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BETWEEN ARITHMETIC AND ALGEBRA: IN THE SEARCH OF A MISSING LINK THE CASE OF EQUATIONS AND INEQUALITIES

Abstract. Following the claims about the operational/structural duality of mathematical co nceptions, (Sfard, 1991) we notice that the majority of mathematical notions draw their meaning from two kinds of processes: the *primary* processes, namely the processes from which the given notion originated, and *secondary* processes – those for which instances of this notion serve as an input. Abstract *objects* act as a link between these two kinds of processes, thus seem to be crucial for our understanding of the corresponding notions. *Pseudostructural* conceptions are the conceptions which develop when the student, unable to think in the terms of abstract objects , uses symbols as things in themselves and, as a result, remains unaware of the relations between the secondary and primary processes. In the case of equations (or inequalities), which in this paper are used as an illustration for the above claims, the primary processes are the arithmetic operations encoded in the formulae, the secondary processes are those which one must perform on equations in order to solve them, and the abstract objects behind the symbols are the truth-sets. Pseudostructural thinking is witnessed whenever there is an evidence that the propositional formulae are conceived just as strings of semantically void symbols, for which the formal transformations used to find the solution are the only source of meaning. This approach to algebraic symbolism seems deceitfully close to the views on algebra endorsed by such mathematicians as Peacock, deMorgan and Hilbert. The difference between this and our students' positions is thus carefully studied and explained. Our empirical study carried out among secondary school pupils has shown that in the algebra, pseudostructural conceptions may be more widely spread than suspected.

When secondary school students deal with variables and parameters, when they look at such expressions as $2x(5-x^2)$ or $\sqrt{a-b}$, when they solve standard equations or inequalities – what are the objects that they see with their mind's eye and manipulate in their heads, what are the reasons for the decisions they make?

It is our aim in this paper to make a step toward a better understanding of students' understanding of algebra. The questions we ask are not as straightforward as one may think, and we will not satisfy ourselves with the simple answers that may be concocted out of what can be found in student's notebooks and on standard tests. Student's thinking is much more complex than can be deduced from the circumstantial evidence of written records. To make our point, let us begin with a brief account of what happened one day between a certain teacher and his pupil.

A 16 year old girl – let us call her Ella – was asked to solve a standard quadratic inequality:

$$x^2 + x + 1 < 0$$

At this stage, Ella could solve any linear inequality and was quite familiar with quadratic functions and their graphs. The girl approached the problem eagerly and within a few minutes produced the following written account of her efforts:

(1) $$x_{1,2} = \frac{-1 + \sqrt{1 - 4} \cdot 1 \cdot 1}{2} = \frac{-1 + \sqrt{-3}}{2}$$

(2) $$T = \{\}$$

(according to the notation used in schools, T signified the truth set of the inequality, namely the set of all the numbers the substitution of which instead of x turns the inequality into a true proposition).

There can be little doubt about the correctness of Ella's solution. Indeed, the fact that the roots x_1 and x_2 cannot be found implies that the parabola $f(x) = x^2 + x + 1$ does not intersect the x-axis. Since its vertex is the minimum of the function, the whole curve is placed above the x-axis and therefore the inequality $f(x) < 0$ does not hold for an x.

Was the written solution the only source of teacher's insight into Ella's thinking, he would certainly reward her efforts with a high score. As it happened, however, he talked and listened to Ella when she was working on the problem, and the things he heard prevented him from praising her. Let us have a look at a fragment of this dialogue.

Ella: [After she wrote line (1) above] There will be no solution for x, because here [points to the number under the $\sqrt{}$ sign] I've got a negative number.

Teacher: O.k., so what about the inequality?

E.: So the inequality isn't true. It just cannot be...

T.: Do you know how to draw the parabola..?

E.: The parabola of this [expression]? But there is no y here... how can one draw parabola when there is no y?

T.: Do you know the relationship between a parabola and the solutions of such an inequality as this?

E.: Of an inequality? No. Only of an equation. But maybe it is the same. Let's suppose that this is equal zero [points to the inequality symbol and makes a movement as if she was writing "=" instead of "<"]. But how can there be a parabola if there is no result here [points to the expression she wrote in (1)], no solution?

T.: So what is your final answer ? What is the solution of the inequality?

E.: There is no solution.

We believe this little example is quite enough to convince anybody that there is much more to student's thinking than implied by his or her written solutions to standard problems. Thus, we can only applaud Davis (1989), for his critique of the current research on learning algebra: "Many – really most – studies focus on what student writes and largely ignore what that student thinks. Yet, what student thinks is much more fundamental than what the student writes". In this paper, we shall try to listen very carefully to what the student has to say while asked explicitly about the meaning of such basic algebraic notions as "solution", "admissible operation" or "equivalent equations". Then, in an attempt to have a glimpse into what is going on in the learner's head while he or she is engaged in an algebraic activity, we shall take a detailed account of his or her reactions to both routine and non-standard problems.

1. Preliminary reflections on understanding algebra

If the teachers had time to talk to the students in the way we did while preparing the material for this paper, they would soon realize that even the most successful of their pupils may be less than satisfied with their own understanding of algebra. Here is an excerpt of a dialogue we led with a sixteen year old Rina, after she had given an expert performance in solving a system of linear equations.

Inteviewer: Okey, you did it beautifully. Now, let me ask you something. You said you multiplied the second equation by 2 and you subtracted the result from the first one. Why is it permitted to do such thing?

Rina: ...I don't know.

I.: Make an effort, say whatever you think.

R.: I never really thought about it.

I.: Didn't it bother you?

R.: It did, but I still don't know.

I.: Maybe you can now think about something. Any idea?

R.: [Remains silent; then, with an embarrassed smile:] No...

The above conversation proves that Rina's understanding was *instrumental* rather than *relational*. While introducing these terms in 1976, Skemp explained that instrumental understanding should be interpreted as "having rules without reasons" – as the kind of comprehension which expresses itself in a technical proficiency (which Rina obviously possessed) not accompanied by the ability to explain the algorithms in any way (which was also the case with Rina). In contrast, relational understanding was described as the ability to produce some kind of justification to the rules at hand.

Before we proceed any further, let us stop for a moment to ask two preliminary questions. First, if the student is able to solve any kind of equation or inequality that he or she is ever likely to tackle, how important is it that he or she can also justify the procedure? Second, if we insist on relational understanding of algebra, what kind of comprehension of algebraic concepts should we be prepared to accept as satisfactory?

It is relatively easy to give an answer to the first question. Except for a long list of arguments against "rules-without-reasons" listed by Skemp himself, let us point out to what seems to us the most obvious shortcoming of this kind of understanding: without an ability to give some kind of explanation to the formal algebraic procedures, the students are not very likely to be able to cope either with non-standard questions or with more advanced algebraic ideas which will be introduced to at least some of them in the future. In the following sections, we will illustrate this claim with many examples.

Answering the second question is a much more demanding task, and we will devote the remaining part of this paper to a discussion of this problem.

1.1. Explaining algebra as building links between primary and secondary processes

While speaking in favor of relational understanding of algebra we imply that we want our students to be able to relate the formal algebraic procedures to the previously developed system of concepts. This is exactly what one expects when he or she requires an *explanation* to a given rule or notion. In the case of algebra, the connection must be made between the algebraic manipulations and the underlying arithmetical processes. Indeed, at the secondary school level the only way to justify the operations we perform on equations is to ground the formal transformations in the numerical computations which they symbolize and generalize. For instance, the transition from, say, $3x + 7 = 2x - 5$ to $3x = 2x - 12$ can only be explained by saying that *whatever number is substituted instead of* x, the first equality holds if and only if the other holds, and therefore subtracting 7 from both sides of the equation does not alter its solution. The manipulation we performed on the equation is an *algebraic* operation, while the fact that the equality relation is preserved under subtraction of a number is a property of *arithmetical* processes. The meaning and soundness of the algebraic procedure is thus inherited from the underlying numerical calculations.

Let us pause for a moment to put the things into a broader context. Let us think about the construction of mathematical knowledge in general. The relationship between algebraic manipulation and numerical computations exemplifies the way in which more advanced mathematical concepts usually relate to those from which they evolved. As was argued in much more detail elsewere (Sfard, 1991; Sfard and Linchevski, 1993), mathematics may be viewed as a hierarchical structure in which new layers are often constructed by subjecting some well-known computational procedures to more general, higher-level processes. This mechanism may be observed time and again both in history and in individual learning. Thus, when focusing our sight on a given mathematical idea, we may usually make a distinction between *primary* and *secondary* processes. For instance, such process as division of an integer by an integer is primary with respect to the idea of a rational number, while the arithmetical operations on rationals are secondary processes. When the notion of function is considered, the sequence of numerical operations necessary to compute the values of a function are primary processes, whereas procedures which may be applied to a function as a whole (e.g. adding or composing, deriving or integrating) are secondary processes. In the case we are now dealing with we will use the term primary processes when referring to arithmetical procedures hiding behind the formulae, whereas the algebraic manipulations themselves will be called secondary processes. In all these cases it is clear that in order to apply the secondary processes in a meaningful way, one must be able to relate them to the primary processes.

The above distinction will help us now to put our finger on the abilities that constitute mastery in algebra. When we scrutinize the way an expert deals with formulae, equations, and inequalities, we soon realize that his or her *capacity for focusing on the right kind of processes and the deftness in making transitions from one level to another* is at the core of his or her fluency in the formal language of algebra.

As strange as it may seem at the first glance, *the ability to temporarily act in an authomatic, "unthinking" mode*, namely to perform secondary processes without constantly worrying about their justification is what makes symbolic algebra so powerful a tool for solving complex computational problems. Or, as Whitehead (1911, p. 59) forcefully put it,

> It is a profoundly erroneous truism, repeated by all copybooks and by eminent people when they are making speaches, that we should cultivate the habit of thinking about what we are doing. The precise opposite is the case. Civilisation advances by extending the number of important operations which we can perform without thinking about them.

In the case of algebra, performing "without thinking" means doing the formal manipulations without constantly keeping in mind their deeper, arithmetical interpretations. The advantages of this mode of action are obvious: it loosens the cognitive stress and vastly increases the capacity for solving complex problems (think about solving a very complicated equation when constantly thinking about the primary meaning of the symbols – the numeric computations represented by the formulae). The necessity, however, to be able to act solely at the secondary (formal) level is only one side of the story. True, the genuine "operations of thought" may sometimes be postponed for a very long time. They are "like cavalry in a battle – they are strictly limited in number, they require fresh horses, and must only be made at decisive moments" (ibid). Nevertheless, even though rare, they are indispensable. Thus, when the suspension of primary meaning becomes permanent, when no return is ever made to the primary processes, the advantage turns into handicap. When a person becomes a captive of the "automatic" mode, when he or she looses his or her ability of referring to the primary processes when such reference would be appropriate, his or her performance displays all the characteristics listed by Skemp as typical of instrumental understanding.

1.2 Abstract objects as links between primary and secondary processes

In this section some thought will be given to the nature of the links through which the back and forth movements between the primary and secondary processes become possible.

When a student tries to solve a problem by performing formal algebraic operations, a question may be asked what are the entities that are being manipulated. The simplest answer would be that algebraic procedures are directed at formulae, at symbols. Obviously, there is some truth to such statement, but contrary to the belief held by many students, it is certainly not the whole truth. Were it the formal expression and that expression alone which dictates the actions to be taken, how could we account for the fact that on different occasions the same formulae are manipulated in different ways? How could we explain why an equation such as $px + 1 - q = 3x + 2$ will sometimes lead to the answer "$x = (q+1)/(p-3)$ for any q and any $p \neq 3$", and sometimes to the claim that p must be equal 3 and q must be equal -1? The difference stems from the diversity of interpretations that may be given to the same expressions. In the above equation, we may refer to the component formulae, $px + 1 - q$ and $3x + 2$, in at least two different ways: we may treat them as expressing certain unknown *numbers*, and we may interpret them as representing two linear *functions*. In the first case we are asking about the value of x (expressed in terms of parameters p and q) which makes the equality hold; in the second we are looking for the values of the parameters p and q for which the two functions are equal.

The meaning we confer on algebraic formulae is what binds together the primary (arithmetical) and secondary processes (symbol manipulations). Whatever the interpretation of a symbolic expression given to it in the course of formal manipulations, the referents we point to are some kinds of *abstract objects*. Indeed, whether we view the formula as denoting a certain (albeit unknown) number or as representing a function, we are referring to a *permanent entity* which, on one hand, is a product of arithmetical operations and, on the other hand, may serve as an input to an algebraic procedure. We may say, therefore, that abstract objects act as links between primary and secondary processes.

We are now in the position to add a new element to our list of skills which constitute mastery in algebra. In the last section we stressed the importance of automatization of the secondary processes, namely of the ability to temporary suspend the primary meaning for the sake of effective manipulations. At the same time we explained why the primary processes cannot be forgotten altogether and must be brought back to one's mind from time to time in the process of problem solving. Now we can complete the picture. Since abstract objects are the mental devices which mediate between the primary and secondary processes, they certainly must play a central role in algebraic problem solving. It is through them that the (secondary) operations performed on formal algebraic expressions become meaningful. Indeed, how could we justify the fact that we subtract $2x$ from both sides of the equation $15 + 2x = 6x - 1$ if we were not able to view $15 + 2x$, $6x - 1$, and $2x$ not only as short prescriptions for certain computations but also as the results of these computations? And in the case of $px + 1 - q = 3x + 2$, how could we account for the operations which led us to the result $p = 3, q = -1$, if we were not able to assume the functional approach to the expressions $px + 1 - q$ and $3x + 2$? Examples may be brought (see e.g. Sfard and Linchevski, 1994) showing that to solve one problem, all the possible approaches may be necessary. While coping with equations and inequalities, a person must be able to go back and forth between *operational approach*, when his or her thought concentrates on processes (those represented by algebraic expressions or those performed on them), and *structural approach*, when he or she focuses on the abstract objects hiding behind the symbols. Thus, the next important component of mastery in algebra is the *flexibility* of approach – the ability to quickly alternate between different modes of thinking and different interpretations of algebraic expressions (see also Sfard, 1991; Gray & Tall, 1991; Sfard & Linchevski, 1994; Moschkovich, et al., 1993).

1.3 When the link is missing: pseudostructural (semantically debased) conceptions

The flexibility of student's algebraic thinking develops gradually (see Sfard & Linchevsky, 1994). When a person is introduced to algebraic symbolism for the first time, his or her understanding of the symbols is far from being as versatile as that of an expert. Many studies (see also Sfard, 1987; Filloy & Rojano, 1985, 1989) have shown that the beginners tend to conceive algebraic expressions in a purely operational way, namely as concise prescriptions for certain computations. It must usually take quite a while before the student is able to think about a formula also in a structural way. Seeing such an expression as 3x+2 as both a computational process and the product of this process is only the beginning. After the ability to view a string of symbols as a name for a number has been developed, the students have still a long way to go until they can address the letters in the formulae as variables rather than unknowns and until they can see the functions hiding behind the formulae. In other words, of the two structural ways of dealing with algebraic expressions, the functional approach is more advanced than the "fixed-value" approach.

Although this fact may often escape teacher's attention, the realization that an algebraic formula can be treated not only as a chain of computational operations but also as a product of these operations may not come easy to a young learner. When we come to think about it, we find out that the difficulty is hardly surprising. To use our favorite metaphor, asking the pupils to treat a prescription for a computational procedure as a result of this procedure is almost like an attempt to convince them that a receipt for a cake is a cake itself! The term *reification* was introduced to denote the switch in pupil's conception which is necessary to turn a process into an object (the word *encapsulation*, used by some other writers, seems to have a similar meaning; see e.g. Dubinsky, 1991). A steadily growing bulk of empirical findings confirms what can easily be explained on a theoretical basis (see Sfard, 1991): that reification is inherently difficult, and that many students never develop a fully-blown structural conception of the most important mathematical concepts taught at schools, the concept of function being probably the most problematic of all.

A failure to see the abstract objects behind algebraic formulae would often turn into a serious handicap for a learner. In the absence of the elements which are necessary to give a deeper meaning to symbol manipulations, the rules of algebra are doomed to be perceived as arbitrary and having no reasons and student's understanding can only be instrumental. The pupil, unable to fathom the nature of the abstract entities which serve as inputs and as outputs to the procedures he or she performs, would often develop conceptions which we once decided to call *semantically debased* or *pseudostructural* (see also Sfard, 1992). When the signs on the paper do not seem to stand for any conceivable entity different from the signs themselves, *the signifier becomes the signified.* In other words, the student focuses on symbolic expressions as such, without looking for their hidden sense. There is a long list of behaviors which can be regarded as indicative of such direct, one-dimensional, approach to algebraic formulae. Here is a collection of phenomena which constitute the syndrome called pseudostrucutral conception.

1. When algebraic symbols are treated as things in their own right, not standing for anything else, *the form of the expression becomes the sole basis for judgments and decisions.* To justify his or her choices, the student should have recourse to the undrlying rules of arithmetic; instead, she or he would lean heavily on the external features of the formula at hand.

EXAMPLE. Here is an excerpt from our conversations with a sixteen year old student whom we asked to check an equivalence of different pairs of equations. The boy was requested to decide whether the two equations, $(x-2)^2 = 0$ and $4x - 11 = 2x - 7$ were equivalent or not. Here is what he said to us: "I try to see whether I have here the same elements... I open the brackets [in the first equations; obtains $x^2 - 4x + 4 = 0$]. I have 4 here, which is $4x$, and x^2... and there will be nothing like that here [points to the second equation]. So these two equations are not equivalent."

2. For those who cannot see beyond the symbols, *the secondary operations would seem arbitrary and unjustified.* The disciplined student will accept them as the rules of the game played by mathematicians and by those who are supposed to behave like ones. This is certainly what can be inferred from the statements of Rina whom we quoted in section 1 above. The study presented in the next sections abounds in additional evidence for pupils inability to see algebraic techniques in a more meaningful way.

3. If a sign serves also as its own referent, there is little hope that the student will be able to see different representations of the same mathematical concept as equivalent. One of the most obvious symptoms of this kind of weakness is the the well documented and widely deplored *difficulty with graphical interpretation of algebraic expressions.* Many researchers have pointed out to the fact that high percentages of students are reluctant to use visual means while solving algebraic problems. More often than not, it seems that the pupils are totally unaware of the relation between analytical and graphical representations of functions (Markovitz et al. 1986; Dreyfus & Eisenberg, 1987; Even, 1988; Schoenfeld et al, 1993). The behavior of Ella, whom we quoted in the opening of this paper, aptly illustrates our present claim. The following episode shows how much confused a student may become when asked to link his knowledge of linear functions and their graphical representations with what she knows about linear equations.

EXAMPLE. At the time we talked to the sixteen year old Orly, she was already supposed to be skillful in drawing the graphs of linear functions. Indeed, when we presented her with an equation $y = kx - 1$ and asked for an example of a shape which can be obtained from it (in the Cartesian plane) by choosing a certain value of k, she draw a straight line with -1 and -0.5 as y- and x-intercepts, respectively. Here is an excerpt form the dialogue which followed the production of the graph.

Interviewer: Can you explain the relationship between this drawing and our equation?

Orly: This [the picture] is a graphical representation of this [the equation].

I.: What does it mean?

O.: That all the x's... That every number we substitute here [in the equation] must be one of the points on the graph. For example, if the graph intersects here [points to the x-intercept], in -0.5, and here [points to the y-intercept] in -1, then this must give a true [!] solution to the equation.

I.: What do you mean? Could you explain? Where do we substitute -1 and -0.5?

O.: Instead of the x and the y. Here we have -0.5 [points to the x-intercept], and here we have the -1 [points to the y-intercept], so we put -0.5 instead of x and -1 instead of y.

I.: But these two numbers are not... They come from two different points...

O.: What do you mean? So what that these are two different points?

It didn't take much time to bring Orly to realize the problem. After a while she was able to straighten things up and to decide that one point on the line rather than coordinates of two different points should be considered as a solution of the equation. Nevertheless, our brief exchange shows how superficial was her understanding of the nature of the bond between the equation and the graph of its truth set.

4. It was stressed several times that mathematical objects – these elusive figments of the human mind – are vitally important for our mathematical thinking. As we explained in our earlier writings, one way of describing their role is to mention the fact that with the help of abstract objects many pieces of knowledge may be brought together to form a unified compact whole. To put it differently, mathematical objects tie together facts, concepts and rules which otherwise would be stored in separate compartments of our memory. From a mathematical object, like from a root, a tree-like scheme would arise. Into this scheme new facts and problems can easily be incorporated on the force of certain common patterns which link them to the entity in question. In the absence of abstract objects, such scheme cannot be constructed and, as a result, *the student would not know how to handle non-routine problems*, even if he or she has already learned the relevant facts and the appropriate methods of solution. He or she just would not recognize the connection. Hence, the pupil would feel that new method must be devised to cope with the situation. Since inventing new techniques is not an easy task, many students would rather slip into a simpler mode of action: he or she would just ignore the differences between the problem at hand and those standard tasks he or she tackled in the past. The method used in the former cases will now be applied to the new kind of situation in a mechanical way, while superficial futures of the symbols guide the student in his or her choice of an algorithm for solution. We observed this kind of behavior in the Ella's case: the girl applied the formula for the roots of quadratic equation automatically, only because they were brought to her mind by certain external features of the inequality she was presented with. Sometimes, even most obvious discrepancies and absurdities would not make the students realize the inadequacy of the method they have been using.

EXAMPLE. There is the phenomenon we witnessed time and again in our interviews (we have no doubt it is known only too well to every teacher): a student, when confronted with the inequality $x^2 - x - 6 > 0$, would often give the following answer: $x_{1,2} > 3, -2$. Pupil's failure to explain the result would not shake his or her belief that this was the correct answer.

5. The symbols would not always suffice as a substitute for their abstract referents. In some situations, the student would feel that certain intangible things must be summoned up if one wants to make sense of the problem at hand. For instance, it is rather difficult to talk about equivalence of equations if one does not focus on the invariants of algebraic manipulations – on those mathematical objects which are requested to remain unaffected by the transition from one of the equations to another. When no appropriate abstract being is available, a total confusion may result. Student's bewilderment would express itself in messy statements, in which different kinds of entities are mentioned at random and mistaken for each other. We call this kind of confusion *out-of-focus phenomenon* (OOF, for short).

EXAMPLES. In our interviews we listened very carefully to the students when they solved equations and inequalities, when they tackled questions about equivalence of equations, and when they tried to define such terms as "solution of an equation" and "equivalent equations". The blurred language used by a big proportion of our interlocutors disclosed their inability to focus on the right kind of mathematical entities.

The pupils would often say "equation" when what they really meant was one of the component formulae (like in "For this [inequality] to be true, one *equation* must be bigger than the other), they would define a truth *set* as "the x" (instead of saying it is the *set of the values of* x which turn the formula into a true proposition), they would define the solution process as a procedure in the end of which "one gets a *true proposition*" (instead of saying that what is found is *the substitution* which turns the equation into a true proposition), etc.

We may talk about an out-of-focus behavior also when an object at hand is described in terms which do not seem adequate to the given context.

EXAMPLE. Sixteen year old Dina was solving a singular system of equations. Some of her utterances clearly showed how uncertain she was about the nature of the mathematical object she was supposed to handle. When "the x disappeared" from one of the equations (she was left with the expression $-1 = 4$) and she was asked to explain the implications, the girl used the expressions "equation exists", and "equation is true" as if they both were synonymous with the claim "equation has a solution."

Dina: The system disappears.

I.: What do you mean?

D.: That it doesn't exist. It is not true.
I.: But it is written here, so what does it mean that it doesn't exist?
D.: It is written, but it is not true.

2. Probing students' understanding of algebra: can they see the links between primary and secondary processes?

After the above theoretical considerations, the most natural thing to do is to ask how flexible is student's understanding of algebra in practice, and how common are pseudostructural conceptions after several years of schooling. As we already observed in the introductory remarks, such questions can only be answered through "fine-grained analysis" (see Schoenfeld et al., 1993) of students utterances and by close inspection of the ways in which the learners tackle algebraic problems.

To have a close look at student's understanding of algebra we decided to combine several methods of investigation. Three questionnaires were prepared in which the subjects were asked either to answer direct questions on the meaning of basic algebraic concepts or to solve series of non-routine problems (we decided to avoid the regular textbook exercises to prevent students from giving automatic answers from which very little may usually be learned about the respondent's conceptions). The questionnaires were applied to 280 students in three integrative secondary schools in Jerusalem (for more detailed description of the examined population, see Figure 1). To have a closer look at the conclusions obtained from this triple study and to check additional conjectures which we were able to formulate on the grounds of our results, we followed the written tests with a series of clinical interviews.

Before the study and its results are presented, let us give some background information about the way algebra is taught in Israeli schools.

FIG. 1: THE POPULATION

Grade	Age	Number of groups	Number of students
NINTH	14-15	4	97
TENTH	15-16	5	112
ELEVENTH	16-17	3	71
	TOTAL	12	280

The path that must be followed by the student is roughly presented in Figure 2.

FIG. 2: HOW ALGEBRA IS TAUGHT IN ISRAELI SCHOOL – BASIC CONCEPTS (after Maschler, M., 1976, 1978)

CONCEPT	DEFINITION	EXAMPLES
NUMERICAL FORMULA (NC)	Combination of numerals, variables, operators, brackets, and other symbols such that if the variables are substituted with numbers, a number results.	i) $3a$ ii) $3x^2 - 5(x+2)$ iii) $(x^7 - 2)/(2x-1)$
PROPOSITIONAL FORMULA (PF)	Combination of numerals, variables, operators, brackets, equality and inequality signs, other symbols, and words, such that when variables are substituted instead of numbers, a proposition (true or false) results.	i) $3x > 12$ ii) $(a+b)^2 = a^2+2ab+b^2$ iii) $x^2+5x+6=0$ iv) $x^2+1=0$
TRUTH SET (TS)	The set of all the substitutions (numbers, pairs of numbers, triples of numbers, etc.) that turn the given PF into a true proposition.	In the above examples: i) $\{x : x > 4\}$ ii) R iii) $\{-3, -2\}$ iv) $\{\}$
EQUIVALENCE OF PFs	Two Pfs (equations, inequalities, systems of equations or inequalities) are equivalent if they have the same variables and the same truth set.	i) $3x+5=x-1$ and $3x = x-6$ ii) $-7x > -14$ and $x < 2$
PERMISSIBLE OPERATIONS	An operation on PF which turns it into an equivalent PF.	i) subtraction of a number from both sides of PF (like in example i above) ii) division of both sides of an inequality by the same negative number and reversion of the inequality sign (like in ii above)

As can be seen from this concise description, equations and inequalities are introduced as two different, but closely related, instances of a single mathematical notion: propositional formula (PF, from now on). The idea of PF is introduced as early as seventh grade, and equations and inequalities are then brought and dealt with simultaneously. Every PF has its truth-set (TS, for short), namely the set of all the substitutions that turn this PF into a true proposition. Any two PFs with the same truth sets are called equivalent. Solving equation or inequality means finding its TS. As a consequence of this approach, even the solving procedures are described in set-theoretic terms: to solve, say, an equation E, one must find the simplest possible PF which is equivalent to E. To summarize, this is a good example of a structural approach: a mathematical notion (PF) is explained in terms of abstract objects (truth-sets).

This modern method has, no doubt, much appeal for those who are able to appreciate the unifying power of propositional formula. Indeed, this simple idea ties together a large bulk of definitions and procedures and thus organizes the whole of basic algebra into a neat, coherent, elegant whole. On the face of it, such top-down (from general to particular) approach should be easier for the learner than the alternative bottom-up method.

We should not forget, however, that the notion of propositional formula, just because of its generality and great abstractness, does not yield easily to the kind of interpretation the student may need in order to have a good grasp of the idea. More often than not, abstract mathematical concepts become meaningful only in relation to those mathematical ideas which they are supposed to generalize. In the above scenario, the ideas which justify the concept of propositional formula and which make it significant appear later than the concept itself. In a sense, therefore, our teaching sequence reverses what seems to be a 'natural' order. This impression becomes even stronger when we look at the history of algebra. The notion of propositional formula appeared only at the advanced stage in the development of the domain, and it served as a means for summarizing and simplifying the existing knowledge about equations and inequalities, rather than as a point of departure for building this knowledge. This leads to the question whether our 'upside down', structural-to-operational approach may be really meaningful for the students. We shall explore this question on the following pages.

2.1 First enquire: drawing a general picture of students' conceptions

With the help of three different questionnaires we intended to draw a first sketchy picture of students' understanding of algebra.

First study: students check equivalence of equations and inequalities

Since our objective was to track down pseudostructural conceptions, it seemed the right move to draw the bead on the concept of equivalence. Although no procedure is mentioned in its definition, we expected that in order to decide whether two PFs are equivalent, some students would look for transformations by which one of the PFs could be turned into the other. In a previous study devoted to this notion (Steinberg et al., 1991) the researchers reported that usually, "the students could assess the equivalence quickly by observing that one equation was derived from the other by some transformation." By itself, this result cannot yet be regarded as an evidence for pseudostructural conceptions. Such conclusion would become justified only if we could show that the student uses the criterion of transformation automatically and never returns to the underlying processes and abstract objects in order to verify his conclusions. We decided, therefore, that as a tool for spotting pseudostructural conceptions we should use non-standard pairs of PFs which, in the case of such automatic behavior, would lead to inconsistency with the definition of equivalence.

To construct the set of items presented in Figure 3, we looked for four pairs of equations (and four pairs of inequalities) that would represent all the possible combinations of two parameters: *equivalence* according to the structural definition on one hand, and, on the other hand, possibility to *transform* one of the PFs in into the other by help of symbolic manipulations. Let us have a closer look at each of the categories.

FIG. 3: THE FIRST QUESTIONNAIRE AND ITS RESULTS

	TRANSFORMABLE (T)				NON-TRANSFORMABLE (-T)			
	item	item	IA	NA	item	item	IA	NA
EQUI-VALENT (E)	a	$4x-11=2x-7$ $4x=2x+4$	9	2	c	$4x-11=2x-7$ $(x-2)^2=0$	68	15
	b	$5x+4<11(x+2)$ $4<6x+22$	18	11	d	$5x+4<11(x+2)$ $4x+5>x-4$	54	19
NOT EQUI-VALENT	e	$(3x-1)(2x-5)=x(3x-1)$ $2x+5=x$	28	17	g	$7x+2=3x+1$ $4x=5$	8	17
(-E)	f	$4x^2>9$ $2x>3$	45	43	h	$3x+2<1-7x$ $5(x-1)>6$	9	35

IA: % of answers which are inconsistent with the definition of equivalence
NA: % of students who gave no answer

While (E,T) and $(-E,-T)$ (items a,b and g,h, respectively) consist of quite standard examples (pairs which either satisfy or do not satisfy both the requirement of equivalence and that of formal transformability), the remaining two groups were expected to pose a difficulty for some pupils.

In $(-E,T)$ (items e,f), the PFs in a pair are not equivalent in spite of the fact that one of them may be formally transformed into the other. Clearly, there is a contradiction between these two conditions, so at least one of them must only *seem* to be satisfied. Indeed, neither the division of both sides by $3x - 1$ (example e), nor the extraction of the square root from both sides of inequality (*f*) is a permitted operation. Nevertheless, our experience as teachers taught us that some students do use this kinds of operations without the necessary precautions. Such behavior can be interpreted as an indication of student's inability to go back to the primary processes in order to verify their decisions.

The category $(E,-T)$ (items c,d) is also non-standard, and to some people may seem counterintuitive: the PFs are equivalent according to the criterion of equal truth-sets, but no "natural" sequence of elementary operations would transform one of them into the other.

For a researcher, curiosities and non-standard examples create rare opportunity for probing student's understanding of different concepts. By exposing the student to such deceptive examples like those in category $(-E,T)$ and to such unexpected (some would say unnatural) ones like those in $(E,-T)$, we hoped to assess their readiness to go beyond standard procedures and to think in terms of the underlying processes and abstract objects. In this context, their answers to the question about equivalence were less important than the verbal explanation they were required to give in order to justify their decisions.

The questionnaire was administered to our sample of 280 students of different ages (see Figure 1). Although there were some subtle differences between the results obtained in various subgroups, all the findings clearly indicated the same tendency. Because of this, and because of space limitations, we shall report here only the general results. By the time the study was carried out, all the pupils have already had quite a long experience with the topics on which our questions were focused. For all of them, solving equations and inequalities was a basic skill, an indispensable ingredient of their everyday mathematical activity. Even so, in items c, d, f, and g, relatively high percentages of the respondents gave answers which were inconsistent with the definition of equivalence (see Figure 3). Thus, according to our expectations, the students' behavior in categories $(-E,T)$ and $(E,-T)$ indicated that for many of them, the formal transformability was practically the only criterion for equivalence. Moreover, the answers to the questions e and f showed that the decisions whether a given transformation is permissible or not had often been quite arbitrary, and certainly had not been based on any requirements regarding underlying processes and objects (it should be noticed that in item e, the percentage of answers inconsistent with the definitions was substantially lower than in c, d, and f; this can probably be explained by the fact that careless division of both sides by an expression containing the variable is one of those common mistakes against which teachers repeatedly warn their students).

The arguments with which the students justified their answers are summarized in Figure 4. The findings seem to reinforce the impression that for many respondents, an equation or inequality was nothing more than a string of symbols which can be manipulated

FIG. 4: ARGUMENTS GIVEN BY THE RESPONDENTS TO SUPPORT THEIR ANSWERS

	EQUATIONS						INEQUALITIES					
	item	T	S	F	OOF	NA	item	T	S	F	OOF	NA
(E,T)	a	30	22	11	5	31	b	20	28	9	2	41
$(E,-T)$	c	52	29	4	4	12	d	47	19	4	3	27
$(-E,T)$	e	40	16	9	2	33	f	41	16	9	7	26
$(-E,-T)$	g	20	23	16	3	38	h	20	18	6	3	53

T: argument based on an attempt to formally Transform one PF into the other
S: argument based on full Solution of PFs and comparison of the results
F: argument based on the similarity or differences in the Form of both PFs
OOF: Out-Of-Focus response
NA: No Argument

according to certain arbitrary rules. Of those pupils who did explain their decisions, the majority used the transformability as a criterion. Many others leaned on purely external features of the PFs, such as partial similarity and partial difference between their component formulae. Under the title "out-off-focus arguments" we have collected all the responses in which different mathematical entities have been confused (for example, two sides of the same equation has been compared in order to answer the question about the equivalence between this equation and another). Although some of the above arguments could be given also by a student who fully adopted the structural approach, in majority of cases they may only be interpreted as indicative of pseudostructural conceptions. Indeed, more often than not, they have been brought to support an incorrect claim about equivalence of two PFs. The possibility that student's understanding was merely instrumental cannot be dismissed even in those cases in which the respondents solved both equations and compared the solutions. Although this is exactly what should be done according to the definition of equivalence, the respondent's actions could sometimes be dictated by a habit rather than by the deep relational comprehension.

All this shows students' inability to relate permissible operations to the truth set of an equation and, in consequence, to the primary processes underlying these concepts. This is in a perfect agreement with the results obtained by Steinberg, Sleeman, and Katorza (1991), according to which "many students are not sure that an equation that has been derived by a valid transformation has the same solution or are unable to recognize when an equation has been transformed in a way that does not alter the answer."

Second study: students talk about the meaning of algebra

Since no questioning technique seems to stand alone as a method of discovering the ways students think about abstract mathematical concepts, we decided to supplement the non-standard equivalence problems with two other types of questionnaire. Both of them were answered by the same 280 secondary-school pupils who participated in the first part of our study.

In the investigation which will be presented in this section, the respondent was directly interrogated on the meaning of such basic algebraic notions as solving an equation, permissible operation, equivalence of equations. The questionnaire consisted of four types of sentences which had to be completed by the student. Each of these sentence-types was first applied to equations and then to inequalities (see Figure 5). Let us have a quick glance on the categories into which students' responses were classified. First, all of them were crudely divided into two groups: non-informative answers on the one hand, and the answers which seemed to convey a reasonably clear message as to the way the respondent thinks about propositional formulae, on the other hand. In the first category, except for the cases in which there was no answer at all (NA), two sub-categories have been distinguished: tautological statements, namely the answers which added no information (T) and the out-of-focus statements (OF) in which the respondent confused several mathematical entities. The informative responses had been divided into sub-categories according to the central idea through which the students tried to explain the concept of question. Three such ideas were identified: truth set (TS), formal transformations (FT), and "the answer" – the expression or number which is produced by the solving procedure (AN).

According to the numbers presented in Figure 5, the high occurrence of the non-informative answers is probably the most significant of our findings. This phenomenon cannot be explained just by saying that the respondents' effort was not sincere enough – in all but two of the items, more than two thirds of the students (and sometimes as much as 90%) did try to give an answer. Nevertheless, they were not able to produce more than out-of-focus utterances such as "Two equations are equivalent when there is an equality between two sides" (item 4E); or tautological statements such as "The operations are permitted because without them we wouldn't be able to solve inequalities" (2I) or just "This is the nature of mathematics" (2E). All this implies once again that for the majority of students, solving equations and inequalities is not a very meaningful activity. The numbers show that this claim applies to inequalities even better than to equations.

FIG. 5: THE DISTRIBUTION OF ANSWERS (in %) TO THE QUESTIONNAIRE ON THE BASIC CONCEPTS RELATED TO EQUATIONS AND INEQUALITIES

N = 280, E = equation, I = inequality		FOCUS ON			NON-INF.		
ITEM		TS	FT	AN	OF	TA	NA
1. To solve an equation [inequality] means.......	E	3	62	7	7	8	13
	I	1	1	15	50	–	30
2. Such operations as adding the same number to both sides of an equation [inequality] are permitted because....	E	–	19	12	11	37	21
	I	–	12	7	14	23	44
3. When we solve an equation [inequality], in the end we arrive at.....	E	4	1	63	10	12	10
	I	2	–	19	38	8	33
4. Two equations [inequalities] are called equivalent if	E	4	1	45	35	2	13
	I	2	1	21	36	1	39

Let us now try to decipher the message conveyed by the responses which we classified as informative. Perhaps the next most striking finding is the very low occurrence of strictly structural answers (TS) – the answers which define an equivalence of PFs as an equality of their truth sets. Since this is the way the subject has been taught to our respondents at school, it did not seemed unreasonable to expect this would be a frequent, if not the leading, kind of answer.

In some of the items, and especially in the first, the students preferred to focus on formal transformations. For example, they claimed that to solve an equation or inequality means "to play with both its sides" or "to simplify it as much as possible" (1E). Thus, they clearly identified the solution with an algebraic process (with what must be done) rather than with its product (with the result we want to get). At the same time it should be noted that no evidence was found to show that the pupils had more than a superficial understanding of the secondary processes. In response to the question why the formal operations on PFs are permitted (item 2), many of the pupils gave such answers like "because they are performed on both sides" or "because they make the equation simpler and easier to solve". No real justification of the "permissible" operations was suggested, so in the eyes of the pupil the laws of equations solving were clearly not more than arbitrary "rules of the game".

This impression becomes even stronger when we consider yet another type of utterances – those that focus on "getting the answer" or on "finding the x" (category AN). Although such statement as "When we solve an equation, we arrive at x" (item 3E) may be regarded as based on the pre-Vitean way of understanding letters in equations (as *unknowns* rather than as *variables* instead of which any number may be substituted), the other results suggest an alternative interpretation. Indeed, in the answers grouped in AN category, the students never tried to explain the nature of "the x" ("the answer"). For example, none of them mentioned that what is found is the number for which the equality holds. Thus, it seems plausible that for at least some of the respondents the regular elementary formulae, namely the expressions of the form "x =number" or "x >number", were not more than "halting signals", mere signs that the process of solving an equation or inequality came to its end.

Third study: students tackle singularities

Our next step was to test the tentative conclusions from the former study by watching more closely the ways the pupils apply the knowledge in some special situations. Our supposition that the learners interpret an expression of the form "x =number" just as a sign which signalizes the completion of a solution process gave rise to the hypothesis that the pupils will not be able to cope with singular PFs – the PFs in which the variable disappears at a certain stage of the solution process. Indeed, if the students are "programmed" to see a problem as solved only when a certain expression is obtained, then in a situation in which such expression does not appear at all they will feel lost and helpless.

An experienced teacher does not need a systematic research to know that this conjecture is probably true. An exemplary evidence was provided to us recently by a teacher who reported less than 15% rate of success on a test consisting of three items – all of them systems of linear equations with truth set equal to {} (no roots) or to the set of all the possible substitutions. The 20 tenth-graders to whom the test was administered were otherwise quite successful. When faced with the singularities, some of them wrote sentences like "There is no logic to it" or "Something went wrong here", and stressed their exasperation with many exclamation marks. Reportedly, this was *not* their first encounter with singularities.

To get a clearer picture of the situation, we designed a test of our own in which the respondents were asked to decide whether certain pairs of PFs were equivalent or not

(Figure 6). All the PFs were singular, with TS equal to or to R. Since any two PFs belonging to the same pair had identical truth set, all the pairs should be regarded as equivalent. In the light of our previous findings, however, we expected that when the singular PFs are concerned, our respondents would be more inclined than ever to look for the possibility of transforming one PF into the other (they just would not be able to base their responses on "final answers" of the usual type). An adequate transformation could easily be found in only half of the cases.

The results presented in Figure 6 seem to confirm our suppositions with particular force. The percentage of the answers consistent with the defition of equivalence never exceeded 65%, and in three out of the four cases of non-transformable couples the scores were as low as 10-11%. These numbers are even lower than those obtained in our first study, which was a priori assessed as harder.

FIG. 6: THE QUESTIONNAIRE ON SINGULAR PFs AND ITS RESULTS (in %)

N = 280	TRANSFORMABLE				NON-TRANSFORMABLE			
		ITEM	IA	NA		ITEM	IA	NA
$\{\}$	a_1	$(x+2)^2=x$ $x^2+3x+4=0$	51	10	c_1	$x^2+1=0$ $x^2+5=0$	82	8
					c^2	$x^2+3x+4=0$ $2(x+1)=2x+5$	60	29
	a_2	$5(3x-1)>15x+7$ $15x>15x+12$	24	11	c_3	$(x+3)^2+2<0$ $5x-2>5x$	61	28
∞	b_1	$6x-2=3(2x-5)+13$ $6x-15=3(2x-5)$	14	25	d_1	$6x-2=3(2x-5)+13$ $(x-2)(x+2)=x^2-4$	32	16
	b_2	$x(x+1)+3>x$ $x^2+3>0$	19	20	d_2	$x^2+3>0$ $2x+5>2(x+1)$	59	24

IA - the answers which are inconsistent with the definition of equivalence
NA - no answer

2.2 Second enquire: A closeup on students' conceptions

So far, our studies have shown what the students cannot do rather than what they think and imagine while dealing with equations and inequalities. From what we saw we concluded that in spite of the carefully designed curriculum, the learners do not seem to follow the path they are supposed to make while building their conceptions, and they do not understand algebraic constructs and procedures in the way dictated by textbook definitions. First and foremost, many of them do not see how the secondary processes grow out of the rules of arithmetic.

When describing students' perception of the secondary processes, we said it was "not very meaningful". It is our goal now to make a revision of this utterance. Saying that any kind of mathematical activity is not very meaningful in the eyes of a student is neither revelational nor informative. In fact, it is not even quite correct. To use Davis' (1988) statement, "students usually do deal with meanings, and when instructional programs fail to develop appropriate meanings, students develop their own meanings – meanings that sometime are not appropriate at all." (p. 9) In other words, there must be some inner logic, some consistency, to the actions performed by the learners and to the decisions they make while solving equations and inequalities. In a series of interviews that followed the studies presented in the previous section we tried to fathom the nature of students' idiosyncratic algebra. To be more precise, we aimed at finding the meanings conferred by the students on algebraic procedures. As we stated already more than once, for the majority of learners, algebraic manipulations do not draw their justification from being generalized laws of arithmetic. If so, the question arises what kind of alternative links glue algebraic concepts and procedures into a coherent whole.

Through the interviews with 14-16 year olds we were able to put our fingers on a few salient traits of algebraic conception shared by many learners. In the remainder of this section we shall present what is probably the most popular vision of secondary processes.

1. Arbitrariness

Historically, algebra emerged as a generalization of arithmetic. This is also the way we try to present it to a student. In order to make the rules of algebra meaningful to the learner, we reach outside the algebra itself, to a more primitive world of numerical computations. As was shown above, this attempt to justify one system with the help of another is often far from successful. Many students tend to view algebra as a world in itself, with all its objects and procedures subjected to certain internal laws, existing only within the boundaries of this world, independently of any external factors.

Although in these circumstances the origins of the rules of algebra may seem arbitrary, once they are established and accepted they create a consistent system. Algebraic manipulations are considered as ways of making formal expressions simpler. For example, in the case of an equations with a variable x, the procedure we choose is aimed at obtaining an expression of the form "x =number". It does not matter very much what is the meaning of this final formula. As we already observed, its main significance stems from the fact that it signalizes the end of the solving procedure. In the study reported in the last section we saw that many students could not cope with equations which do not lead to this typical "halting signal". In our interviews we could have a closer glimpse at this phenomenon. We noticed that whenever x disappeared before an elementary expression was reached, the students tended to declare non-existence of a solution regardless of the exact shape of the x-free formula obtained in the end.

EXAMPLE. While faced with the system of equations with parameter k:

$$kx - y = 1$$
$$x - y = 3$$

15 year old Mariella declared that "k cannot be equal 1 because then this [x and y] will disappear and there will be no solution."

Needless to say, interpreting the disappearance of of the variable as an absence of solutions would often result in a false answer.

EXAMPLE. When asked about the equivalence of the following two equations:

$$(x-3)(x+3) = x^2 - 9 \quad \text{and} \quad (x+2)^2 = x$$

16 year old Ronnen (R) opened the brackets of the first equation and obtained $x^2 - 9 = x^2 - 9$.

R.: The different elements cancel each other.... There is no... there are no... let's see whether in the second equation we get the same.

The boy opened the brackets in the second equation, brought it to the canonical form $x^2 + 3x + 4 = 0$, tried to solve and found out that it had no solutions.

R.: There are no solutions to this equation. So, these two are equivalent. It seems to me that both of them have no solutions. They both are cancelled, in the end.

In such cases as the those presented above a functional approach is needed to interpret the result of transformations. Indeed, one must think about an equation as a comparison of functions in order to realize that its truth set is R. Both Mariella and Ronnen evidently lacked the necessary flexibility of thinking.

2. *Intuitive acceptability*

Although the rules of algebra seem arbitrary, they are intuitively acceptable. Students often justify secondary operations by saying that they don't change the equation.

Indeed, many people feel that the permissible operations leave equation "unchanged" and they explain it by stating that whatever is done to both sides of PF "preserves the balance". The learner may be unable to explain the nature of this balance or to pinpoint the aspects which remain unaffected by the permissible operations. Here, she or he may run into difficulties similar to those experienced by a person who tries to explicate the principles of face recognition. The inability to explain, however, does not necessarily undermine the strong intuitive belief that as long as the same operation was performed on both sides of an equation, the equation remained "the same". This intuition is often based on an *analogy* with the rules of arithmetic rather than on a their conscious generalization.

EXAMPLE. 15 year old Naomi (N) added a number to both sides of an equation. She stated that the resulting equation was equivalent to the original.

I.: Why are these two equations equivalent?

N.: When I add a number to both sides I don't change anything, because it is balanced... It's like when I have a fraction, say 4/8. If I divide [the nominator and denominator] by 4, I'll get the same value: $1/2 = 4/8$. It's the same, it's equivalent.

The conviction that any operation "preserves the balance" as long as it is performed on both sides of a PR is, in a sense, primary and does not seem to require further justification. Naturally, it would often lead to a faulty judgment.

EXAMPLE. Dina (see the last example) was judging the equivalence of $4x^2 > 9$ and $2x > 3$.

D.: I think that they are equivalent, because if we take a root from this one [$4x^2 > 9$], we get this one [$2x > 3$]. And I think that it's o.k. to do this, so they are equivalent.

I.: What does it mean that they are equivalent? Could you explain?

D.: That in the beginning they were the same, this equation was exactly the same as this one. Some operation was performed that turned this one into something else. But it *is*, in fact, the same equation.

She chose to point to the "sameness" of the two equations to justify the transformation, even though she was well aware of the "official" reason for the equivalence:

I.: So this is the meaning of the notion of equivalence?

D.: No. Equivalence means that the unknown is the same.

This last example shows the power of intuition with particular clarity. Dina's own conception of equivalence was so strong that she never felt a need to verify her intuitive judgments with the criteria suggested by the formal definition.

3. Justification by purpose

When "the rules of the game" are established and accepted, it is possible to to find a more "tangible" justification of the operations performed on PFs. One way of doing this is to say that the transformations lead to the purpose we have in mind while solving equations: they simplify the PF and make us closer to an expression of the form "x =number." If so, any operation which does not result in a simpler formula would usually be dismissed.

EXAMPLE. 13 year old Danna (D), who just learned to solve simple linear equations, was presented with the problem:

$$112 = 12x + 28$$

and said that she will subtract 28 from both sides. An interesting exchange with the interviewer (I) followed:

I.: Why do you want to subtract 28 and not, say $12x$?

D.: You can't subtract $12x$. How would you do this? If you had here [on the left side] $17x$, for example, and here [on the right] $12x$, then you could subtract $12x$ from both sides. You can only subtract $12x$ from both sides when there is x on both sides. When there is x only on one side we can't [do it, because] we won't reach any result.

I.: What do you mean by "we can't". That it is not allowed or just that it would not be helpful?

D.: [After a long pause] Perhaps we can do it... maybe it is not forbidden.

For the same reason, any operation that would simplify a PF if performed on its both sides runs a good chance of being accepted as permissible. Here is a representative example showing the common consequences of the confusion between the legality of the a transformation and its effectiveness.

EXAMPLE. 15 year old Erez (E), was solving an equation

$$(3x - 1)(2x + 5) = x(3x - 1)$$

E.: Perhaps I could divide [both sides of the equation]. Say, I'd divide by $3x - 1$... Yes, it's fine.

I.: What are the rules of dividing [an equation]?

E.: We can do this when there is a multiplication in the equation.

4. Relativity of the final result

Maybe the most striking of our findings was the discovery that in the eyes of some of our interviewees the final solution of an equation might not be uniquely determined by the equation itself and could be dependent on the procedure chosen by the solver. Student's belief in the correctness of the "balance-preserving" operations and in the necessity of the regular "halting signal" might be stronger than any other considerations. Sometimes, the student would accept a possibility of two different answers to the same question rather than reject a procedure which he intuitively regards as correct.

EXAMPLE. Erez (see the last example) was asked to check the equivalence of the following two equations:

$$2x + 5 = x \quad \text{and} \quad (3x - 1)(2x + 5) = x(3x - 1)$$

He solved the first equation and obtained $x = -5$. Then, to solve the second equation, he decided to divide its both sides by $3x - 1$. The resulting linear equation yielded the number -5 again. Erez concluded that the two equations were equivalent. Then he was asked by the interviewer to open the brackets in the second equation and solve it again. This time he found two solutions, 1/3 and -5. Once again, he was asked to assess the equivalence of the original equations.

E.: I'll substitute -5 here [in $2x - 5 = x$], and I'll get.. $-10 + 5 = -5$... that's o.k.

I.: What about 1/3?

E.: With 1/3? [substitutes in $2x - 5 = x$ and obtains $2/3 + 5 = 1/3$]. No, for this x it's not true.

I.: So, are these two equations equivalent or not?

E.: So they are not equivalent.

I.: You said before that they are equivalent. So what is your final answer? Are they or aren't they equivalent?

E.: It's true that they are equivalent as long as I don't solve it [the second equation] with the formula [for the roots of quadratic equations]. But the moment I used the formula, it is true only for one x, -5.

The acceptance of the idea that several procedures leading to different answers may all be correct was particularly evident in the case of singularities. Some of our interviewees interpreted the disappearance of the variables from an equation as a "lack of solutions", but at the same time expressed their belief that a different procedure could lead to a different result.

EXAMPLE. Sixteen year old Dina (D) was solving the singular system of linear equations:

$$2(x - 3) = 1 - y$$

$$2x + y = 7.$$

She arrived at an equation $0 = 0$.

D.: This means that it is true.

I.: What do you mean? What are the solutions of this equation? How many solutions are there?

D.: I don't know. I didn't manage to arrive at them.

I.: Are there any solutions here at all? What do you think?

D.: In the way I know, there are none.

One way to interpret this phenomenon is to say that it is the whole procedure which transforms an equation into an elementary expression, rather than its final outcome, that is viewed by a student as a solution to the problem.

3. Conclusions and implications

Not surprisingly, our empirical studies have shown a clear tendency toward conceptions which have very little to do with the structural definition of equivalence offered by the textbooks. More often than not, students' answers were inconsistent with this definition – the phenomenon that could only be explained by the fact that the respondents based their decisions on the existence or absence of formal transformations rather than on the equality or inequality of truth sets. The pupils leaned heavily on *secondary* processes, apparently feeling no need to justify these procedures by any factors external to algebra itself. They viewed the rules of algebra as arbitrary, even if intuitively acceptable, and justifiable only by their purpose. Since no references had been made by the pupils to the underlying primary processes, we were inclined to interpret these findings as indicative of pseudostructural conceptions.

At the first glance, what was found in our investigations brings to mind the views promoted by some leading nineteenth and twentieth century mathematicians. According to the formalist school, which was introduced to algebra by Peacock and deMorgan (see e.g. [1], [3]) and was later transferred to a much broader context by Hilbert and his followers, the mathematical symbols, although interpretable in many different ways, have no meaning of their own. From the assumption of "semantical emptiness" Peacock soon arrived at a complete de-arithmetization of algebra: since the meaning of symbols can no longer be expected to come from their non-existent designata, it must be sought in the way the formulae are transformed and combined with each other. These transformations, in their turn, are the basic elements in which all the algebra takes its roots – and they are totally arbitrary.

On the face of it, this is exactly the kind of conception that was displayed by the participants of our questionnaire. The similarity, however, may be deceitful. Indeed, what for the British mathematicians was a result of a deliberate and well calculated move, may be potentially dangerous for a mathematically unsophisticated student. Peacock's request to strip algebraic symbols from their initial semantic load originated in a conscious decision of a person who knew exactly what he was going to give up, and who was perfectly able to go back to the renounced meanings whenever appropriate. As we stressed more than once in this paper, such occasional returns seem to be indispensable for successful problem-solving. Whether today's students posses such flexibility of thinking is a crucially important question, and it was the purpose of our investigation to provide it with an answer.

A closer look at our findings made us realize that the similarity of students' conceptions to the views expressed by Peacock and his colleagues is very superficial indeed. The differences are more significant than the common traits. For the majority of pupils, it seems, an equation and inequality are meaningless strings of symbols to which certain well-defined procedures are routinely applied. The beliefs about the nature of these procedures is where the formalists and the today's students part. Although both the mathematicians and the pupils view the formal operations as arbitrary, for the formalist such approach is a matter of a deliberate choice, while for the student it is an inevitable outcome of his or her basic inability to link algebraic rules to the laws of arithmetic. This inability is evidenced by the fact that the pupils cannot cope with problems which require flexibility of thinking and do not yield to the standard solving algorithms. By a "standard solving algorithm" we mean a chain of simplifying transformations which ends when no further simplification can be made. The last element of such chain is considered to be the solution of the equation or the inequality. Pupils' ability to translate this final expression into a truth set seems to be very limited. This is probably why our respondents' performance was considerably worse for inequalities than for equations (the expression "$x < a$" does not define a concrete number, thus is more difficult to interpret than "$x = a$"), and practically disastrous for singular PFs.

In short, an equation or inequality seems to be for a student a thing in itself, for which the formal manipulations are the only source of meaning. Some ideas regarding the possible ways of fighting pseudostructural conceptions are now being tested in an ongoing study.

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