpha \neq 0$, and $B \in \frac{1}{2}H^2(S,\mathbb{Z}) \subset H^2(S,\mathbb{Q})$ a B-field representing α.*

a) In case $2 \not| d$ there are three isomorphism classes of lattices $T_\alpha(S)$.

i) $Bh \equiv 0 \mod \mathbb{Z}$, in this case there is a unique isomorphism class, of order $2^{20} - 1$, with discriminant group $\mathbb{Z}/2d\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$,

ii) $Bh \equiv 1/2 \mod \mathbb{Z}$, $B^2 \equiv 0 \mod \mathbb{Z}$, in this case there is a unique isomorphism class, of order $2^9(2^{10} + 1)$, with discriminant group $\mathbb{Z}/8d\mathbb{Z}$,

iii) $Bh \equiv 1/2 \mod \mathbb{Z}$, $B^2 \equiv \frac{1}{2} \mod \mathbb{Z}$, in this case there is a unique isomorphism class, of order $2^9(2^{10} - 1)$, with discriminant group $\mathbb{Z}/8d\mathbb{Z}$.

b) In case $2|d$ there are three isomorphism classes of lattices $T_\alpha(S)$.

i) $Bh \equiv 0 \mod \mathbb{Z}$, $B^2 \equiv 0 \mod \mathbb{Z}$, in this case there is a unique isomorphism class of lattices, of order $2^9(2^{10} + 1) - 1$, with discriminant group $\mathbb{Z}/2d\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$,

ii) $Bh \equiv 0 \mod \mathbb{Z}$, $B^2 \equiv \frac{1}{2} \mod \mathbb{Z}$, in this case there is a unique isomorphism class of lattices, of order $2^9(2^{10} - 1)$, with discriminant group $\mathbb{Z}/2d\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$,

iii) $Bh \equiv \frac{1}{2} \mod \mathbb{Z}$, in this case there is a unique isomorphism class of lattices, of order 2^{20}, with discriminant group $\mathbb{Z}/8d\mathbb{Z}$.

A non-trivial Brauer class $\alpha \in Br(S)_2$ is of K3-type if $B_\alpha h \equiv 1/2 \mod \mathbb{Z}$ and $B_\alpha^2 \equiv 0 \mod \mathbb{Z}$ ([vG05, Corollary 9.4], the latter is significant only if $2 \not| d$).

In case $d = 1$, let $C_6 \subset \mathbb{P}^2$ be the (smooth) degree six branch curve of the double cover $S \to \mathbb{P}^2$. Then the Brauer classes in $Br(S)_2$ correspond to ([vG05], [IOOV17]):

i) If $B_\alpha h \equiv 0$, α corresponds to a point of order two $p \in Jac(C_6)$.

ii) If $B_\alpha h \equiv \frac{1}{2}$ and $B_\alpha^2 \equiv 0$, α corresponds to an even theta characteristic on C_6.

iii) If $B_\alpha h \equiv \frac{1}{2}$ and $B_\alpha^2 \equiv \frac{1}{2}$, α corresponds to an odd theta characteristic on C_6.

1.4. Brauer classes, sublattices and invariants: the case $p > 2$

In case the Picard rank of the degree $2d$ K3 surface S is one, the lattices $T_\alpha(S)$ for $S \in Br(S)_p$ and $p > 2$ are classified up to isometry by their discriminant groups, of order $2p^2d$, $d(T_\alpha(S), q)$, where $q = q_\alpha$ is a quadratic form with values in $\mathbb{Q}/2\mathbb{Z}$. In case the discriminant group is cyclic, we denote by v a generator of the discriminant group, $d(T_\alpha(S)) = \langle v \rangle$.

To give the classification, we fix an isomorphism $H^2(S,\mathbb{Z}) = U^3 \oplus E_8(-1)^2$ such that $h = (1, d) \in U$, the first copy of U in the lattice. Then

$$T(S) = h^\perp = \mathbb{Z}v \oplus \Lambda', \quad \text{with } v = (-1, d) \in U \ .$$

Let $w := (0, -1) \in U$. Then $wv = 1$ and any $\alpha \in \mathrm{Hom}(T(S), \frac{1}{p}\mathbb{Z}/\mathbb{Z})$ is determined by a B-field $B_\alpha = \frac{1}{p}(i_\alpha w + \lambda_\alpha) \in \frac{1}{p}H^2(S,\mathbb{Z})$ where $i_\alpha \in \mathbb{Z}$ and $\lambda_\alpha \in \Lambda'$:

$$\alpha : T(S) \longrightarrow \tfrac{1}{p}\mathbb{Z}/\mathbb{Z}, \qquad \alpha(zv + \lambda') = B_\alpha \cdot (zv + \lambda') = \tfrac{1}{p}(i_\alpha z + \lambda_\alpha \cdot \lambda') \ .$$

We define $c_\alpha := -\lambda_\alpha^2/2 \in \mathbb{Z}$ and we observe that

$$B_\alpha \cdot h = -\tfrac{1}{p} i_\alpha, \qquad \lambda_\alpha^2 = -2c_\alpha \ .$$

LEMMA 2. *([MSTV17, Theorem 9]) Let S be a K3 surface such that $Pic(S) = \mathbb{Z}h$, $h^2 = 2d > 0$. Let $p > 2$ be a prime number, $\alpha \in Br(S)_p$ and $B_\alpha \in \frac{1}{p}H^2(S,\mathbb{Z}) \subset H^2(S,\mathbb{Q})$ a B-field representing α.*

a) *In case $p \nmid d$, there are three isomorphism classes of lattices $T_\alpha(S)$,*

   i) *the discriminant group is cyclic, so isomorphic to $\mathbb{Z}/2p^2d$, and $-2dp^2q(v) \mod p$ is a quadratic residue, there are $\frac{1}{2}p^{10}(p^{10}+1)$ such lattices;*

   ii) *the discriminant group is cyclic, so isomorphic to $\mathbb{Z}/2p^2d$, and $-2dp^2q(v) \mod p$ is not a quadratic residue, there are $\frac{1}{2}p^{10}(p^{10}-1)$ such lattices;*

   iii) *there is a unique isomorphism class of lattices with discriminant group $\mathbb{Z}/2d\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ and there are $(p^{20}-1)/(p-1)$ such sublattices.*

   A Brauer class is of K3-type if the discriminant group is cyclic and $-2dp^2q(v) \mod p$ is a square in $\mathbb{Z}/p\mathbb{Z}$.

b) *In case $p|d$, there are four isomorphism classes of lattices $T_\alpha(S)$,*

   i) *$B_\alpha h \equiv 0 \mod \mathbb{Z}$, the discriminant group is $\mathbb{Z}/2d\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z}$ and $c_\alpha \mod p$ is a quadratic residue, there are $\frac{1}{2}p^9(p^{10}-1)$ such sublattices;*

   ii) *$B_\alpha h \equiv 0 \mod \mathbb{Z}$, the discriminant group is $\mathbb{Z}/2d\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z}$ and $c_\alpha \mod p$ is not a quadratic residue, there are $\frac{1}{2}p^9(p^{10}-1)$ such sublattices;*

iii) $B_\alpha h \equiv 0 \mod \mathbb{Z}$, *there is a unique isomorphism class of lattices with discriminant group is* $\mathbb{Z}/2d\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$, *there are* $(p^9+1)(p^{10}-1)/(p-1)$ *such lattices;*

iv) $B_\alpha h \not\equiv 0 \mod \mathbb{Z}$, *in this case there is a unique isomorphism class of lattices with discriminant group* $\mathbb{Z}/2p^2d\mathbb{Z}$, *there are* p^{20} *such sublattices.*

A Brauer class is of K3-type if the discriminant group is cyclic.

1.5. Remarks

The intersection number $B_\alpha h \in \frac{1}{p}\mathbb{Z}/\mathbb{Z}$ is an invariant of the Brauer class α only in case $p|d$. In fact, B_α and $B_\alpha + \frac{1}{p}h$ define the same α but $(B_\alpha + \frac{1}{p}h)h = B_\alpha h + \frac{1}{p}2d$ which is congruent to $B_\alpha h$ only if $p|d$.

Moreover, if $p|d$ and $B_\alpha h \equiv 0 \mod \mathbb{Z}$, one obtains the invariant $B_\alpha^2 \in (\frac{1}{p})^2\mathbb{Z}/\frac{1}{p}\mathbb{Z}$ since any other representative is given by $B_\alpha + \frac{1}{p}D + c$ with $D = ah \in Pic(S)$ and $c \in H^2(S,\mathbb{Z})$.

If $p \not| d$, we see that there is a choice of the B-field representative such that $B_\alpha h = 0$, that is, $B_\alpha \in \frac{1}{p}T(S)$, any other such representative is then given by $B_\alpha + c$ with $c \in T(S)$.

1.6. Vanishing Brauer classes

Let S be a K3 surface with $Pic(S) = \mathbb{Z}h$ and $h^2 = 2d$. We consider a specialization of S to a K3 surface S' with Picard lattice

$$Pic(S') = \left(\mathbb{Z}h \oplus \mathbb{Z}k, \begin{pmatrix} h^2 & hk \\ hk & k^2 \end{pmatrix} = \begin{pmatrix} 2d & b \\ b & 2c \end{pmatrix}\right) \quad \text{for some } \ b, c \in \mathbb{Z} \ .$$

We can identify $H^2(S',\mathbb{Z}) = H^2(S,\mathbb{Z})$ and we have $T(S') \subset T(S)$. Thus, there is a restriction map on the p-torsion subgroups of the Brauer groups:

$$Br(S)_p \longrightarrow Br(S')_p \ .$$

A non-zero element in the kernel of this map is a *vanishing Brauer class* ([GvG24]).

In [GvG24, Proposition 2.1.2 and Corollary 2.1.3] we exhibited a B-field representative of a vanishing Brauer class $\alpha_{van} \in Br(S)_2$ which can be easily generalized. To do so we identify $H^2(S,\mathbb{Z})$ with $H^2(S',\mathbb{Z})$ such that $h \in Pic(S)$ specializes to the element with the same name in $H^2(S',\mathbb{Z})$.

PROPOSITION 1. *Let p be a prime number. We denote by $\alpha_{van} \in Br(S)_p$ a vanishing Brauer class for the specialization of* (S,h) *to* S' *as above.*

i) There is an α_{van} *with B-field representative given by*

$$B_{van} := \tfrac{1}{p}k\,(\in \tfrac{1}{p}H^2(S,\mathbb{Z}))\ .$$

ii) For this $\alpha_{van} \in Br(S)_p$ *we have*

$$B_{van}h \equiv \tfrac{1}{p}b \mod \mathbb{Z}, \quad B_{van}^2 \equiv 2c(\tfrac{1}{p})^2 \mod \mathbb{Z}\ .$$

Proof. Notice that $\frac{1}{p}k \not\in \frac{1}{p}Pic(S) + H^2(S,\mathbb{Z})$, hence it defines a non-trivial class in $Br(S)_p$. But $\frac{1}{p}k \in \frac{1}{p}Pic(S')$ hence $\frac{1}{p}k$ defines the trivial class in $Br(S')_p$. Therefore the Brauer class with B-field representative $\frac{1}{p}k$ is the vanishing Brauer class α_{van}. □

2. K3 surfaces of degree $2d$ and $2p^2d$

2.1. The Mukai lattice and moduli spaces of sheaves

Let (X,H) be a polarized K3 surface, with $H \in Pic(X)$ primitive, of degree $H^2 = 2p^2d$ where $d > 0$ and p is a prime number. As in [MSTV17, §2.6, §3] we consider the Mukai vector

$$v := (p,H,pd) \in \tilde{H}(X) := H^0(X,\mathbb{Z}) \oplus H^2(X,\mathbb{Z}) \oplus H^4(X,\mathbb{Z})\ .$$

The Mukai lattice $\tilde{H}(X)$ has the bilinear form

$$(r,c,s)(r',c',s') := -(rs'+sr') + c\cdot c', \qquad \text{so} \quad v^2 = 0\ ,$$

where $c \cdot c'$ is the intersection product of $c, c' \in H^2(X,\mathbb{Z})$ and H^0, H^4 are naturally identified with $\mathbb{Z}$. The Mukai lattice has the weight two Hodge structure defined by the one on $H^2(S,\mathbb{Z})$. The sublattice of integral $(1,1)$-classes is thus generated by the summands H^0, H^4 and $Pic(S)\,(\subset H^2)$. In particular, v is of type $(1,1)$.

From the work of Mukai [Mu84] it follows that the moduli space $M_X(v)$ of sheaves $\mathcal{E}$ with

$$v = v(\mathcal{E}) := \Big(\mathrm{rank}(\mathcal{E}), c_1(\mathcal{E}), \mathrm{rank}(\mathcal{E}) + (1/2)c_1(\mathcal{E})^2 - c_2(\mathcal{E})\Big)$$

is a K3 surface S. It is the unique K3 surface for which there is a Hodge isometry $H^2(S,\mathbb{Z}) \cong v^\perp/v$. This implies that the image of the transcendental lattice $T(X)$ of X under the map $T(X) \hookrightarrow v^\perp \to v^\perp/v$ has finite index in $T(S)$. The Picard ranks of X and S are thus the same. The sublattice generated by $H^0(X,\mathbb{Z}), H^4(X,\mathbb{Z})$ and $\mathbb{Z}H$ intersects $v^\perp$ in a rank two sublattice whose image in $v^\perp/v$ has rank one. Then (S,h), where h is a generator of this rank one lattice, is a polarized K3 surface of degree $2d$ (cf. the proof of Theorem 1).

2.2. A map between moduli spaces

This defines a finite map

$$\kappa = \kappa_v : \mathcal{M}_{2dp^2} \longrightarrow \mathcal{M}_{2d}, \qquad (X, H) \longmapsto (M_X(v), h)$$

where $\mathcal{M}_e$ is the coarse moduli space of K3 surfaces of degree e. The case $d = 1$ was used by Kondo in [Ko93].

The geometry behind this map is well understood in the case that $d = 1$, $p = 2$: a general K3 surface (X, H) of degree eight determines a K3 surface (S, h) of degree two as follows. The line bundle H gives an embedding of X as a complete intersection of three quadrics in $\mathbb{P}^5$. The surface S is the double cover of the $\mathbb{P}^2$ that parametrizes the quadrics which is branched over discriminant curve $C_6 \subset \mathbb{P}^2$ which parametrizes the singular quadrics ([Kh05], [IKh13], [IKh15], [KS18], [MSTV17, 3.2]).

THEOREM 1. *Let, for a given $d > 0$ and prime number p, $(X_{b,c}, H)$ be a K3 surface of degree $2p^2d$ with Picard lattice*

$$Pic(X_{b,c}) = \left(\mathbb{Z}H \oplus \mathbb{Z}K, \begin{pmatrix} 2p^2d & b \\ b & 2c \end{pmatrix}\right).$$

Let $S_{b,c} := \kappa(X_{b,c})$. The K3 surface $(S_{b,c}, h)$ of degree $2d$ has Picard lattice

$$Pic(S_{b,c}) = \begin{cases} \left(\mathbb{Z}h \oplus \mathbb{Z}k, \begin{pmatrix} 2d & b \\ b & 2cp^2 \end{pmatrix}\right) & \textit{if } p \not| b\,, \\[2ex] \left(\mathbb{Z}h \oplus \mathbb{Z}k, \begin{pmatrix} 2d & b/p \\ b/p & 2c \end{pmatrix}\right) & \textit{if } p | b\,. \end{cases}$$

Proof. First of all we show that the general $S_{b,c}$ has a polarization of degree $2d$. The sublattice of $(1,1)$ classes in $\tilde{H}(X_{b,c})$ contains the primitive sublattice N generated by $(1,0,0), (0,H,0), (0,0,1)$. One easily finds that

$$N \cap v^{\perp} = \langle \alpha := (-1, 0, d),\ \beta := (2p, H, 0)\rangle, \qquad v = p\alpha + \beta\,.$$

Therefore $(N \cap v^{\perp})/v \cong \mathbb{Z}h$ where h is represented by α and $h^2 = \alpha^2 = 2d$, and h is primitive in $v^{\perp}/v = H^2(S_{b,c}, \mathbb{Z})$, of type $(1,1)$ and $(S_{b,c}, h)$ defines a point in $\mathcal{M}_{2d}$.

The Picard lattice of $S_{b,c}$ is the image of $N + \mathbb{Z}(0, K, 0)$, the sublattice of all $(1,1)$ classes in $\tilde{H}(X_{b,c})$, in $H^2(S_{b,c}, \mathbb{Z})$.

$$(N + \mathbb{Z}(0, K, 0)) \cap v^{\perp} = \langle \alpha,\ \beta,\ \gamma\rangle \quad \text{with} \quad \gamma := \begin{cases} (0, pK, b), & p \not| b, \\ (0, K, b'), & b = pb'. \end{cases}$$

As $v = p\alpha + \beta$, the image of this sublattice is generated by the images h, k of α and γ respectively and one finds the Gram matrices as in the theorem. □

COROLLARY 1. *If the K3 surfaces $X_{b,c}$ and $S_{b,c}$ are isomorphic, then the prime number p does not divide b.*

Proof. If the surfaces are isomorphic, the determinants of the Gram matrices of the Picard groups must be the same. This is the case only if p does not divide b. (In general it is not the case however that if p does not divide b then X and S are isomorphic, nor that their Picard lattices are isomorphic.) □

3. The map κ and Brauer groups

3.1. The cyclic subgroup $C \subset Br(S)$ determined by X

For a general (X,H), the transcendental lattice $T(X) = H^{\perp}$ maps to a sublattice of index p in $h^{\perp}$, in fact $H^2 = p^2h^2$. In particular there is an isomorphism $T(S)/T(X) \cong \mathbb{Z}/p\mathbb{Z}$ and hence there is a surjective map $T(S) \to \frac{1}{p}\mathbb{Z}/\mathbb{Z}$ whose kernel is $T(X)$. Thus $S = \kappa(X)$ comes with a subgroup $C = C_X \subset Br(S)_p$ of order p.

The following theorem identifies the subgroup $C \subset Br(S)_p$. It also determines the vanishing Brauer class for a specialization of an (S,h) with Picard rank one to $S_{b,c}$. We recall that one can choose the isomorphism $H^2(S,\mathbb{Z}) \cong U^3 \oplus E_8(-1)^2$ in such a way that h maps to the vector $(1,d)$ in the first copy of U. Then $T(S) = h^{\perp}$ is the sublattice

$$T(S) = \mathbb{Z}t_S \oplus U^2 \oplus E_8^2 \qquad (\subset \Lambda_{K3} := U^3 \oplus E_8(-1)^2). \qquad t_S := \begin{pmatrix} -1 \\ d \end{pmatrix} .$$

In case X has higher Picard rank, let $H \cdot Pic(X) = \gamma\mathbb{Z}$. A result of Mukai implies that the index of $T(X)$ in $T(S)$ is $GCD(p,\gamma)$, as we verify below for the Picard rank two case.

THEOREM 2. *Let (X,H) be a K3 surface of degree $2p^2d$ with $Pic(X) = \mathbb{Z}H$ and let $(S,h) = \kappa(X,H)$. Then there is an isomorphism $H^2(S,\mathbb{Z}) \cong U^3 \oplus E_8(-1)^2 = U \oplus \Lambda'$ such that*

$$h \longmapsto \left(\begin{pmatrix} 1 \\ d \end{pmatrix}, 0\right), \quad T(S) \xrightarrow{\cong} \begin{pmatrix} -1 \\ d \end{pmatrix}\mathbb{Z} \oplus \Lambda' ,$$

and

$$T(X) = \ker(\alpha_X : T(S) \longrightarrow \tfrac{1}{p}\mathbb{Z}/\mathbb{Z}, \quad t \longmapsto B_X \cdot h \mod \mathbb{Z}),$$

where the B-field is

$$B_X := \tfrac{1}{p}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, 0\right) \in (U \oplus \Lambda') \otimes \mathbb{Q} .$$

In the specialization of S to $S_{b,c}$, the subgroups of $Br(S)_p$ generated by α_X and α_{van} coincide if and only if b is odd.

In case $p = 2$, the invariants of B_X are $B_Xh = 1/2$ and $B_X^2 = 0$. If also $d = 1$, S is a double plane, the Brauer class corresponds to an even theta characteristic and X is *a K3 surface of degree eight, moreover, for the specialization of S to $S_{b,c}$ the vanishing Brauer class α_{van}:*

i) is α_X and corresponds to an even theta for b odd,

ii) corresponds to a theta characteristic for $b \equiv 2 \mod 4$ which is even if c is even, but $\alpha_{van} \neq \alpha_X$, and is odd otherwise,

iii) corresponds to a point p of order two in the Jacobian $Jac(C_6)$ for $b \equiv 0 \mod 4$.

In case $b \equiv 0 \mod 4$ the theta characteristic $\alpha_{van} + \alpha_X$ is even/odd exactly when c is even/odd.

Proof. Up to isometry, there is a unique embedding of $Pic(X_{b,c})$ in the K3-lattice $\Lambda_{K3} = U \oplus \Lambda'$ with $\Lambda' = U^2 \oplus E_8(-1)^2$ ([Ni80, Thm. 1.14.4]). We choose the isometry such that, for some $K' \in \Lambda'$,

$$H = ((1, p^2 d), 0) \in U \oplus \Lambda', \qquad K = ((0, b), K') \in U \oplus \Lambda', \quad K^2 = (K')^2 = 2c\ .$$

As $Pic(X) = \mathbb{Z}H$ we get

$$T(X) = H^\perp = \mathbb{Z}t_X \oplus \Lambda', \quad t_X := \begin{pmatrix} -1 \\ p^2 d \end{pmatrix}\ .$$

Notice that $\tilde{H} = (H^0(X) \oplus U \oplus H^4(X)) \oplus \Lambda'$, let

$$\tilde{U} := H^0 \oplus U \oplus H^4 \subset \tilde{H}(X, \mathbb{Z})\ ,$$

where U is the first summand of Λ_{K3}. Then $v = (p, H, pd) \in \tilde{U}$. As $\Lambda' \subset v^\perp$ and $\langle v \rangle \cap \Lambda' = \{0\}$, this unimodular lattice maps isomorphically to the sublattice $\Lambda' \subset v^\perp / v = H^2(S, \mathbb{Z})$. To find the image of $T(X)$, it remains to find the image in $v^\perp / v$ of $(0, (-1, p^2 d), 0) \in \tilde{U} \cap v^\perp$.

With the notation in the proof of Theorem 1,

$$\tilde{U} \cap v^\perp = \langle \alpha = (1, 0, -d),\ \beta_1 = (0, \tbinom{1}{0}, pd),\ \beta_2 = (0, \tbinom{0}{p}, 1) \rangle\ .$$

Notice that

$$v = p\alpha + \beta_1 + pd\beta_2, \qquad (0, (-1, p^2 d), 0) = -\beta_1 + pd\beta_2\ .$$

Hence $(\tilde{U} \cap v^\perp)/v$ is generated by the images h, k' of α and β_2 whereas β_1 maps to $-ph - pdk' = -p(h + dk')$. The intersection products are $h^2 = 2d$, $hk' = -1$, $(k')^2 = 0$. The sublattice $\langle h, k' \rangle \subset H^2(S, \mathbb{Z})$ is isomorphic to U:

$$\langle h, k' \rangle \xrightarrow{\simeq} U, \qquad h \longmapsto (1, d), \quad k' \longmapsto (0, -1)\ .$$

Then $(0, (-1, p^2 d), 0) = -\beta_1 + pd\beta_2$ maps to $p(h + dk') + pdk' = p(h + 2dk')$ which maps to $p(1, -d) \in U$. So we can choose the isomorphism between $H^2(S, \mathbb{Z})$ and

Λ_{K3} in such a way that $h \mapsto (1,d)$ and the image of $(0,(-1,p^2d),0)$ maps to $p(1,-d)$. The image of $T(X)$ in $H^2(S,\mathbb{Z})$ is then the sublattice

$$T(X) \xrightarrow{\cong} p\binom{-1}{d}\mathbb{Z} \oplus \Lambda' \quad (\subset T(S))\ .$$

Notice that for $B_X = (0,1) \in U$, we have $B_X(1,-d) = -1$ which implies that $\ker(B_X) = T(X)$.

A simple computation shows that $B_X h = \frac{1}{p}(0,1)\cdot(1,d) = \frac{1}{p}$ and $B_X^2 = 0$.

The subgroups generated by α_X and α_{van} are the same if and only if the inclusion $T(X_{b,c}) \subset T(S_{b,c})$ is an equality, which is equivalent to these lattices having the same discriminants. This is again equivalent to the determinants of the Picard lattices of $X_{b,c}$ and $S_{b,c}$ being the same and by Theorem 1 we see that this happens if and only if $p \not| b$.

The invariants of α_{van} are determined by the second column of a Gram matrix of $Pic(S_{b,c})$ by Proposition 1. A Gram matrix is given in Theorem 1 and (i)-(iii) follow.

In case B_{van} corresponds to a point of order two, the sum $B_s := B_X + B_{van}$ corresponds to a theta characteristic. The parity of this characteristic is determined by $B_s^2 \mod \mathbb{Z}$. We find $B_X^2 = 0$ and, by Proposition 1, $B_{van} = (1/2)k$ with k as in Theorem 1, hence $B_{van}^2 = (1/4)(2c) = c/2$. It remains to compute $2B_X \cdot B_{van}$ which we claim is 0, so that $B_s^2 = c/2$ and the last statement of the theorem is proven.

To verify the claim, we recall that k has representative $\gamma \in v^\perp$ and since $b \equiv 0 \mod 4$ we have $\gamma = (0,K,b')$ where $b = 2b'$. With our choice of embedding, $K = ((0,b),K')$ and then $\gamma = (0,(0,b),b') + (0,K',0) = b'\beta_2 + (0,K',0)$ which maps to $b'(0,-1) + K' \in U \oplus \Lambda' = H^2(S,\mathbb{Z})$. Hence $B_X \cdot B_{van} = (0,1)\cdot(0,-b'/2) = 0$ since $B_X \cdot K' = 0$. □

3.2. An intrinsic description of the map κ

To define $\kappa : \mathcal{M}_{2p^2d} \to \mathcal{M}_{2d}$, we used a Mukai vector v. Here we give another way to define the map, where we use some of the notation from the (proofs of the) previous results.

Let (X,H) be a polarized K3 surface of degree $2p^2d$. Recall that in $H^2(X,\mathbb{Z})$ we have the sublattices $\mathbb{Z}H$ and $H^\perp = \mathbb{Z}t_X \oplus \Lambda'$, their direct sum has index $2p^2d$ in $H^2(X,\mathbb{Z})$. To get all of the second cohomology group one has to add the 'glue vector' $(H+t_X)/2p^2d$. Since the discriminant group $(H^\perp)^*/H^\perp$ of $H^\perp$ is cyclic, there is a unique overlattice, denoted by $h^\perp$, such that $H^\perp \subset h^\perp$ has index p. This overlattice is generated by $H^\perp$ and a $t_S \in h^\perp$ with $pt_S = t_X$. Let $\mathbb{Z}h$ be the rank one lattice with $h^2 = 2d$. Then the overlattice of $\mathbb{Z}h \oplus h^\perp$ defined by the glue vector $(h+t_S)/2d$ is an even unimodular lattice and hence is isometric to Λ_{K3}. This lattice has the Hodge structure induced by the one on $T(X) \subset h^\perp$ and hence defines a unique polarized K3 surface $(S,h) := \kappa((X,H))$ by surjectivity of the period map and the Torelli theorem.

3.3. FM partners

We consider the fibers of the map κ in the case $d = 1, p = 2$. Given a general K3 surface $(S,h) \in \mathcal{M}_2$, by Theorem 2 an (X,H) in the fiber over it determines an order 2 subgroup $C = \ker(\alpha_X) \subset Br(S)_2$. Moreover, the unique non-trivial Brauer class $\alpha \in C$ corresponds to an even theta characteristic on C_6. Given an even theta characteristic on a general C_6, this invertible sheaf has no non-trivial global sections and using [Be00] one obtains a K3 surface X of degree 8 from a resolution of this sheaf. Since there are $2^9(2^{10}+1)$ even theta characteristics on the genus 10 curve C_6, this number is also the degree of $\kappa : \mathcal{M}_8 \to \mathcal{M}_2$.

For a general $d \geq 1$ and a prime number p however, the order p subgroup $C = \ker(\alpha_X) \subset Br(S)_p$ only determines the sublattice

$$T_C = T_{\alpha_X}(S) := \ker(\alpha_X : T(S) \longrightarrow \tfrac{1}{p}\mathbb{Z}/\mathbb{Z}) ,$$

with the induced Hodge structure from $T(S)$. To obtain a K3 surface X, one must embed T_C primitively into a K3 lattice, which can be done only if α_X, and thus C, is of K3-type, and even then the embedding need not be unique up to isometry.

PROPOSITION 2. *Let $(S,h) \in \mathcal{M}_{2d}$ be a polarized K3 surface with $Pic(S) = \mathbb{Z}h$ and $h^2 = 2d$. For a prime number p, the cardinality of the fiber of $\kappa : \mathcal{M}_{2p^2d} \to \mathcal{M}_{2d}$ over (S,h) is*

$$\sharp\kappa^{-1}(S,h) = \begin{cases} \frac{1}{2}p^{10}(p^{10}+1) & \text{if} \quad d = 1, \\ p^{10}(p^{10}+1) & \text{if} \quad p \not| d, d > 1, \\ p^{20} & \text{if} \quad p|d\ . \end{cases}$$

A polarized K3 surface (X,H) is in the fiber $\kappa^{-1}(S,h)$ if and only if $T(X)$ is Hodge isometric to a sublattice of index p of $T(S)$.

Proof. We give two proofs. The first using lattices and the Torelli theorem, the second uses volumes and was shown to us by I. Barros.

If $(X,H) \in \kappa^{-1}(S,h)$ then $T(X)$ is an index p sublattice of $T(S)$, hence it is defined by an order p subgroup of K3-type of $Br(S)_p$. Conversely such a subgroup C defines a sublattice T_C of index p that can be primitively embedded into the K3 lattice, that is, this sublattice is isometric to $H^\perp$ for some (X,H) if C is of K3-type.

The 'forgetful' map that associates to a polarized K3 surface (X,H), with $H^2 = 2p^2d$, the Hodge structure $H^\perp$ defines a map from $\mathcal{M}_{2p^2d} \to \mathcal{M}^T_{2p^2d}$ which is finite and has degree equal to the number of FM partners of X for a K3 surface X with $Pic(X) = \mathbb{Z}H$. This number is ([Og02, Proposition 1.10])

$$|FM(X)| = 2^{\tau(p^2d)-1},$$

where $\tau(p^2d)$ is the number of prime factors of p^2d. Thus, given the Hodge structure T_C, there are $2^{\tau(p^2d)-1}$ K3 surfaces X with $T(X) \cong T_C$. However, there are also $2^{\tau(d)-1}$ K3 surfaces S' with $T(S') \cong T(S)$.

In particular, if $p|d$ then $\tau(d) = \tau(p^2d)$ and thus, given T_C, each S' with $T(S') \cong T(S)$ determines a unique X' with $T_C \cong T(X')$. The same is true if $d = 1$: $\tau(1) = \tau(p^2) = 1$.

If however $p\not| d$ and $d > 1$ then $\tau(p^2d) = \tau(d)+1$ and thus S' determines two K3 surfaces X'.

The degree of the map, for $p = 2$, $p > 2$ then follows from [vGK23, Theorem 2.3] and [MSTV17, §2.6] respectively, which gives the number of subgroups C of K3-type.

For the second proof, we use the Hirzebruch-Mumford volumes of the moduli spaces computed in [GHS07, §3.5]:

$$\mathrm{vol}_{HM}(\mathcal{M}_{2d}) = (\frac{d}{2})^{10}\prod_{p|d}(1+p^{-10})\cdot\frac{|B_2B_4\dots B_{20}|}{20!!}, \qquad (d>1)$$

and for $d = 1$ one has the same formula multiplied by 2. Then the degree of κ is:

$$\frac{\mathrm{vol}_{HM}(\mathcal{M}_{2p^2d})}{\mathrm{vol}_{HM}(\mathcal{M}_{2d})} = \begin{cases} \frac{1}{2}p^{10}(p^{10}+1) & \text{if} \quad d=1, \\ p^{10}(p^{10}+1) & \text{if} \quad p\not| d, d>1, \\ p^{20} & \text{if} \quad p|d\ . \end{cases}$$

(In [MSTV17, §2.7, Remark], it is stated that, based on a result of Kondo, for $d = 1$ the degree of κ is $p^{10}(p^{10}+1)$, but this is not correct.) □

4. Examples

4.1. K3 surfaces with a line

We consider some special cases of $\kappa : X_{2p^2d} \mapsto S_{2d}$ with Picard rank two. In the literature we found the cases

$$(d,p) = (1,2), \quad (1,3), \quad (2,2), \quad (2,3), \quad (3,2)\ .$$

We consider these cases where moreover X has a line. Then $X = X_{1,-1}$ in the notation of Theorem 1, so $b = 1$. The Picard lattices of $X_{1,-1}$ and $S_{1,-1}$ have the same determinant by Corollary 1. We then consider whether the K3 surfaces $X_{1,-1}$ and $S_{1,-1}$ are isomorphic, which is not always the case as we will see.

Since the determinants are the same, we have the equality of the Brauer classes $\alpha_X = \alpha_{van} \in Br(S)$, that is, the Bauer class $\alpha_X \in Br(S_{1,-1})$ defining $X_{1,-1}$ (up to possible FM partners) is trivial. It would be interesting to see explicitly a rational section of the specialization of the conic bundles with Brauer class α_X on a general S to $S_{1,-1}$ in this case. Such conic bundles are quite explicitly known in some of the examples.

4.2. X_8 and S_2 : square determinants

We consider first the K3 surfaces $X_{b,c}$ of degree $8 = 2p^2 d$ with $d = 1, p = 2$, such that the Picard lattice has determinant D which is a square: $D = -\det(Pic(X_8)) = b^2 - 16c$. In that case $X_{b,c}$ has a genus one fibration given by a divisor class E with $E^2 = 0$.

In [KS18, Lemma 3.10] it is shown that there are infinitely many $D = -\det(Pic(X_8)) = b^2 - 16c$ for which $X_{b,c}$ and $S_{b,c}$, with b odd, are not isomorphic, even if the Picard lattices have the same determinant. For this they consider the case $D = m^2$ for an odd integer m, for example take $b = m$ and $c = 0$. There is an isomorphism $X_{b,c} \cong S_{b,c}$ if and only if $r^2 - Ds^2 = \pm 8$ has an integer solution according to [MN03]. This can now be written as $r^2 - (ms)^2 = \pm 8$ and if one takes $ms > 3$ then $|r^2 - (ms)^2| > 8$ for any $r \neq \pm ms$, hence the result.

A simple geometric example where these surfaces are not isomorphic is thus the case that X is a smooth complete intersection of three quadrics that contains a rational normal cubic curve. Then $X = X_{3,-1}$ and K is the class of the cubic curve:

$$Pic(X_{3,-1}) = \left(\mathbb{Z}H \oplus \mathbb{Z}K, \begin{pmatrix} 8 & 3 \\ 3 & -2 \end{pmatrix}\right), \qquad D = -\det(Pic(X_8)) = 25,$$

and $X_{3,-1} \ncong S_{3,-1}$, in fact, the Picard lattices are not isometric. From Theorem 1 we have

$$Pic(S_{3,-1}) = \left(\mathbb{Z}h \oplus \mathbb{Z}k, \begin{pmatrix} 2 & 3 \\ 3 & -8 \end{pmatrix}\right) \cong \left(\mathbb{Z}h \oplus \mathbb{Z}e, \begin{pmatrix} 2 & 5 \\ 5 & 0 \end{pmatrix}\right), \quad e := h + k\,.$$

If the Picard lattices were isometric, there should also be a (-2)-class, like K, in $Pic(S_{3,-1})$. However, $(xh + ye)^2 = 2x^2 + 10xy = 2x(x + 5y)$ and this is -2 only if either $x = 1$, $x + 5y = -1$ or $x = -1$, $x + 5y = 1$, however both are impossible for $(x, y) \in \mathbb{Z}^2$.

On the other hand, in the cases $D = 1, 9$, one can take $X_{1,0}$ and $X_{3,0}$ respectively and these are isomorphic to $S_{1,0}$, $S_{3,0}$ respectively since $(r,s) = (\pm 3, \pm 1)$, $(r,s) = (\pm 1, \pm 3)$ give solutions to $r^2 - Ds^2 = \pm 8$. These two cases are well-known.

For $D = 1$ the two Picard lattices are both isomorphic to the hyperbolic plane U and $X_{1,0} \cong S_{1,0}$ since the glueing of U to $T(X) = T(S)$ is just a direct sum. These surfaces have a unique elliptic fibration. (see [MN03, Proposition 3.2.1]).

In the case $D = 9$ the surface $X_{3,0} \cong S_{3,0}$ is a $K3$ surface of bidegree $(2,3)$ in $\mathbb{P}^1 \times \mathbb{P}^2$. See [MN03, Proposition 3.2.1], [vG05, §5.8], [Be22]), [IKh13, Proposition 3.7]).

The classical association $X_8 \mapsto S_2$, already mentioned in §3.3, is studied in (the list is surely not complete) [IKh13], [IKh15], [Kh05], [KS18], [MN03], [MSTV17].

4.3. X_{16} with a line and S_4

For $d = 2$, $p = 2$ one has $X_{2p^2d} = X_{16}$. We assume that this surface contains a line L. Then $X_{16} = X_{1,-1}$ and with K the class of the line:

$$Pic(X_{1,-1}) = \left(\mathbb{Z}H \oplus \mathbb{Z}K,\ M_{16} = \begin{pmatrix} 16 & 1 \\ 1 & -2 \end{pmatrix}\right).$$

From Theorem 1 one finds

$$Pic(S_{1,-1}) = \left(\mathbb{Z}h \oplus \mathbb{Z}k, \begin{pmatrix} 4 & 1 \\ 1 & -8 \end{pmatrix}\right).$$

The two Picard lattices are isomorphic:

$$\left\{\begin{array}{rcl} h & = & H-2K, \\ k & = & -H+3K, \end{array}\right. \qquad \left\{\begin{array}{rcl} H & = & 3h+2k, \\ K & = & h+k, \end{array}\right.$$

and $\det(Pic(X_{16})) = \det(Pic(S_4)) = -33$. To show that $X_{1,-1} \cong S_{1,-1}$ it suffices to show that the glueing of the Picard lattice to the transcendental lattice is unique up to isomorphisms. That again follows from the surjectivity of the map $O(Pic(X_{1,-1})) \to O(D_P)$ where D_P is the discriminant group of $Pic(X_{1,-1})$. In fact, let

$$S := \begin{pmatrix} 19 & 64 \\ 8 & 27 \end{pmatrix}, \qquad \text{then} \quad SM_{16}S^t = M_{16}\ ,$$

so that $S \in O(Pic(X_{16}))$. The discriminant group of the Picard group is generated by $\delta := (2,1)/33$ and one finds that $\delta S \equiv 23\delta$. Since $\mathbb{Z}/33\mathbb{Z} \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/11\mathbb{Z}$, with $23 \mapsto (-1,1)$, we see that $-I, S \in O(Pic(X_{16}))$ generate $O(D_P)$.

Geometrically, the isomorphism $X_{1,-1} \to S_{1,-1}$ is given by the 'double projection' from the line $L \subset X_{1,-1} \subset \mathbb{P}^9$. First one projects from the line: $\phi_{H-K} : X_{16} \to X'_{12} \subset \mathbb{P}^7$, notice that $(H-K)^2 = 12$. The image of L is a rational normal curve of degree $(H-K)K = 3$ which spans a $\mathbb{P}^3 \subset \mathbb{P}^7$. Projection from the span of the normal curve induces the map $\phi_{H-2K} : X_{16} \to S_4 \subset \mathbb{P}^3$, the image of L is a quintic rational curve in the quartic surface S_4 since $(H-2K)K = 5$.

See [IR05], [IR07], [MSTV17, §3.4], [vGK23, 5.3] for geometrical aspects of the map $X_{16} \mapsto S_4$.

4.4. X_{18} with a line and S_2

For $d = 1$, $p = 3$ one has $X_{2p^2d} = X_{18}$ and $S_{2d} = S_2$. Assume that X contains a line, then $X = X_{1,-1}$ and

$$Pic(X_{1,-1}) = \left(\mathbb{Z}H \oplus \mathbb{Z}K, \begin{pmatrix} 18 & 1 \\ 1 & -2 \end{pmatrix}\right)\ ,$$

therefore

$$Pic(S_{1,-1}) = \left(\mathbb{Z}h \oplus \mathbb{Z}k, \begin{pmatrix} 2 & 1 \\ 1 & -18 \end{pmatrix}\right).$$

The two Picard lattices are isomorphic and have determinant -37, for example an isomorphism is:

$$\left\{\begin{array}{rcl} h & = & 2H-5K, \\ k & = & -5H+13K, \end{array}\right. \qquad \left\{\begin{array}{rcl} H & = & 13h+5k, \\ K & = & 5h+2k\ . \end{array}\right.$$

The two $K3$'s are also isomorphic since the (sufficient) conditions in [MN04, Theorem 3.1.5] are satisfied. More precisely, there exists $h_1 \in Pic(X_{18})$ such that $h_1^2 = 2p$ and $h_1 H \equiv 0 \mod p$ (here $p = 3$), for example $h_1 = H + 3K$.

Another way to see this is to notice that, since the order of the discriminant groups is 37, a prime number, the orthogonal group of the discriminant lattice of the Picard groups is $\{\pm 1\}$. Thus the glueing of the Picard lattice to the transcendental lattice is unique and the surfaces are isomorphic.

See also [MSTV17, §3.3] for the map $X_{18} \mapsto S_2$.

4.5. X_{24} with a line and S_6

For $d = 3$, $p = 2$ one has $X_{2p^2d} = X_{24}$ and $S_{2d} = S_6$. Assume that X contains a line, then $X = X_{1,-1}$ and the Picard lattices of X and $S_{1,-1}$ have the Gram matrices

$$P_{24} = \begin{pmatrix} 24 & 1 \\ 1 & -2 \end{pmatrix}, \qquad P_6 = \begin{pmatrix} 6 & 1 \\ 1 & -8 \end{pmatrix}.$$

The determinants are -49, so we are in the case of square determinants as in §4.2. Let $e = h + k$ in $Pic(S_{1,-1})$, then $e^2 = (h+k)^2 = 6 + 2 - 8 = 0$. Then h, e is a basis of the Picard lattice of $S_{1,-1}$ and

$$(xe + yh)^2 = x^2e^2 + 2xyeh + y^2h^2 = 14xy + 6y^2 .$$

Notice that there is no (-2)-vector in this lattice since $y(7x+3y) = -1$ has no integer solutions. Therefore the Picard lattices are not isometric and hence $X_{1,-1} \ncong S_{1,1}$.

The case $X_{24} \mapsto S_6$ was studied in detail in the recent paper [KM23].

4.6. X_{36} with a line and S_4

For $d = 2$ and $p = 3$ one has $X_{2p^2d} = X_{36}$, which has genus 19, and $S_{2d} = S_4$. This case was considered in [KM23, §1.2], see also [BBFM23, Remark 4.16]. Assume that X contains a line, then $X = X_{1,-1}$ and the Picard lattices of X and $S_{1,-1}$ have the Gram matrices

$$P_{36} = \begin{pmatrix} 36 & 1 \\ 1 & -2 \end{pmatrix}, \qquad P_4 = \begin{pmatrix} 4 & 1 \\ 1 & -18 \end{pmatrix}.$$

The determinants are -73. The Gram matrices are equivalent:

$$SP_{36}S^t = P_4, \qquad S = \begin{pmatrix} 57 & 272 \\ 136 & 649 \end{pmatrix},$$

hence the Picard lattices are isomorphic. Since the orthogonal group of the discriminant group, which is $\mathbb{Z}/73\mathbb{Z}$, is $\{\pm 1\}$, the glueing of the Picard lattice to the transcendental lattice is unique. Thus $X_{1,-1} \cong S_{1,-1}$.

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