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HAMILTONIAN STRUCTURES ON MULTISYMPLECTIC MANIFOLDS

Abstract. After a brief review of some basic notions of multisymplectic geometry and a discussion of some examples, special attention is paid to the concepts of Hamiltonian multivector field and Hamiltonian form on a multisymplectic manifold. In particular, it is shown that the space of equivalence classes of Hamiltonian forms, modulo closed forms, can be equipped with a graded Lie algebra structure. Next, it is demonstrated that the tangent bundle of a multisymplectic manifold is also multisymplectic, and that a locally Hamiltonian vector field is determined by a Lagrangian section of this "tangent multisymplectic structure".

1. Introduction

Apart from his fundamental work on the symplectic approach to statics and to nonrelativistic and relativistic particle dynamics, W.M. Tulczyjew has also made important contributions to the symplectic formulation of field theories (cf. [11, 20, 21]). His ideas on the subject have also been inspiring for others working in this area and have brought the notions of multisymplectic structure and multiphase space into prominence (see e.g. [9, 10]).

In the context of the covariant approach to Hamiltonian field theory, the concept of multisymplectic structure essentially refers to the canonical structure living on a bundle of exterior forms on a (fibred) manifold (see, among others, [2, 5, 6, 7, 12, 20]). This structure, which can be regarded as the field-theoretical analogue of the canonical symplectic structure on the phase space of a classical mechanical system, is defined by a closed (more precisely, exact) differential form which is nondegenerate in the sense that its characteristic distribution consists of the zero-vector field only. (An explicit expression of this form will be given in Section 3).

Just as the notion of symplectic manifold is much more encompassing than that of a cotangent bundle, we may extend the notion of multisymplectic manifold (or structure) in order to include any manifold equipped with a differential form which is closed and

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*Senior Research Associate at the Fund for Scientific Research – Flanders (Belgium) nondegenerate. As such, the class of multisymplectic manifolds contains as special subclasses those of the symplectic and the orientable manifolds (a volume form being multisymplectic in the above sense). Properties of general multisymplectic structures therefore in particular apply to the symplectic and orientable cases. Conversely, it is also of interest to find out which properties of the latter two structures admit an extension to the more general multisymplectic setting. The previous arguments already suffice to say that, from a purely mathematical point of view, multisymplectic geometry is a topic worth investigating in its own right. Inspired by some interesting work on the subject by G. Martin ([16, 17]), we have embarked in [1] on a systematic study of the geometry of multisymplectic manifolds. In the present contribution we will mainly concentrate on certain aspects of Hamiltonian structures (i.e. Hamiltonian (multi-)vector fields and Hamiltonian forms) on multisymplectic manifolds.

The scheme of this paper is as follows. In the next section we will briefly recall some of the basic concepts introduced in [1], and Section 3 will be devoted to some illustrative examples of multisymplectic manifolds. Section 4 then deals with the notions of Hamiltonian multivector field and Hamiltonian form. The main result thereby consists in the identification of a particular graded Lie algebra structure on the space of (equivalence classes of) Hamiltonian forms, modulo closed forms. Finally, in Section 5, extending a known result from symplectic geometry, we show that the tangent bundle TM of a multisymplectic manifold M admits a natural multisymplectic structure, and that a locally Hamiltonian vector field on M can be fully characterised in terms of a Lagrangian submanifold of TM (with respect to this induced multisymplectic structure).

Throughout this paper, we confine ourselves to real, finite dimensional smooth manifolds. The bundle of exterior k-forms and the bundle of k-vectors (i.e. contravariant skew-symmetric k-tensors) on a manifold M will be denoted by $\bigwedge^k M$ and $V_k M$, respectively. The fibres of these bundles over a point $x \in M$ are given by $\bigwedge^k(T_x^* M)$ and $\bigwedge^k(T_x M)$, respectively. For $k = 1$ we have $\bigwedge^1 M = T^* M$ and $V_1 M = TM$. The modules of differential k-forms and k-vector fields on M, will be denoted by $\Omega^k(M)$ and $V_k(M)$, respectively. The contraction of a vector (field) X and an exterior (differential) form α will be written as $i_X \alpha$.

2. General definitions

For more details concerning the material presented in this section, we refer to [1].

Starting with the linear case, we say that a *multisymplectic vector space* of order $k+1$ $(1 < k+1 \leq \dim V)$ is a pair (V, ω) consisting of a (real) vector space V and an exterior $(k+1)$-form ω on V such that for $v \in V$, $i_v \omega = 0$ iff $v = 0$. For any subspace W of V and for each ℓ, with $1 \leq \ell \leq k$, the ℓ–th *orthogonal complement of* W with respect to ω is the linear subspace of V defined by

$$W^{\perp,\ell} = \{v \in V \mid i_{v \wedge w_1 \wedge \cdots \wedge w_\ell} \omega = 0, \text{for all}\, w_i \in W, i = 1, \ldots, \ell\}.$$

In particular, we have the following filtration of orthogonal complements

$$W^{\perp,1} \subseteq W^{\perp,2} \subseteq \cdots \subseteq W^{\perp,k}.$$

Obviously, $W^{\perp,\ell} = V$ whenever $\ell > \dim W$ and

$$W \cap W^{\perp,k} = \ker(\omega|_W) = \{w \in W \mid i_w(\omega|_W) = 0\}.$$

A subspace W of a multisymplectic vector space (V,ω) will be called *ℓ-isotropic* if $W \subset W^{\perp,\ell}$, *ℓ-coisotropic* if $W^{\perp,\ell} \subset W$ and *ℓ-Lagrangian* if $W = W^{\perp,\ell}$. For $k = 1$ (i.e. ω a 2-form) the previous notions all reduce to the familiar ones from linear symplectic geometry.

The transition from the linear to the differentiable setting is easily made. A *multisymplectic manifold* of order $k+1$ is a pair (M,ω) consisting of a smooth (real, finite dimensional) manifold M and a $(k+1)$-form ω on M which is closed, i.e. $d\omega = 0$, and nondegenerate in the sense that for a vector field X on M,

$$i_X\omega = 0 \quad \text{iff} \quad X = 0.$$

Any differential form verifying both properties (i.e. being closed and nondegenerate) is called a multisymplectic form.

Given a multisymplectic manifold (M,ω) of order $k+1$ $(1 < k+1 \leq \dim M)$, the form ω induces k vector bundle homomorphisms

$$\hat{\omega}_j : V_j M \longrightarrow \bigwedge^{k+1-j} M, \quad U \in \bigwedge^{j}(T_xM) \longmapsto i_U\omega(x), \tag{2.1}$$

for $j = 1,\dots,k$. The nondegeneracy of ω in particular implies that $\hat{\omega}_1 : TM \to \bigwedge^k M$ is injective and $\hat{\omega}_k : V_k M \to T^*M$ is surjective.

A submanifold N of a multisymplectic manifold (M,ω) will be called *ℓ-isotropic* (resp. *ℓ-coisotropic*, *ℓ-Lagrangian*), for $1 \leq \ell \leq k$, if for each $x \in N$, T_xN is a ℓ-isotropic (resp. ℓ-coisotropic, ℓ-Lagrangian) subspace of the multisymplectic vector space $(T_xM, \omega(x))$.

Given two multisymplectic manifolds (M_1,ω_1) and (M_2,ω_2) of equal order, a *multisymplectomorphism* (or, multisymplectic diffeomorphism) is a diffeomorphism $\Psi : M_1 \to M_2$ such that $\Psi^*\omega_2 = \omega_1$. It is known that symplectic and volume forms belong to the category of geometric structures which, in the spirit of Klein's Erlangen programme, are fully characterised by their respective groups of automorphisms. Therefore, it would certainly be of interest to find out whether such a property also applies to more general multisymplectic structures. Work along these lines is in progress.

3. Some examples

(i) Let E be an arbitrary n-dimensional smooth manifold and consider the bundle $\bigwedge^k E$ of exterior k-forms on E, with natural projection $\rho_k : \bigwedge^k E \to E$. Local coordinates on E are denoted by $(q^1,\dots,q^n)$ and the induced bundle coordinates on $\bigwedge^k E$ are denoted by $(q^i, p_{i_1 \ldots i_k})$, with $1 \le i_1 < \cdots < i_k \le n$. On $\bigwedge^k E$ there exists a canonical k-form Θ_E which can be defined in a way completely similar to the Liouville 1-form on a cotangent bundle (see e.g. [1]), and which in canonical coordinates is given by

$$\Theta_E = p_{i_1 \ldots i_k} dq^{i_1} \wedge \ldots \wedge dq^{i_k},$$

with summation running over all $1 \le i_1 < \cdots < i_k \le n$. It is then found that the exact $(k+1)$-form $\Omega_E = d\Theta_E$, which in coordinates reads

$$\Omega_E = dp_{i_1 \ldots i_k} \wedge dq^{i_1} \wedge \ldots \wedge dq^{i_k}, \tag{3.1}$$

is nondegenerate and, consequently, $(\bigwedge^k E, \Omega_E)$ is a multisymplectic manifold of order $k+1$. In addition, one can show that the fibres of $\bigwedge^k E$ are 1-Lagrangian submanifolds and that E (identified with the zero section of p_k) and, more generally, the image of any closed k-form defined on E, determines a k-Lagrangian submanifold of $(\bigwedge^k E, \Omega_E)$ (cf. [1]). This clearly yields an extension of the properties of a cotangent bundle.

Next, assume that E is fibred over some manifold M, with projection $\pi : E \to M$. For any positive integer r, with $1 \le r \le k-1$, let $\bigwedge^k_r E$ denote the bundle of $(k-r)$-horizontal k-forms on E, i.e. those k-forms which vanish whenever $r+1$ of its arguments are vertical tangent vectors (with respect to π). This bundle is a subbundle of $\bigwedge^k E$ and it is not difficult to verify that the pull-back of Ω_E to $\bigwedge^k_r E$ still determines a multisymplectic structure (see [1]). For $k = \dim M$ and $r = 1$, this is precisely the multisymplectic structure appearing in first-order field theory (see e.g. [2, 5, 11]).

It should be emphasised that, except in the symplectic case, a general multisymplectic manifold need not even be locally isomorphic to a bundle of exterior forms. Stated otherwise, an arbitrary multisymplectic form need not have a local representative of the form (3.1). In fact, in order for there to be a kind of Darboux theorem in the multisymplectic setting, one has to narrow the concept of multisymplectic manifold by imposing additional conditions (cf. [1,16]).

(ii) A volume form ν on an n-dimensional orientable manifold M yields a multisymplectic strucure of order n. In this case, the induced bundle homomorphisms $\hat{\nu}_j : V_j M \to \bigwedge^{n-j} M$, defined by (2.1), are all isomorphisms. Moreover, any ℓ-dimensional submanifold of (M, ν) is ℓ-Lagrangian (cf. [1]).

(iii) A symplectic manifold (M, ω) is obviously multisymplectic (of order 2). Moreover, if $\dim M = 2n$, the symplectic form ω induces $n-1$ additional multisymplectic structures on M of even order, respectively, $4, 6, \ldots, 2n$, namely:

$$\omega \wedge \omega; \quad \ldots, \quad \overbrace{\omega \wedge \cdots \wedge \omega}^{n},$$

whereby the last one is (up to a numerical factor) the standard volume form generated by ω.

(iv) Let G be a compact semisimple Lie group and let $< \, , \, >$ denote the left invariant metric on G induced by the Cartan-Killing metric of its Lie algebra. One can then define the following 3-form on G:

$$\omega(X,Y,Z) =< X,[Y,Z] > + \text{cycl.}$$

where, on the right-hand side, the cyclic sum is taken with respect to the vector fields X,Y,Z. Choose a basis θ^i of left invariant 1- forms on G and denote the stucture constants and the components of the Cartan-Killing form (with respect to the dual basis of the Lie algebra) by c^i_{jk} and g_{ij}, respectively. The form ω then reads

$$\omega = g_{ij}c^j_{k\ell}\theta^i \wedge \theta^k \wedge \theta^\ell.$$

It is straightforward to check that ω is closed and nondegenerate and, hence, defines a multisymplectic structure of order 3 on G.

To close this section we still mention, without going into details, that multisymplectic structures may also appear, for instance, in the study of manifolds with transversally orientable foliations, where the multisymplectic form may be related to the so-called Godbillon-Vey invariant of the foliation (see e.g. [3, 18] for a definition of this invariant). A simple but illustrative example is given by the following. Consider the Lie group $Sl(2,\Re)$ and let Γ be a discrete subgroup such that the qoutient space $M = Sl(2,\Re)/\Gamma$ is compact. It has been pointed out in [3] that M admits a co-dimension 1-foliation, induced by (the projection of) a left-invariant 1-form on $Sl(2,\Re)$, and a volume form, the (nontrivial) cohomology class of which is an invariant of the foliation.

4. Hamiltonian (multi-)vector fields and Hamiltonian forms

Let (M,ω) be a multisymplectic manifold of order $k+1$. A vector field X on M will be called *locally Hamiltonian* if $L_X\omega = 0$ (where L_X denotes the Lie derivative with respect to X). In particular, X will be locally Hamiltonian iff its flow consists of (local) multisymplectomorphisms. Note in passing that if M is orientable and ω a volume form (i.e. $k+1 = \dim M$), the locally Hamiltonian vector fields are precisely the divergence free vector fields on M.

If for a vector field X the k-form $i_X\omega$ is exact, i.e. if $i_X\omega = d\alpha$ for some $(k-1)$-form α on M, then X is said to be *Hamiltonian* and α is called a *Hamiltonian form* (of order $k-1$). Obviously, the set of Hamiltonian vector fields is a subset of the set of locally Hamiltonian vector fields. In analogy with the symplectic case, one can in fact show that the locally Hamiltonian vector fields on a multisymplectic manifold constitute a Lie algebra (for the Lie bracket of vector fields) and that the set of Hamiltonian vector fields is an ideal of this Lie algebra.

Let us now introduce some notations. We will use $Z^\ell(M)$ to denote the space of closed ℓ-forms on M (for any $\ell \geq 0$). The set of Hamiltonian forms of order $k-1$ will be denoted by $\mathcal{H}^{k-1}(M)$, i.e.

$$\mathcal{H}^{k-1}(M) = \{\alpha \in \Omega^{k-1}(M) \mid i_X\omega = d\alpha, \text{ for some vector field } X\},$$

and the set of Hamiltonian vector fields by $\mathcal{V}^H_1(M)$. In particular, $\mathcal{H}^{k-1}(M)$ is a real vector space consisting of those $(k-1)$-forms on M the exterior derivative of which are sections of the subbundle $\hat{\omega}_1(TM)$ of $\bigwedge^k M$. In case ω is either a symplectic form ($k = 1$) or a volume form ($k + 1 = \dim M$), the vector bundle homomorphism $\hat{\omega}_1$ is an isomorphism and $\mathcal{H}^{k-1}(M)$ then coincides with $C^\infty(M)$ and with $\Omega^{k-1}(M)$, respectively.

From the nondegeneracy of ω it follows that for a given $\alpha \in \mathcal{H}^{k-1}(M)$, the corresponding Hamiltonian vector field X is uniquely determined, and we denote it by X_α. Conversely, given a Hamiltonian vector field X, a corresponding Hamiltonian form is only determined up to a closed $(k-1)$-form. Since, obviously, $Z^{k-1}(M)$ is a subspace of $\mathcal{H}^{k-1}(M)$, we can introduce the quotient space

$$\tilde{\mathcal{H}}^{k-1}(M) = \mathcal{H}^{k-1}(M)/Z^{k-1}(M),$$

and the above argument shows that the spaces $\mathcal{V}_1^H(M)$ and $\tilde{\mathcal{H}}^{k-1}(M)$ are isomorphic (as linear spaces). Elements of $\tilde{\mathcal{H}}^{k-1}(M)$ will be denoted by $\tilde{\alpha}$, with representative $\alpha \in \mathcal{H}^{k-1}(M)$, i.e. $\tilde{\alpha}$ is the set of all $(k-1)$-forms differing from α by a closed form.

On $\mathcal{H}^{k-1}(M)$ one can define an algebra structure, with product denoted by $\{\ ,\ \}$, as follows. Take any two Hamiltonian forms α, β with associated (uniquely determined) Hamiltonian vector fields X_α, X_β. One immediately finds that

$$i_{[X_\alpha, X_\beta]}\omega = d(i_{X_\alpha} i_{X_\beta}\omega),$$

i.e. $i_{X_\alpha} i_{X_\beta}\omega$ is a representative of the equivalence class of Hamiltonian forms corresponding to the Hamiltonian vector field $[X_\alpha, X_\beta]$. Now, putting

$$\{\alpha, \beta\} = i_{X_\alpha} i_{X_\beta}\omega \tag{4.1}$$

we infer from the above that $[X_\alpha, X_\beta] = X_{\{\alpha,\beta\}}$. One easily verifies that the bracket $\{\ ,\ \}$ is a $\Re$-bilinear skew-symmetric operator (with $\Re$ the field of real numbers) which satisfies the relation

$$\{\alpha, \{\beta, \gamma\}\} + \text{cycl.} = -d i_{X_\alpha} i_{X_\beta} i_{X_\gamma}\omega.$$

Within the framework of the Hamilton-Cartan approach to the calculus of variations, a similar algebraic structure on a class of differential forms has been considered, for instance, in [4]. In the symplectic case, (4.1) yields the Poisson bracket on $C^\infty(M)$. In the general multisymplectic case it is readily seen that $Z^{k-1}(M)$ belongs to the kernel of $(\mathcal{H}^{k-1}(M), \{\ ,\ \})$ and, hence, the bracket $\{\ ,\ \}$ passes to the quotient and induces a genuine Lie algebra structure on $\tilde{\mathcal{H}}^{k-1}(M)$ with

$$\{\tilde{\alpha}, \tilde{\beta}\} \overset{\text{def}}{=} \widetilde{\{\alpha, \beta\}}.$$

The Lie algebra $(\tilde{\mathcal{H}}^{k-1}(M), \{\ ,\ \})$ is isomorphic to the Lie algebra $(\mathcal{V}_1^H(M), [\ ,\ ])$, with isomorphism given by $\mathcal{V}_1^H(M) \to \tilde{\mathcal{H}}^{k-1}(M), X_\alpha \mapsto \tilde{\alpha}$.

The above framework for Hamiltonian vector fields and forms can now be extended to multivector fields and differential forms of various order. Let (M, ω) again be an arbitrary multisymplectic manifold of order $k+1$. A $(k-\ell)$-vector field U on M, with $0 \leq \ell \leq k-1$, is called *locally Hamiltonian* if the $(\ell+1)$-form $i_U\omega$ is closed, and *Hamiltonian* if this form is exact, i.e. if there exists an ℓ-form α such that $i_U\omega = d\alpha$. The form α is then called a *Hamiltonian form* of order ℓ, and the set of these forms is a linear space, denoted by $\mathcal{H}^{\ell}(M)$. Note in particular that, since the bundle homomorphism $\hat{\omega}_k : V_k(M) \to T^*M$ is surjective (cf. Section 2), we have $\mathcal{H}^0(M) = C^{\infty}(M)$.

For a given Hamiltonian multivector field, the corresponding Hamiltonian form is again determined up to a closed form. Conversely, the Hamiltonian multivector field which can be associated with a given Hamiltonian form of order ℓ, is only determined up to a $(k-\ell)$-vector field whose contraction with ω vanishes. The space of Hamiltonian $(k-\ell)$-vector fields will be denoted by $\mathcal{V}^H_{k-\ell}(M)$.

Closed forms are trivially Hamiltonian and, therefore, it is convenient to think of Hamiltonian forms as being defined modulo closed forms. To make this more formal, we consider the quotient spaces

$$\tilde{\mathcal{H}}^{\ell}(M) = \mathcal{H}^{\ell}(M)/Z^{\ell}(M),$$

for $\ell = 0, \ldots, k-1$, with elements denoted by $\tilde{\alpha}$. In case M is connected, $Z^0(M) = \Re$ and then $\tilde{\mathcal{H}}^0(M) = C^{\infty}(M)/\Re$. Note that the elements of $\tilde{\mathcal{H}}^{\ell}(M)$ are in one-to-one correspondence with the equivalence classes of Hamiltonian $(k-\ell)$-vector fields, whereby two elements of $\mathcal{V}^H_{k-\ell}(M)$ are called equivalent if they differ by a $(k-\ell)$-vector field which is annihilated by ω. For later use, we also introduce the direct sum spaces

$$\mathcal{H}^*(M) = \bigoplus_{\ell=0}^{k-1} \mathcal{H}^{\ell}(M), \quad \text{and} \quad \tilde{\mathcal{H}}^*(M) = \bigoplus_{\ell=0}^{k-1} \tilde{\mathcal{H}}^{\ell}(M).$$

Now, let α_1 and α_2 be two Hamiltonian forms of order ℓ_1 and ℓ_2, respectively, with associated Hamiltonian multivector fields $U_1 \in \mathcal{V}^H_{k-\ell_1}(M)$ and $U_2 \in \mathcal{V}^H_{k-\ell_2}(M)$. Let $[U_1, U_2]$ denote the Schouten-Nijenhuis bracket of these multivector fields (see e.g. [23] for a definition and for the properties of this bracket). In particular, $[U_1, U_2]$ is a multivector field of order $2k - (\ell_1 + \ell_2 + 1)$. Using the properties of the Schouten-Nijenhuis bracket, and taking into account that $d\omega = 0, i_{U_1}\omega = d\alpha_1$ and $i_{U_2}\omega = d\alpha_2$, one finds

$$i_{[U_1,U_2]}\omega = -d i_{U_1 \wedge U_2}\omega. \tag{4.2}$$

This shows that $[U_1, U_2]$ is again a Hamiltonian multivector field with representative Hamiltonian form $-i_{U_1\wedge U_2}\omega$. The latter, of course, trivially vanishes whenever $\ell_1+\ell_2+1 < k$. Note that the form $-i_{U_1\wedge U_2}\omega$ is independent of the choice of the multivector fields U_1 and U_2 associated with the given Hamiltonian forms α_1 and α_2. This prompts us to introduce the following bracket of Hamiltonian forms:

$$\{\alpha_1, \alpha_2\} = -i_{U_1\wedge U_2}\omega. \tag{4.3}$$

From the previous discussion it follows that this bracket is well-defined and determines a unique element of $\mathcal{H}^{(\ell_1+\ell_2+1)-k}(M)$, or equals zero in case $\ell_1 + \ell_2 + 1 < k$. From the definition (4.3) one infers that $\{\,,\,\}$ induces a $\Re$-bilinear operation on $\mathcal{H}^*(M)$ and that for any $\alpha_i \in \mathcal{H}^{\ell_i}(M)$ $(i = 1, 2)$

$$\{\alpha_1, \alpha_2\} = (-1)^{(k-\ell_1)(k-\ell_2)}\{\alpha_2, \alpha_1\}.$$

In addition, combining (4.1) and (4.3) it is seen that for any three Hamiltonian forms $\alpha_i \in \mathcal{H}^{\ell_i}(M)$, with corresponding Hamiltonian multivector fields $U_i \in \mathcal{V}^H_{k-\ell_i}(M)$ $(i = 1, 2, 3)$, $d\{\{\alpha_1, \alpha_2\}, \alpha_3\} = i_{[[U_1, U_2], U_3]}\omega$. Using the graded Jacobi identity for the Schouten-Nijenhuis bracket (cf. [23]) it is found that the cyclic sum $(-1)^{(k-\ell_1)(k-\ell_3)}\{\{\alpha_1, \alpha_2\}, \alpha_3\} +$ cycl. yields a closed form.

By construction, the bracket of Hamiltonian forms vanishes whenever one of its arguments is a closed form. Consequently, the algebraic structure induced by $\{\ ,\ \}$ on $\mathcal{H}^*(M)$ passes to the quotient space $\tilde{\mathcal{H}}^*(M)$, with product defined by

$$\{\tilde{\alpha}, \tilde{\beta}\} = \widetilde{\{\alpha, \beta\}}.$$

This bracket satisfies, among others, the graded Jacobi identity

$$(-1)^{(k-\ell_1)(k-\ell_3)}\{\{\widetilde{\alpha_1}, \widetilde{\alpha_2}\}, \widetilde{\alpha_3}\} + cycl. = 0,$$

for arbitrary $\widetilde{\alpha_i} \in \tilde{\mathcal{H}}^{\ell_i}(M)$. We now put $\mathcal{K}^s(M) = \tilde{\mathcal{H}}^{k-s-1}(M)$, so that $\tilde{\mathcal{H}}^*(M)$ can be rewritten as

$$\tilde{\mathcal{H}}^*(M) = \bigoplus_{s \geq 0} \mathcal{K}^s(M),$$

whereby we adopt the convention that $\mathcal{K}^s(M) = \{0\}$ whenever $s > k - 1$. We then introduce the following grading on $\tilde{\mathcal{H}}^*(M)$: for every nonzero homogeneous element $\tilde{\alpha} \in \mathcal{K}^s(M)$ let

$$|\tilde{\alpha}| = s = k - (\text{order of } \alpha) - 1. \tag{4.4}$$

Next, we slightly modify the product on $\tilde{\mathcal{H}}^*(M)$ by defining a new bracket $\{\ ,\ \}^*$ according to

$$\{\tilde{\alpha}, \tilde{\beta}\}^* = (-1)^{|\tilde{\alpha}|}\{\tilde{\alpha}, \tilde{\beta}\}.$$

Putting the various pieces together, it is now a staightforward matter to verify that the bracket $\{\ ,\ \}^*$, apart from being a $\Re$- bilinear operator, satisfies the following properties:

$$|\{\widetilde{\alpha_1}, \widetilde{\alpha_2}\}^*| = |\widetilde{\alpha_1}| + |\widetilde{\alpha_2}|,$$

i.e. the grading operator $|\,.\,|$ acts additively with respect to the bracket operation on $\tilde{\mathcal{H}}^*(M)$, and for arbitrary $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} \in \tilde{\mathcal{H}}^*(M)$ we have

$$\{\tilde{\alpha}, \tilde{\beta}\}^* = -(-1)^{|\tilde{\alpha}||\tilde{\beta}|}\{\tilde{\beta}, \tilde{\alpha}\}^*,$$

and

$$(-1)^{|\tilde{\alpha}||\tilde{\gamma}|}\{\{\tilde{\alpha}, \tilde{\beta}\}, \tilde{\gamma}\} + (-1)^{|\tilde{\beta}||\tilde{\alpha}|}\{\{\tilde{\beta}, \tilde{\gamma}\}, \tilde{\alpha}\} + (-1)^{|\tilde{\gamma}||\tilde{\beta}|}\{\{\tilde{\gamma}, \tilde{\alpha}\}, \tilde{\beta}\} = 0.$$

Summarising, we have thus shown that the following result holds.

THEOREM 4.1. *$(\bigoplus_{s \geq 0} \mathcal{K}^s(M), \{\ ,\ \}^*)$ is a graded Lie algebra (with grading defined by (4.4)). Moreover, the subalgebra consisting of the equivalence classes of Hamiltonian forms of order $k - 1$ (i.e. the elements $\tilde{\alpha}$ for which $|\tilde{\alpha}| = 0$), is a Lie algebra.*

5. The tangent multisymplectic structure

It is well-known that, given a symplectic manifold (M,ω), the symplectic form induces a symplectic structure on the tangent bundle TM. This fact plays a crucial role in Tulczyjew's approach to Hamiltonian dynamics ([19, 22]). In particular, he has shown that a (locally) Hamiltonian vector field can be fully characterised by a Lagrangian section of the tangent bundle with respect to this induced symplectic structure. More generally, an arbitrary Lagrangian submanifold of the tangent bundle of a symplectic manifold can be interpreted as a "generalised" (or implicit) Hamiltonian dynamical system (see e.g. [15, 19]). The "tangent symplectic structure" can be introduced in several equivalent ways. For our present needs it suffices to point out that it can be defined as the complete lift ω^c of the given symplectic form ω. (We refer the reader to [13, 24] for the basic definitions and properties of complete and vertical lifts of vector fields and forms on a manifold.)

In this section we will show that the above picture extends to Hamiltonian vector fields on a general multisymplectic manifold. We start with the following important result.

PROPOSITION 5.1. *Let (M,ω) be a multisymplectic manifold of order $k+1$. Then, the complete lift ω^c defines a multisymplectic structure of order $k+1$ on TM.*

Proof. First, since the complete lift operation of forms commutes with exterior differentiation, it immediately follows that $d\omega^c = 0$. Next, assume $i_{\tilde{X}}\omega^c = 0$ for some vector field $\tilde{X}$ on TM. We then have to prove that $\tilde{X} = 0$, and we will do this pointwise.

At any given point of TM, say $u \in T_xM$, $\tilde{X}(u)$ can always be written as

$$\tilde{X}(u) = X_1^c(u) + X_2^v(u) \tag{5.1}$$

where X_1^c and X_2^v denote the complete and vertical lifts of some (local) vector fields X_1 and X_2 on M. Taking into account that $i_{X_1^c}\omega^c = (i_{X_1}\omega)^c$ and $i_{X_2^v}\omega^c = (i_{X_2}\omega)^v$, we find

$$0 = (i_{\tilde{X}}\omega^c)(u) = (i_{X_1}\omega)^c(u) + (i_{X_2}\omega)^v(u). \tag{5.2}$$

We now let this act on k arbitrary vertical tangent vectors at u, which can always be represented as the vertical lifts $(u_1)^v, \dots, (u_k)^v$ of k vectors $u_i \in T_xM$. Since the vertical lift of a differential form on M yields a basic form on TM (the vertical lift of forms being essentially the pull-back under the tangent bundle projection) it follows that

$$\omega(X_1(x), u_1, \dots, u_k) = 0,$$

and this should hold for any k tangent vectors $u_i \in T_xM$. The nondegeneracy of ω then entails $X_1(x) = 0$ and, hence, $X_1^c(u)$ is vertical. In view of (5.1), the latter implies that $\tilde{X}(u)$ is vertical, i.e. $\tilde{X}(u) = Y^v(u)$ for some vector field Y on M. With (5.2) we now find that $(i_Y\omega)^v(u) = 0$, from which it readily follows that $Y(x) = 0$. By definition of the vertical lift operation, this finally yields $\tilde{X}(u) = 0$, which completes the proof. ∎

The multisymplectic structure defined by ω^c on the tangent bundle of a multisymplectic manifold (M,ω) will be called the *tangent multisymplectic structure*. If, in local coordinates (x^i) on M, ω is given by $\omega = a_{i_1 \dots i_{k+1}} dx^{i_1} \wedge \cdots \wedge dx^{i_{k+1}}$, then, in the natural bundle coordinates (x^i, v^i) on TM, ω^c reads

$$(5.3)\quad \omega^c = \frac{\partial a_{i_1 \dots i_{k+1}}}{\partial x^j} v^j \, dx^{i_1} \wedge \cdots \wedge dx^{i_{k+1}} + \sum_{j=1}^{k+1} a_{i_1 \dots i_{k+1}} dx^{i_1} \wedge \cdots \wedge dv^{i_j} \wedge \cdots \wedge dx^{i_{k+1}}.$$

In analogy with the symplectic case, we now have the following geometrical characterisation of locally Hamiltonian vector fields on multisymplectic manifolds.

THEOREM 5.2. *A vector field X on a multisymplectic manifold (M, ω) of order $k+1$ is locally Hamiltonian iff its image $X(M)$ is a k-Lagrangian submanifold of the tangent multisymplectic structure (TM, ω^c).*

Proof. For an arbitrary vector field X on M we have

$$(5.4)\qquad L_X \omega = X^* \omega^c,$$

and since $X^* \omega^c$ can be identified with the pull-back of ω^c to $X(M) = \mathrm{Im} X$, this already tells us that if $\mathrm{Im} X$ is k-Lagrangian (and thus, in particular, $X^* \omega^c = 0$), then X is locally Hamiltonian.

Conversely, assume $L_X \omega = 0$. From (5.4) we then learn that the pull- back of ω^c to $\mathrm{Im} X$ vanishes and, hence, $\mathrm{Im} X$ is necessarily k-isotropic. By definition of k-Lagrangian submanifold (cf. Section 2) it therefore remains to be shown that at each point $u \in \mathrm{Im} X$, $(T_u(\mathrm{Im} X))^{\perp, k} \subset T_u(\mathrm{Im} X)$. This part of the proof can be most easily accomplished in coordinates. Assume $u = X(x)$ and let (x^i) denote local coordinates in a neighbourhood of $x \in M$. Consider then the following tangent vectors at u:

$$A_i = X_* \left(\left. \frac{\partial}{\partial x^i} \right|_x \right) = \left. \frac{\partial}{\partial x^i} \right|_u + \frac{\partial X^j}{\partial x^i}(x) \left. \frac{\partial}{\partial v^j} \right|_u, \quad B_i = \left. \frac{\partial}{\partial v^i} \right|_u.$$

The vectors A_i constitute a basis of $T_u(\mathrm{Im} X)$ and (A_i, B_i) is a basis of $T_u(TM)$. In terms of the latter, an arbitrary tangent vector $C \in T_u(TM)$ reads $C = a^i A_i + b^i B_i$, for some real constants a^i, b^i. Using the local expression (5.3) for ω^c, and taking into account the fact that ω itself is nondegenerate, a straightforward computation reveals that $C \in (T_u(\mathrm{Im} X))^{\perp, k}$ implies $b^i = 0$ for all i and thus, $C \in T_u(\mathrm{Im} X)$. This completes the proof of the theorem. ∎

It is to be expected that for the study of the geometry of multisymplectic manifolds one will have to invoke the full algebra of Hamiltonian multivector fields. In that respect, it will be of interest, among others, to find an appropriate extension of the results described in this section to general Hamiltonian multivector fields. This, and other aspects concerning the Hamiltonian structures on a multisymplectic manifold, will be discussed in forthcoming work.

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